Topologia Algebrica 1. Teorie d'omologia. Note.

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Homology Theories. Notes

1. Singular homology

1.1. The singular cubical set of a space

- **Top**: the category of topological spaces and continuous mappings (= maps).
- $\mathbf{I} = [0, 1]$: the *standard interval*, with euclidean topology.
- Basic structure: two *faces* (δ^0, δ^1) and a *degeneracy* (ε), linking it with the singleton $\mathbf{I}^0 = \{*\}$

(1) $\delta^\alpha : \{*\} \rightleftarrows \mathbf{I} : \varepsilon$ $\qquad(\alpha = 0, 1)$,

$\delta^0(*) = 0, \qquad \delta^1(*) = 1, \qquad \varepsilon(t) = *.$

- *Faces* and *degeneracies* of the standard cubes $\mathbf{I}^n$ (for $\alpha = 0, 1;\ i = 1,..., n$)

(2) $\delta^\alpha_i = \mathbf{I}^{i-1}\times\delta^\alpha\times \mathbf{I}^{n-i}: \mathbf{I}^{n-1} \to \mathbf{I}^n,$ $\qquad \delta^\alpha_i(t_1,..., t_{n-1}) = (t_1,..., t_{i-1}, \alpha,..., t_{n-1}),$

$\varepsilon_i = \mathbf{I}^{i-1}\times\varepsilon\times\mathbf{I}^{n-i}: \mathbf{I}^n \to \mathbf{I}^{n-1},$ $\qquad \varepsilon_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n).$

- They satisfy the *co-cubical* relations

(3) $\delta^\beta_j.\delta^\alpha_i = \delta^\alpha_i.\delta^\beta_{j-1}\ (i < j),$ $\qquad \varepsilon_i.\varepsilon_j = \varepsilon_{j-1}.\varepsilon_i\ (i < j),$

$\varepsilon_j.\delta^\alpha_i = \delta^\alpha_{i-1}.\varepsilon_j\ (j < i),$ $\quad$ *or* $\ \mathrm{id}\ (j = i),$ $\quad$ *or* $\ \delta^\alpha_i.\varepsilon_{j-1}\ (j > i).$

- This produces, for every topological space X, a *cubical set* $\square X = ((\square_n X), (\partial^\alpha_i), (e_i))$

(4) $\square_n X = \mathbf{Top}(\mathbf{I}^n, X),$ $\qquad$ the set of *singular n-cubes* $a: \mathbf{I}^n \to X$ of the space X,

$\partial^\alpha_i = \partial^\alpha_{ni}: \square_n X \to \square_{n-1}X,$ $\qquad \partial^\alpha_i(a) = a.\delta^\alpha_i: \mathbf{I}^{n-1} \to X,$

$e_i = e_{ni}: \square_{n-1}X \to \square_n X,$ $\qquad e_i(a) = a.\varepsilon_i: \mathbf{I}^n \to X,$ $\qquad (\alpha = 0, 1;\ i = 1,..., n).$

- In general: a *cubical set* $K = ((K_n), (\partial^\alpha_i), (e_i))$ is a sequence of sets K_n $(n \geq 0)$, together with mappings, called *faces* (∂^α_i) and *degeneracies* (e_i)

(5) $\partial^\alpha_i = \partial^\alpha_{ni}: K_n \to K_{n-1},$ $\qquad e_i = e_{ni}: K_{n-1} \to K_n$ $\qquad (\alpha = 0, 1;\ i = 1,..., n).$

satisfying the *cubical* relations

(6) $\partial^\alpha_i.\partial^\beta_j = \partial^\beta_{j-1}.\partial^\alpha_i\ (i < j),$ $\qquad e_j.e_i = e_i.e_{j-1}\ (i < j),$

$\partial^\alpha_i.e_j = e_j.\partial^\alpha_{i-1}\ (j < i),$ $\quad$ *or* $\ \mathrm{id}\ (j = i),$ $\quad$ *or* $\ e_{j-1}.\partial^\alpha_i\ (j > i).$

Elements of K_n are called n-*cubes*; *vertices* and *edges* for $n = 0$ or 1, respectively. Every n-cube $a\in K_n$ has 2^n vertices: $\partial^\alpha_1\partial^\beta_2\partial^\gamma_3(a)$ for $n = 3$.

A *morphism* of cubical sets $f = (f_n): K \to L$ is a sequence of mappings $f_n: K_n \to L_n$ commuting with faces and degeneracies. Cubical sets and their morphisms form a category **Cub**.

- The functor $\square : \mathbf{Top} \to \mathbf{Cub}$ acts as follows on the map $f: X \to Y$

(7) $\square f\colon \square X \to \square Y$, $(\square f)_n\colon a \mapsto f.a\colon \mathbf{I}^n \to Y$.

1.2. The chain complex of a cubical set and the singular chain complex of a space

- Degenerate elements of a cubical set K: all elements of type $e_i(a)$

(1) $\mathrm{Deg}_n K = \cup_i \mathrm{Im}(e_i\colon K_{n-1} \to K_n)$, $\mathrm{Deg}_0 K = \emptyset$.

- Because of the cubical relations, we have (for i = 1,..., n)

(2) $a \in \mathrm{Deg}_n K \;\Rightarrow\; (\partial_i^\alpha a \in \mathrm{Deg}_{n-1} K$ *or* $\partial_i^- a = \partial_i^+ a)$, $e_i(\mathrm{Deg}_{n-1} K) \subset \mathrm{Deg}_n K$.

- The cubical set K determines a (*normalised*) *chain complex* $C_*(K)$, i.e. a sequence of abelian groups and homomorphisms (called *boundaries*, or *differentials*)

(3) $$\ldots \longrightarrow C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \ldots \longrightarrow C_1(K) \xrightarrow{\partial_1} C_0(K)$$

$$\partial_n.\partial_{n+1} = 0 \quad (n > 0),$$

defined as follows:

(4) $C_n(K) = (\mathbf{Z}K_n)/(\mathbf{Z}\mathrm{Deg}_n K) = \mathbf{Z}\bar{K}_n$ $(\bar{K}_n = K_n \setminus \mathrm{Deg}_n K)$,

$\partial_n\colon C_n(K) \to C_{n-1}(K)$, $\partial_n(\hat{a}) = \Sigma_{i,\alpha} (-1)^{i+\alpha} (\partial_i^\alpha a)^\wedge$ $(a \in K_n)$,

($\mathbf{Z}S$ is the free abelian group on the set S; $\hat{a}$ is the class of the n-cube a up to degenerate cubes; but we will write the normalised class $\hat{a}$ as a, identifying all degenerate cubes with 0.)

Hint. To prove that $\partial_n.\partial_{n+1} = 0$ one uses the cubical relations for faces: $\partial_i^\alpha.\partial_j^\beta = \partial_{j-1}^\beta.\partial_i^\alpha$ $(i < j)$. □

- In general: a *chain complex* $A = ((A_n), (\partial_n))$ of abelian groups is a sequence as above, with $\partial_n.\partial_{n+1} = 0$ $(n > 0)$. A *morphism* $\varphi\colon A \to B$ of chain complexes is a sequence of homomorphisms $\varphi_n\colon A_n \to B_n$ commuting with differentials: $\partial_n.\varphi_n = \varphi_{n-1}.\partial_n$ $(n > 0)$. They form the category $C_*\mathbf{Ab}$ of chain complexes of abelian groups.

- The functor $C_*\colon \mathbf{Cub} \to C_*\mathbf{Ab}$ acts on the morphism $f = (f_n)\colon K \to L$ by linear extension

(5) $f_\# = C_*(f)\colon C_*(K) \to C_*(L)$, $f_{\#n}(a) = f_n(a)$.

- Composing with the functor $\square\colon \mathbf{Top} \to \mathbf{Cub}$, we get the *singular chain complex* of a space, or *complex of singular chains*, written again C_*

(6) $C_*\colon \mathbf{Top} \to C_*\mathbf{Ab}$, $C_*(X) = C_*(\square X)$, $f_{\#n}(a) = f.a$ $(a\colon \mathbf{I}^n \to X)$.

1.3. Homology

- The homology functor of chain complexes: the group of n-*cycles* modulo the group of n-*boundaries*

(1) $H_n\colon C_*\mathbf{Ab} \to \mathbf{Ab}$ $(n \geq 0)$,

$H_n(A) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$, $H_n(\varphi)[z] = [\varphi_n z]$.

- Composing with the previous functors, we have the *singular homology* of a space

(2) $$\mathbf{Top} \xrightarrow{\square} \mathbf{Cub} \xrightarrow{C_*} C_*\mathbf{Ab} \xrightarrow{H_n} \mathbf{Ab}$$

H_n: **Top** $\to$ **Ab** $\qquad H_n(X) = H_n(C_*(\square X)) \qquad (n \geq 0)$,

$H_n(f) = f_{*n}, \qquad f_{*n}[\Sigma_i \lambda_i a_i] = [\Sigma_i \lambda_i (fa_i)]$.

REMARKS. The category **Cub** has all limits and colimits and is cartesian closed.

- It is the presheaf category of functors $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* 2^n, together with the maps $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates.

1.4. Elementary results

- $H_n(X) \cong \oplus_{i\in I} H_n(X_i)$, where $(X_i)_{i\in I}$ is the family of path-connected components of the space X.
- $H_n(\varnothing) = 0 \quad (n \geq 0)$,
- $H_0(\{*\}) \cong \mathbf{Z}, \quad H_n(\{*\}) = 0 \quad (n > 0)$.

Proposition. If X is path-connected, non empty: $H_0(X) \cong \mathbf{Z}$, with $\varphi[\Sigma\, \lambda_i.x_i] = \Sigma\, \lambda_i$.

Hint. Use the *augmented* chain complex $\ldots \to C_1(X) \to C_0(X) \to \mathbf{Z}$ where $\partial_0(\Sigma\, \lambda_i.x_i) = \Sigma\, \lambda_i$; prove that ∂_0 is surjective and $\mathrm{Ker}(\partial_0) = \mathrm{Im}(\partial_1)$. Then φ is the induced isomorphism. □

1.5. Homotopy for topological spaces

- Two maps $f_0, f_1: X \to Y$ in **Top** are *homotopic* ($f_0 \simeq f_1$) if there is a map $F: \mathbf{I}\times X \to Y$ such that $F(\alpha, x) = f_\alpha(x)$, for all $x\in X$ ($\alpha = 0, 1$). This relation is a congruence of categories.
- Two spaces X, Y are homotopy equivalent ($X \simeq Y$) if there are maps $f: X \rightleftarrows Y :g$ with $gf \simeq idX$, $fg \simeq idY$.
- A space is said to be *contractible* if it is homotopy equivalent to $\{*\}$.

REMARKS. The quotient category Ho**Top** = **Top**/$\simeq$ has, by definition, the same objects and morphisms $[f]: X \to Y$ consisting of homotopy classes of maps; it is called *the homotopy category of spaces*. Two spaces are homotopy equivalent if and only if they are isomorphic objects *in* Ho**Top**.

1.6. Homotopy for chain complexes

- Two maps $\varphi, \psi: A \to B$ in C_***Ab** are *homotopic* ($\varphi \simeq \psi$) if there is a sequence of homomorphisms $\Phi_n: A_n \to B_{n+1}$ ($n \geq 0$) such that $\partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n = -\varphi_n + \psi_n$.
- This relation is a congruence of categories, in C_***Ab**.

Proposition [Homotopy Invariance of algebraic homology]. The functors H_n: C_***Ab** $\to$ **Ab** are *homotopy invariant*: if $\varphi \simeq \psi: A \to B$ then $H_n(\varphi) = H_n(\psi): H_n(A) \to H_n(B)$ (for all $n \geq 0$).

1.7. Homotopy Invariance of singular homology

Theorem. The functors H_n: **Top** $\to$ **Ab** are *homotopy invariant*: if $f \simeq g: X \to Y$ then $H_n(f) = H_n(g): H_n(X) \to H_n(Y)$ (for all $n \geq 0$).

Hint. Given a homotopy $F: \mathbf{I}\times X \to Y$ between $f, g: X \to Y$, one constructs a homotopy between the associated chain morphisms $C_*(X) \to C_*(Y)$

(1) $\Phi_n: C_n(X) \to C_{n+1}(Y)$, $\quad \Phi_n(a) = F.(\mathbf{I}\times a) \quad (a: \mathbf{I}^n \to X)$,

$\partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n = -C_n(f) + C_n(g)$. □

Corollary. If the spaces X, Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$ (for all $n \geq 0$).

Corollary. If the space X is contractible, then $H_n(X) \cong H_n(\{*\})$ (for all $n \geq 0$) and X is path-connected.

2. Computing singular homology

2.1. Exact sequences of abelian groups and chain complexes [*Homological Algebra*]

Definition. A sequence $\ldots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \ldots$ in **Ab** is *exact* at A_n if $\mathrm{Im}(f_{n+1}) = \mathrm{Ker}(f_n)$. It is *exact* if it is exact at every point. Examples:

- A chain complex A is exact at A_n if and only if $H_n(A) = 0$;
- $0 \to A \to 0$ is exact *in* $A \Leftrightarrow A = 0$.
- $0 \to A \xrightarrow{f} B \to 0$ is exact *in* $A \Leftrightarrow f$ is mono; *in* $B \Leftrightarrow f$ is epi; *in* A *and* $B \Leftrightarrow f$ is iso.
- $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence* if it is exact:
  (a) exact in A (f mono); (b) exact in B ($\mathrm{Im}(f) = \mathrm{Ker}(g)$); (c) exact in C (g epi).

- In $C_*\mathbf{Ab}$ we have the same definitions. Kernels and images are defined componentwise:
Given $\varphi: A \to B$ in $C_*\mathbf{Ab}$:

(1) $\mathrm{Ker}(\varphi) = ((\mathrm{Ker}(\varphi_n), (\partial_n))$, $\quad \mathrm{Im}(\varphi) = ((\mathrm{Im}(\varphi_n), (\partial_n))$,

where the differentials are the restriction of the differentials of A.

2.2. The homology sequence of a short exact sequence of chain complexes

[*Homological Algebra*]

Theorem. Given a short exact sequence of chain complexes

(1) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

there is an exact sequence of homology groups

(2) $\ldots \to H_n(A) \xrightarrow{f_{*n}} H_n(B) \xrightarrow{g_{*n}} H_n(C) \xrightarrow{\Delta_n} H_{n-1}(A) \ldots \xrightarrow{\Delta_1} H_0(A) \xrightarrow{f_{*0}} H_0(B) \xrightarrow{g_{*0}} H_0(C) \to 0$

where the *connective* homomorphism $\Delta_n: H_n(C) \to H_{n-1}(A)$ is defined as follows

(3) $\Delta_n[c] = [a]$,

where $c \in Z_n(C)$, $a \in Z_{n-1}(A)$ and $\exists\, b \in C_n$ such that $g_n(b) = c$, $\partial_n b = f_{n-1}(a)$.

The sequence (2) is *natural* for *morphisms* of the sequence (1): a *translation* (u, v, w) of the sequence (1), by commutative squares, induces a *translation* $(..., u_{*n}, v_{*n}, w_{*n},...)$ of the sequence (2), by commutative squares.

Hint. Easy proof, by 'diagram chasing'. □

2.3. Subdivision.

- This is one of the main results, for singular homology.
- Let X be a topological space and $\mathcal{U} = (U_i)$ a 'generalised open cover' of X: $X = \bigcup \mathrm{int}(U_i)$.
- $C_*(X; \mathcal{U})$: denotes the subcomplex of $C_*(X)$ of $\mathcal{U}$-*small chains*

  generated by the cubes $a: \mathbf{I}^n \to X$ whose image is contained in some U_i.

Subdivision Theorem. In these hypotheses, the inclusion $j: C_*(X; \mathcal{U}) \to C_*(X)$ induces isomorphism in homology: $H_n(X; \mathcal{U}) \to H_n(X)$.

Hint. The idea is to subdivide *cubes*, replacing them by $\mathcal{U}$-small *chains*.

(A) We construct the *subdivision operator*, a natural morphism of chain complexes

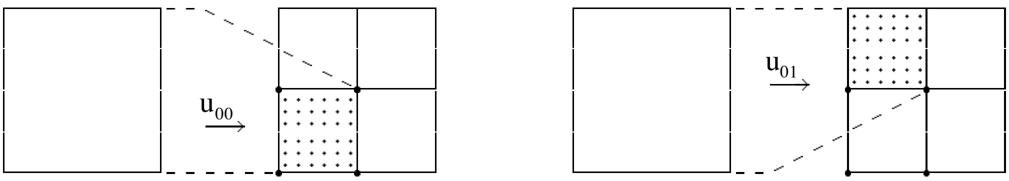
(1) $\mathrm{Sd}: C_*(X) \to C_*(X)$, $\qquad \mathrm{Sd}_n(a) = \Sigma_v\, a.u_v \qquad (v \in \{0, 1\}^n)$,

$u_v: \mathbf{I}^n \to \mathbf{I}^n$, $\qquad u_v(t) = (t + v)/2$,

$\partial_n.\mathrm{Sd}_n = \mathrm{Sd}_{n-1}.\partial_n$,

which subdivides any n-cube into a *chain* of 2^n n-cubes, indexed on the vertices $v \in \{0, 1\}^n$ of $\mathbf{I}^n$

(2)



$u_{(0,0)}(t_1, t_2) = (t_1/2, t_2/2)$, $\qquad u_{(0,1)}(t_1, t_2) = (t_1/2, (t_2 + 1)/2)$.

(B) This morphism Sd is homotopic to the identity, by a chain homotopy $\varphi = (\varphi_n)$

(3) $\varphi_n: C_n(X) \to C_{n+1}(X)$, $\qquad \varphi_n(a) = (-1)^{n+1}\, \Sigma_v\, a.\eta_v \qquad (v \in \{0, 1\}^n)$,

$\mathrm{Sd}_n - \mathrm{id} = \partial_{n+1}\varphi_n + \varphi_{n-1}\partial_n$,

obtained by means of a suitable family of maps $\eta_v: \mathbf{I}^{n+1} \to \mathbf{I}^n$ (cf. Massey [1980]). Note that:

(4) $\varphi_n(C_n(X; \mathcal{U})) \subset C_{n+1}(X; \mathcal{U})$.

(C) The induced homomorphism $j_n: H_n(X; \mathcal{U}) \to H_n(X)$ is *surjective*.

- For every cube $a: \mathbf{I}^n \to X$, consider the following open cover of $\mathbf{I}^n$

(5) $V_i = a^{-1}(\mathrm{int}(U_i)) \qquad (i \in I)$

- Applying the Lebesgue Lemma on open covers of compact metric spaces, there is some $k \in \mathbf{N}$ such that any 'subcube' K of $\mathbf{I}^n$ with edge 2^{-k} is contained in *some* V_{i_K}, whence

(6) $a(K) \subset a(V_{i_K}) \subset \mathrm{int}(U_{i_K}) \subset U_{i_K}$, $\qquad \mathrm{Sd}^k(a) \in C_n(X; \mathcal{U})$.

- Take a cycle $z \in C_n(X)$. For every cube $a: \mathbf{I}^n \to X$ which appears in z, we can proceed as above. There is thus some $k \in \mathbf{N}$ such that $z' = Sd^k(z) \in C_n(X; \mathcal{U})$. The composed chain homotopy $\psi: Sd^k \simeq id: C_*(X) \to C_*(X)$ gives

(7) $z - z' = \partial_{n+1}\psi_n(z) + \psi_{n-1}\partial_n(z) = \partial_{n+1}\psi_n(z)$ (in $C_n(X)$),

$[z] = j_n[z']$, $[z'] \in H_n(X; \mathcal{U})$.

(D) The induced homomorphism $j_n: H_n(X; \mathcal{U}) \to H_n(X)$ is *injective*.

- Take a cycle $z \in C_n(X; \mathcal{U}))$ which annihilates in $H_n(X)$:

(8) $z = \partial c$, for some chain $c \in C_{n+1}(X)$.

- As above: there is some $k \in \mathbf{N}$ such that $c' = Sd^k(c) \in C_{n+1}(X; \mathcal{U}))$.

- The composed chain homotopy $\psi: Sd^k \simeq id: C_*(X) \to C_*(X)$ gives

(9) $c - c' = \partial\psi(c) + \psi\partial(c) = \partial\psi(c) + \psi(z)$ (in $C_{n+1}(X)$),

$z = \partial c = \partial c' - \partial\psi(z)$ is a boundary in $C_n(X; \mathcal{U})$,

because φ takes $C_n(X; \mathcal{U})$ into $C_{n+1}(X; \mathcal{U})$, by (4), whence also its *composite* ψ does. □

2.4. The exact sequence of Mayer-Vietoris

Theorem. Let X be a topological space, U and V subsets of X such that $X = int(U) \cup int(V)$ and $A = U \cap V$. There is an exact sequence of singular homology groups

(1) $$\ldots \to H_n(A) \xrightarrow{h_n} H_n(U) \oplus H_n(V) \xrightarrow{k_n} H_n(X) \xrightarrow{\Delta_n} H_{n-1}(A) \ldots$$

$$\ldots \to H_0(A) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(C) \to A$$

where (writing $i: A \subset U$, $j: A \subset V$, $u: U \subset X$, $v: V \subset X$ the inclusion mappings)

(2) $h_n = (i_{*n}, j_{*n})$, $h_n[z]_A = ([z]_U, [z]_V)$,

$k_n = [u_{*n}, - v_{*n}]$, $k_n([z]_U, [w]_V) = [z]_X - [w]_X = [z - w]_X$,

$\Delta_n[z] = [\partial c]$ $(z \in Z_n(X),\ z = c + c',\ c \in C_n(U),\ c' \in C_n(V))$.

The sequence is natural for continuous mappings $f: X \to X'$, where $X' = int(U') \cup int(V')$ and $f(U) \subset U'$, $f(V) \subset V'$.

Hint. The proof follows from two theorems:

(A) the Subdivision Theorem (2.3), applied to the 'generalised open cover' $\mathcal{U} = (U, V)$ of X;

(B) the homology sequence of a short exact sequence of chain complexes (2.2), applied to:

(3) $$0 \longrightarrow C_n(A) \xrightarrow{h} C_n(U) \oplus C_n(V) \xrightarrow{k} C_n(X; \mathcal{U}) \longrightarrow 0$$

$h_n = (i_{\#n}, j_{\#n})$, $k_n = [u_{\#n}, - v_{\#n}]$. □

2.5. The homology of the spheres; other computations

Theorem A. For $n > 0$: $H_k(\mathbf{S}^n) \cong \mathbf{Z}$ $(k = 0, n)$; $H_k(\mathbf{S}^n) = 0$ (otherwise).

Hint. By induction. Apply Mayer-Vietoris to $\mathbf{S}^n$, with open subsets $U = \mathbf{S}^n \setminus \{S\}$, $V = \mathbf{S}^n \setminus \{N\}$ where: $N = (0,..., 0, 1)$, $S = (0,..., 0, -1)$. □

Theorem B. There is an isomorphism $\Delta_n: H_n(\mathbf{S}^n) \to H_{n-1}(\mathbf{S}^{n-1})$ $(n \geq 0)$ which is natural for maps $f: \mathbf{S}^n \to \mathbf{S}^n$ such that: $f(\mathbf{S}^{n-1}) \subset \mathbf{S}^{n-1}$, $f(N) = N$, $f(S) = S$.

Hint. Use the naturality of the M-V sequence on f, since: $f(U) \subset U$, $f(V) \subset V$.

We have two commutative squares (where $A = U \cap V$; g, h are restrictions of f; $i: \mathbf{S}^{n-1} \subset A$)

$$(1) \qquad \begin{array}{ccccc} H_n(\mathbf{S}^n) & \xrightarrow{\Delta_n} & H_{n-1}(A) & \xleftarrow[\cong]{i_{*n}} & H_{n-1}(\mathbf{S}^{n-1}) \\ {\scriptstyle f_{*n}}\downarrow & & \downarrow{\scriptstyle g_{*n}} & & \downarrow{\scriptstyle h_{*n}} \\ H_n(\mathbf{S}^n) & \xrightarrow[\Delta_n]{} & H_{n-1}(A) & \xleftarrow[i_{*n}]{} & H_{n-1}(\mathbf{S}^{n-1}) \end{array}$$ □

- Other computations: using the Mayer-Vietoris sequence and homotopy invariance, one computes easily the homology of: the torus, the Klein bottle, the projective plane, etc. For some computations one should use the notion of split exact sequence (2.9).

2.6. The degree of an endomap of a sphere

Given a map $f: \mathbf{S}^n \to \mathbf{S}^n$, the associated endomorphism of $H_n(\mathbf{S}^n) \cong \mathbf{Z}$ is the multiplication by a number $\deg(f) \in \mathbf{Z}$

(1) $f_{*n}: H_n(\mathbf{S}^n) \to H_n(\mathbf{S}^n)$, $\qquad [z] \mapsto \deg(f).[z]$.

Properties:

- $\deg(\mathrm{id}\mathbf{S}^n) = 1$, $\qquad \deg(gf) = \deg(g).\deg(f)$,
- $f \simeq g \Rightarrow \deg(f) = \deg(g)$,
- $\deg(T) = (-1)^{n+1}$, where $T: \mathbf{S}^n \to \mathbf{S}^n$ is the *antipodal map* $(T(x) = -x)$,
- if $f(x) \neq Tg(x)$, $\forall\, x \in \mathbf{S}^n$, then $f \simeq g$ and $\deg(f) = \deg(g)$.

2.7. Applications

(A) Theorem (The invariance of dimension). If $\mathbf{R}^m$ and $\mathbf{R}^n$ are homeomorphic, than $m = n$.

Hint. Use the Alexandroff compactification and H_m. □

(B) Theorem (Retracts). The sphere $\mathbf{S}^n$ is not a retract of $\mathbf{R}^{n+1}$ or $\mathbf{B}^{n+1}$.

Hint. Suppose, for a contradiction, that it is a retract and use H_n. □

(C) Theorem (The Brouwer fixed-point theorem). Every map $f: \mathbf{B}^n \to \mathbf{B}^n$ has at least a fixed point.

Hint. Suppose, for a contradiction, that f has no fixed points; construct a retraction of $\mathbf{S}^n \subset \mathbf{B}^{n+1}$. □

(D) Theorem (Vector fields on spheres). If $n > 0$ is *even*, every tangent vector field on $\mathbf{S}^n$ annihilates at least at a point.

Hint. Suppose that $\mathbf{S}^n$ has a tangent vector field which never annihilates. Then, there is a map $f: \mathbf{S}^n \to \mathbf{S}^n$ with $f(x)$ orthogonal to x, everywhere. It follows that $f(x) \neq \pm x$; by 2.6, $f \simeq \mathrm{id}: \mathbf{S}^n \to \mathbf{S}^n$

and $f \simeq T: \mathbf{S}^n \to \mathbf{S}^n$ (where $T(x) = -x$ is the antipodal map). Thus $\deg(T) = \deg(id) = 1$; but we know that $\deg(T) = (-1)^{n+1}$ (2.6), whence n must be odd. □

(E) Remark. If $n > 0$ is *odd*, the following map

(1) $f: \mathbf{S}^n \to \mathbf{S}^n$, $\qquad f(x_1,..., x_{n+1}) = (-x_2, x_1, -x_4, x_3,..., -x_{n+1}, x_n)$.

has $f(x) \perp x$, everywhere. Therefore, there *is* a tangent vector field on $\mathbf{S}^n$ which never annihilates.

(F) Theorem (Intermediate Value Theorem on the Cube). Let $f: \mathbf{I}^n \to \mathbf{I}^n$ be a continuous mapping which sends each (n–1)-dimensional face into itself. Then f is surjective and sends each face (of any dimension) *onto* itself.

Hint. This statement is trivial for $n = 0$ (and amounts to the classical Intermediate Value Theorem for $n = 1$). If the statement holds for $n-1 \geq 0$, it follows that f covers the boundary of $\mathbf{I}^n$. But f is homotopic to the identity; collapsing the boundary, it follows that f induces a map $\mathbf{S}^n \to \mathbf{S}^n$ that is still homotopic to the identity, whence surjective; finally the image of f also covers the interior of $\mathbf{I}^n$. □

(G) Theorem (Intermediate Value Theorem on the Ball). Let $f: \mathbf{B}^n \to \mathbf{B}^n$ be a continuous mapping which sends the boundary $\mathbf{S}^{n-1}$ into itself. If the restriction $f': \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is not homotopic to a constant map (or, equivalently, if its homological degree is non null), then f is surjective.

Hint. Suppose for a contradiction that f is not surjective, and use the fact that $\mathbf{S}^{n-1}$ is a deformation retract of the complement of any internal point in $\mathbf{B}^n$. □

2.8. Exercises (Paths and homology in degree 1)

Let $a, b: \mathbf{I} \to X$ be two path in the topological space X. Then

(a) the path a is a cycle $\Leftrightarrow$ a is a *loop*, i.e. $a(0) = a(1)$ $(\partial_1^0(a) = \partial_1^1(a))$;

(b) if a, b are *homotopic with fixed endpoints*, then $a - b$ is a boundary $(a - b \in B_1(X))$;

(c) if a, b are loops, *homotopic as loops* $\Rightarrow$ $[a] = [b]$ in $H_1(X)$,

(d) if a is a loop, *homotopic as a loop* to a constant loop $\Rightarrow$ $[a] = 0$ in $H_1(X)$,

(e) if the paths a, b are consecutive $(a(1) = b(0))$ $\Rightarrow$ $a + b - a*b$ is a boundary,

(f) if $\tilde{a}$ is the *reversed path* $(\tilde{a}(t) = a(1-t))$ $\Rightarrow$ $a + \tilde{a}$ is a boundary.

2.9. Split exact sequences [*Homological Algebra*]

A short sequence (m, q) is said to *split* if the following equivalent conditions hold:

$$(1) \qquad A \underset{p}{\overset{m}{\rightleftarrows}} B \underset{n}{\overset{q}{\rightleftarrows}} C$$

(a) (m, q) is short exact and the monomorphism m is a *section* ($\exists$ p: pm = idA),

(b) (m, q) is short exact and the epimorphism q is a *retraction* ($\exists$ n: qn = idC),

(c) there exist two homomorphisms p, n such that: pm = idA, qn = idC, mp + nq = idB.

- In this case, B is isomorphic to $A \oplus C$.

- If C is a *free* abelian group, the short exact sequence (1) necessarily splits.

3. Relative singular homology and homology theories

3.1. The Five Lemma [*Homological Algebra*]

Lemma. Given a commutative diagram of abelian groups (or R-modules), with exact rows

$$
\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle v'} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array} \qquad (1)
$$

if u, v, u', v' are isomorphisms, also w is an isomorphism.

Hint. By 'diagram chasing'. □

3.2. Pairs of spaces

- $\mathbf{Top}_2$: the category of *pairs of topological spaces*:
  - a *pair* (X, A) is a space X with a subspace A (the pair is read as: X *modulo* A),
  - a *map* $f: (X, A) \to (Y, B)$ is a map $f: X \to Y$ such that $f(A) \subset B$.
- **Top** is embedded in $\mathbf{Top}_2$ identifying the space X with the pair $(X, \emptyset)$.
- A *homotopy* $F: f_0 \simeq f_1: (X, A) \to (Y, B)$ *between maps of pairs* is a map of pairs such that:

(1) $F: (\mathbf{I}\times X, \mathbf{I}\times A) \to (Y, B)$, $\quad F(\alpha, x) = f_\alpha(x)$, for all $x \in X$ $\quad (\alpha = 0, 1)$;

  - this is equivalent to an ordinary homotopy $F: f_0 \simeq f_1: X \to Y$ such that $F(\mathbf{I}\times A) \subset B$.
- Terms of **Top** (objects, maps, homotopies, homology groups...) are called *absolute*;
- terms of $\mathbf{Top}_2$ are called *relative*.

3.3. Relative Singular Homology (with integral coefficients)

- $C_*: \mathbf{Top}_2 \to C_*\mathbf{Ab}$ (the functor of *relative chains*),
  - $C_*(X, A) = C_*(X)/C_*(A)$ (the *complex of (relative) chains* of the pair (X, A)),
  - $f_\#: C_*(X, A) \to C_*(Y, B)$, $\quad f_\#(\Sigma_i \lambda_i.\overline{a_i}) = \Sigma_i \lambda_i.\overline{(fa_i)}$ $\quad (f: (X, A) \to (Y, B))$.
- Note: a relative chain $\overline{c} \in C_n(X, A)$: is a *cycle* $\Leftrightarrow$ $\partial(\overline{c}) = 0$ $\Leftrightarrow$ $\partial c \in C_{n-1}(A)$,
  is a *boundary* $\Leftrightarrow$ $c \in \partial(C_{n+1}(X)) + C_n(A)$.
- $H_n: \mathbf{Top}_2 \to \mathbf{Ab}$ (*relative singular homology*),
  - $H_n(A, X) = H_n(C_*(X, A))$,
  - $f_{*n}: H_n(X, A) \to H_n(Y, B)$, $\quad f_*[\overline{c}] = [f_\#(\overline{c})]$.
- **Theorem**. This functor is *homotopy invariant*.

Hint. Given a homotopy $F: f \simeq g: (X, A) \to (Y, B)$ between maps of pairs (3.2), the homotopy between the chain morphisms $f_\#, g_\#: C_*(X) \to C_*(Y)$ constructed in 1.7 for the absolute case

(1) $\Phi_n: C_n(X) \to C_{n+1}(Y), \qquad \Phi_n(a) = F.(\mathbf{I}\times a) \qquad (a: \mathbf{I}^n \to X),$

takes $C_n(A)$ into $C_{n+1}(B)$, and induces a homotopy $\Psi: f_\# \simeq g_\#: C_*(X, A) \to C_*(Y, B)$. □

3.4. Theorem (The homology sequence of a pair)

For every pair of topological spaces (X, A), the following sequence is exact and natural

$$(1) \quad ... \to H_n(A) \xrightarrow{u_{*n}} H_n(X) \xrightarrow{v_{*n}} H_n(X, A) \xrightarrow{\Delta_n} H_{n-1}(A) ... \to H_0(X) \to H_0(X, A) \to 0$$

where $u: A \subset X$ is the inclusion, $v: (X, \emptyset) \to (X, A)$ is defined by the identity of X, and the *connective homomorphism* Δ_n is

$$(2) \quad \Delta_n: H_n(X, A) \longrightarrow H_{n-1}(A), \qquad \Delta_n[\bar{c}] = [\partial c] \qquad (\bar{c} \in C_n(X, A)).$$

Hint. By 2.2, the natural short exact sequence of chain complexes

$$(3) \quad 0 \to C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \to 0$$

yields the exact sequence (1), including its naturality and the formula (2). □

3.5. Theorem (Excision)

If X is a topological space, $U \subset A \subset X$ and $\mathrm{cl}(U) \subset \mathrm{int}(A)$, then the inclusion mapping

(1) $u: (X \setminus U, A \setminus U) \to (X, A)$,

induces isomorphism in homology: $u_{*n}: H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$.

Hint. By hypothesis, the family $\mathcal{U} = (X \setminus U, A)$ forms a 'generalised open cover' of X.

- By Subdivision (2.3), the inclusion $C_*(X; \mathcal{U}) \to C_*(X)$ induces an iso in homology.
- Applying the Five Lemma (3.1) to the homology sequences of the following commutative diagram with short exact rows

$$(2) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X; \mathcal{U}) & \longrightarrow & C_*(X; \mathcal{U})/C_*(A) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, A) & \longrightarrow & 0 \end{array}$$

it follows that also the canonical morphism $C_*(X; \mathcal{U})/C_*(A) \to C_*(X, A)$ induces iso in homology.

- Finally, by a Noether isomorphism

$$(3) \quad C_*(X; \mathcal{U})/C_*(A) = (C_*(X \setminus U) + C_*(A))/C_*(A) \cong$$

$$\cong C_*(X \setminus U)/(C_*(X \setminus U) \cap C_*(A)) = C_*(X \setminus U)/(C_*(A \setminus U) = C_*(X \setminus U, A \setminus U). \quad □$$

3.6. Definition of Homology Theories (The axioms of Eilenberg-Steenrod)

An (abstract) *homology theory* consists of the following data:

(a) for each pair of topological spaces (X, A), a sequence $H_n(X, A)$ of abelian groups,

(b) for each map $f: (X, A) \to (Y, B)$, a sequence $f_{*n}: H_n(X, A) \to H_n(Y, B)$ of homomorphisms,

(b) for each pair (X, A), a sequence $\Delta_n: H_n(X, A) \to H_{n-1}(A, \emptyset)$ of homomorphisms, so that the following axioms hold (writing $H_n(X)$ for $H_n(X, \emptyset)$):

- *Functoriality*. The data produce a sequence of functors H_n: **Top**$_2 \to$ **Ab**;
  - in other words: $(id(X, A))_{*n} = idH_n(X, A)$ and $(gf)_{*n} = g_{*n} \circ f_{*n}$ for f, g composable).

- *Naturality*. For f: (X, A) → (Y, B), the following diagram commutes

$$\text{(1)} \qquad \begin{array}{ccc} H_n(X, A) & \xrightarrow{f_{*n}} & H_n(Y, B) \\ \Delta_n \downarrow & & \downarrow \Delta_n \\ H_{n-1}(A) & \xrightarrow[f'_{*n}]{} & H_{n-1}(B) \end{array} \qquad \text{(f': A} \to \text{B is the restriction of f).}$$

- *Exactness*. For every pair (X, A), the following sequence is exact (u, v as in 3.4)

$$\text{(2)} \quad \dots \to H_n(A) \xrightarrow{u_{*n}} H_n(X) \xrightarrow{v_{*n}} H_n(X, A) \xrightarrow{\Delta_n} H_{n-1}(A) \dots$$

- *Homotopy Invariance*. If f, g: (X, A) → (Y, B) are homotopic, then

$$\text{(3)} \quad f_{*n} = g_{*n}: H_n(X, A) \to H_n(Y, B) \qquad (n \geq 0).$$

- *Excision*. If $U \subset A \subset X$ and $cl(U) \subset int(A)$, then the inclusion mapping $(X \setminus U, A \setminus U) \to (X, A)$ induces isomorphism in homology, in every degree.

- *Dimension*. $H_n(\{*\}) = 0$ for all $n \neq 0$.

3.7. Comments

- The abelian group $H_0(\{*\})$ is called: the *group of coefficients* of the theory.

- We have already proved (in 3.3-3.5) that *Relative Singular Homology is a homology theory* (in the previous sense) *with integral coefficients*: its group of coefficients is **Z** (up to isomorphism).

- For every abelian group G, we shall construct a *singular homology theory with coefficients in* G. This requires the use of tensor products (of abelian groups).

4. Tensor products [*Homological Algebra, Multilinear Algebra*]

4.1. Modules on a commutative ring

- R will always be a commutative ring with unit. R-modules and R-homomorphisms form the category R-**Mod**. In particular, R is a module on itself.

- Every abelian group has precisely one structure of **Z**-module; the two notions will be identified.

- If R is a field, modules are called vector spaces; this case will be considered at the end (4.8).

- Exact sequences have an obvious extension to R-modules.

- The *free* R-module on a set I can be constructed as a direct sum of copies of R

$$\text{(1)} \quad F(I) = R^{(I)} = \oplus_{i \in I} R,$$

with the obvious canonical basis: $e_i = (\delta_{ij})_{j\in I}$ $(i \in I)$, often identified with I.

- Exercise. An abelian group A has a structure of vector space on **Q** (rationals) if and only if it is *torsion-free* and *divisible* ($\forall a\in A$, $\forall n\in \mathbf{Z}$: $n \neq 0 \Rightarrow \exists!\, x\in A$: $nx = a$). Then, the structure is *unique*.

- Exercise. A structure of **Z**[X]-module on the abelian group A amounts to a homomorphism $A \to A$.

4.2. Tensor product of modules

- If A, B are R-modules, a mapping $\varphi\colon A\times B \to C$ is said to be *bilinear* (on R) if:

(1) $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$, (2) $\varphi(\lambda.a, b) = \lambda.\varphi(a, b)$,

(3) $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$, (4) $\varphi(a, \lambda.b) = \lambda.\varphi(a, b)$,

for all $a, a' \in A$; $b, b' \in B$; $\lambda \in R$ (this will be understood, below). For $R = \mathbf{Z}$, the properties (2) and (4) are a consequence of (1) and (3).

- The *tensor product* of A, B is an R-module $A \otimes_R B$ *equipped with a bilinear mapping* φ_0

$$(5)\qquad \begin{array}{ccc} A\times B & \xrightarrow{\ \varphi\ } & C \\ {\scriptstyle \varphi_0}\downarrow & \nearrow_{h} & \\ A\otimes_R B & & \end{array}$$

such that, for every bilinear mapping $\varphi\colon A\times B \to C$ there is *one and only one* R-homomorphism h such that $\varphi = h\varphi_0$.

- It is easy to show that the solution is determined up to isomorphism (a unique isomorphism coherent with the structural bilinear mappings).

- A solution exists: $A\otimes_R B = F(A\times B)/H(A, B)$, with $\varphi_0(a, b) = [(a, b)]$, where:

  - $F(A\times B)$ is the free R-module generated by the *set* $A\times B$ *(formal linear combinations of its elements)*
  - $H(A, B)$ is the sub-module of $F(A\times B)$ generated by all the elements of the following types:

(1') $(a + a', b) - (a, b) - (a', b)$, (2') $(\lambda.a, b) - \lambda.(a, b)$,

(3') $(a, b + b') - (a, b) - \varphi(a, b')$, (4') $(a, \lambda.b) - \lambda.(a, b)$.

- We write $a\otimes b = \varphi_0(a, b) = [(a, b)] \in A\otimes_R B$ (for $a\in A$, $b\in B$).

- Then $(a+a')\otimes b = a\otimes b + a'\otimes b$, $(\lambda.a)\otimes b = \lambda.(a\otimes b)$, etc.

- Every element of $A\otimes_R B$ can be written as a (finite) sum $\Sigma_i\, a_i\otimes b_i$, *NOT uniquely*.

4.3. Tensor product of homomorphisms

- The tensor product is a *functor in two variables* (covariant in both): given two R-homomorphisms $f\colon A \to A'$, $g\colon B \to B'$ there is a homomorphism

(1) $f\otimes g\colon A\otimes_R B \to A'\otimes_R B'$, $(f\otimes g)(a\otimes b) = f(a) \otimes g(b)$,

and this construction preserves identities and composition:

(2) $\mathrm{id}A \otimes \mathrm{id}B = \mathrm{id}(A\otimes_R B)$, $(f'\circ f) \otimes (g'\circ g) = (f'\otimes g')\circ(f\otimes g)$.

This functor is *bilinear* (additive and homogeneous in each variable):

(3) $(f+f')\otimes g = f\otimes g + f'\otimes g$, $\quad (\lambda.f)\otimes g = \lambda.(f\otimes g)$,

$f\otimes(g+g') = f\otimes g + f\otimes g'$, $\quad f\otimes(\lambda.g) = \lambda.(f\otimes g)$.

4.4. Exercises (for abelian groups: $R = \mathbf{Z}$)

- If m, n are coprime, then $\mathbf{Z}_m \otimes_{\mathbf{Z}} \mathbf{Z}_n = 0$;

- more generally, if $mA = 0$ and every element of B can be *divided by* m, then $A \otimes_{\mathbf{Z}} B = 0$.

- $\mathbf{Z}_m \otimes_{\mathbf{Z}} \mathbf{Q} = 0$;

- more generally, if T is a *torsion abelian group* and D is *divisible*, then $T \otimes_{\mathbf{Z}} D = 0$.

- Prove that $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is a vector space on $\mathbf{Q}$. The *rank* of an abelian group A is defined as

(1) $\mathrm{rk}(A) = \dim_{\mathbf{Q}} (A \otimes_{\mathbf{Z}} \mathbf{Q})$.

- In particular, a *finitely generated* abelian group A is isomorphic to a direct sum $tA \oplus \mathbf{Z}^n$ (where tA is the torsion part of A), and $\mathrm{rk}(A) = n$ (use 4.5D).

4.5. Basic properties

R is a fixed commutative unital ring. We write $A\otimes B$ for $A\otimes_R B$.

(A) The tensor product is *commutative*. More precisely, there is a canonical isomorphism:

(1) $A\otimes B \to B\otimes A$, $\quad a\otimes b \mapsto b\otimes a$.

(B) The tensor product *has a unit*, the R-module R. Canonical isomorphism:

(2) $A\otimes R \to A$, $\quad a\otimes\lambda \mapsto \lambda.a$, $\quad a \mapsto a\otimes 1_R$.

(C) The tensor product is *associative*. Canonical isomorphism:

(3) $(A\otimes B)\otimes C \to A\otimes(B\otimes C)$, $\quad (a\otimes b)\otimes c \mapsto a\otimes(b\otimes c)$.

(D) The tensor product is *distributive* on direct sums. Canonical isomorphism:

(4) $(\bigoplus_{i\in I} A_i)\otimes B \to \bigoplus_{i\in I} (A_i\otimes B)$, $\quad (a_i)_{i\in I}\otimes b \mapsto (a_i\otimes b)_{i\in I}$.

(E) Corollary. There are canonical isomorphisms:

(5) $R^{(I)}\otimes B \cong B^{(I)} = \bigoplus_{i\in I} B$, $\quad R^m\otimes B \cong B^m$,

$R^{(I)}\otimes R^{(J)} \cong R^{(I\times J)}$, $\quad R^m\otimes R^n \cong R^{m.n}$.

(F) If A, B are R-free with bases $(a_i)_{i\in I}$, $(b_j)_{j\in J}$, then $A\otimes B$ is free with basis $(a_i\otimes b_j)_{(i,j)\in I\times J}$.

4.6. Exact functors (between categories of modules)

(A) A functor F: R-**Mod** $\to$ S-**Mod** is said to be *left exact*: if, given an exact sequence of type (1), also the resulting sequence (2) is exact

(1) $0 \to A \to B \to C$ $\quad$ (2) $0 \to FA \to FB \to FC$.

- Exercise. This is equivalent to saying that F preserves kernels (up to isomorphism).

(B) The functor F is *right exact*: if the same happens with the sequences:

(1') $A \to B \to C \to 0$ (2') $FA \to FB \to FC \to 0$.

- This is equivalent to saying that F preserves cokernels (up to isomorphism).

(C) The functor F is said to be *exact*: if it satisfies the following equivalent properties:

(a) F preserves exact sequences, (b) F preserves short exact sequences,

(c) F preserves kernels and cokernels, (d) F is left and right exact,

(e) F is left exact and preserves epimorphisms, (e) F is right exact and preserves monom.

(D) The functor F is said to be *additive*: if $F(f + g) = F(f) + F(g)$, for all *parallel* homomorphisms f, g (same domain and same codomain).

- *Every additive functor preserves split exact sequences* (by 2.9c).

4.7. Exactness properties of the tensor product

(A) For every module X, the functor $-\otimes_R X$: R-**Mod** $\to$ R-**Mod** is *right-exact*: given an exact sequence of type (1), also the resulting sequence (2) is exact

(1) $A \to B \to C \to 0$, (2) $A\otimes X \to B\otimes X \to C\otimes X \to 0$.

- Exercise. For $R = \mathbf{Z}$

(3) $\mathbf{Z}_m \otimes_{\mathbf{Z}} \mathbf{Z}_n \cong \mathbf{Z}_d$, where $d = \text{g.c.d.}(m, n)$.

- Hint: Apply (A) to the exact sequence $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_m \to 0$ produced by $k \mapsto m.k$.

- Exercise. For $R = \mathbf{Z}$: show that $-\otimes_{\mathbf{Z}} \mathbf{Z}_n$ does not preserve monomorphisms.

(B) The R-module X is said to be *flat* if the functor $-\otimes_R X$: R-**Mod** $\to$ R-**Mod** is *exact*:, i.e. preserves all exact sequences. By 4.6, this is equivalent to saying that $-\otimes_R X$ preserves monomorphisms.

- *Every free module is flat*. (One can prove that an abelian group is flat if and only if it is *torsion-free*.)

(C) For every module X, the functor $-\otimes_R X$: R-**Mod** $\to$ R-**Mod** preserves all *split* exact sequences (because their initial monomorphism has a left inverse; or - also - because $-\otimes X$ is additive).

(D) For $R = \mathbf{Z}$ and every abelian group X, the functor $-\otimes X$: **Ab** $\to$ **Ab** preserves all exact sequences of *free abelian groups* (because they can be subdivide into short exact sequences of free abelian groups, which split.)

4.8. Tensor products of vector spaces

Let us assume that the base ring is a (commutative) field K. K-modules are called vector spaces and have specific properties, essentially deriving from the fact that *all vector spaces are free*.

- In K-**Mod**, every monomorphism (resp. epimorphism) has a left (resp. right) inverse. All short exact sequences in K-**Mod** *split*. Every additive functor F: K-**Mod** $\to$ S-**Mod** is exact (4.6D).

- Therefore, all vector spaces are flat: the functor $-\otimes_K X$ is always exact.

- There is a *canonical homomorphism* (the functor Hom will be studied in Ch. 6)

(1) $i: A \otimes_K B \to \text{Hom}_K(A^*, B)$, $i(a\otimes b)(\alpha) = \alpha(a).b$ (for $\alpha: A \to K$),

where $A^* = \mathrm{Hom}_K(A, K)$ is the *dual* of A.

- Exercise: prove that, *if* A *is finitely generated*, then i is an isomorphism.

- Tensor product of vector spaces *can* be defined using bases (see 4.4F). But then, to define $f\otimes g\colon A\otimes B \to A'\otimes B'$, one has to *choose* bases in A, B and *prove* that $f\otimes g$ is well defined.

- Tensor product of *finitely generated* vector spaces *can* be defined as $A \otimes_K B = \mathrm{Hom}_K(A^*, B)$. This can also be used for vector bundles.

5. Relative singular homology with coefficients in a group

G is an abelian group. Tensor products are on **Z**.

5.1. Main definitions

- The functor $-\otimes G\colon \mathbf{Ab} \to \mathbf{Ab}$ has an obvious extension to chain complexes

(1) $-\otimes G\colon C_*\mathbf{Ab} \to C_*\mathbf{Ab}$,

$A\otimes G = (... \to A_n\otimes G \to A_{n-1}\otimes G \to ...)$, $\quad \partial'_n = \partial_n\otimes G$,

$(f\otimes G)_n = f_n\otimes G\colon A_n\otimes G \to B_n\otimes G$ $\quad$ (for $f\colon A \to B$ in $C_*\mathbf{Ab}$).

- The *singular chain complex* of a space, *with coefficients in* G

(2) $C_*(-; G)\colon \mathbf{Top} \to C_*\mathbf{Ab}$, $\quad C_*(X; G) = C_*(X)\otimes G$,

$C_n(X; G) = C_n(X)\otimes G \cong \bigoplus_a G$ $\quad (a \in \square_n X \setminus \mathrm{Deg}_n X)$,

$f_{\#n}(\Sigma_i \lambda_i.a_i) = \Sigma_i \lambda_i.(fa_i)$ $\quad (\lambda_i \in G,\ a_i\colon \mathbf{I}^n \to X)$,

where $\lambda_i.a_i = (\lambda_a) \in \bigoplus_a G$, with: $\lambda_a = \lambda_i$ for $a = a_i$, $\lambda_a = 0_G$ for $a \neq a_i$.

- Similarly, we have the *singular chain complex* of pair of spaces, *with coefficients in* G

(3) $C_*(-; G)\colon \mathbf{Top}_2 \to C_*\mathbf{Ab}$, $\quad C_*(X, A; G) = C_*(X, A)\otimes G$.

- *Singular Homology* of a pair of spaces, *with coefficients in* G

(4) $H_n(-; G)\colon \mathbf{Top}_2 \to \mathbf{Ab}$ $\quad H_n(X, A; G) = H_n(C_*(X, A; G))$,

$H_n(f) = f_{*n}$, $\quad f_{*n}[\Sigma_i \lambda_i a_i] = [\Sigma_i \lambda_i(fa_i)]$ $\quad (\lambda_i \in G)$.

- For $G = \mathbf{Z}$, we find the previous chain complexes (and homology): $C_*(X, A; \mathbf{Z}) \cong C_*(X, A)$.

5.2. Theorem (Subdivision for homology with coefficients in G)

In the hypotheses of 2.3, the canonical morphism $C_*(X; \mathcal{U})\otimes G \to C_*(X; G)$ induces isomorphism in homology, in every degree.

Hint. We deduce this from the Subdivision Theorem with integral coefficients (2.3).

- The short exact sequence (1) splits in every degree (its components are free abelian group)

(1) $C_*(X; \mathcal{U}) \overset{j}{\rightarrowtail} C_*(X) \overset{p}{\twoheadrightarrow} D_*$ $\qquad$ (2) $C_*(X; \mathcal{U})\otimes G \overset{j\otimes G}{\rightarrowtail} C_*(X; G) \overset{p\otimes G}{\twoheadrightarrow} D_*\otimes G$

whence, applying $-\otimes G$, also the sequence (2) is short exact.

- By the exactness of the homology sequence of (1), where all j_{*n} are iso: $H_n(D_*) = 0$, for all n.

- Thus D_* is an *exact* sequence of *free* abelian groups, and also $D_*\otimes G$ is an *exact* sequence.

- By the exactness of the homology sequence of (2), where $H_n(D_*\otimes G) = 0$: all $(j\otimes G)_{*n}$ are iso. □

5.3. Theorem (Relative Singular Homology with coefficients in G and E-S axioms)

Relative Singular Homology with coefficients in G *is* a homology theory with coefficients in G (in the sense of Eilenberg-Steenrod).

Hint. *Functoriality*: see 5.1.

- *Exactness and Naturality*. The (natural) short exact sequence $C_*(A) \rightarrowtail C_*(X) \twoheadrightarrow C_*(X, A)$ has free components. Therefore also $C_*(A; G) \rightarrowtail C_*(X; G) \twoheadrightarrow C_*(X, A; G)$ is short exact, and its homology sequence is exact (and natural)

$$(1)\quad ... \to H_n(A; G) \xrightarrow{u_{*n}} H_n(X; G) \xrightarrow{v_{*n}} H_n(X, A; G) \xrightarrow{\Delta_n} H_{n-1}(A; G) ... \to H_0(X, A; G) \to 0$$

- *Homotopy invariance*. Let $F: f \simeq g: (X, Y) \to (Y, B)$ be a homotopy of maps of pairs. We have constructed a homotopy $\Psi = (\Psi_n): f_\# \simeq g_\#: C_*(X, A) \to C_*(Y, B)$ (3.3). Applying the additive functor $-\otimes G$ one has a homotopy $(\Psi_n\otimes G): f_\# \simeq g_\#: C_*(X, A; G) \to C_*(Y, B; G)$.

- *Excision*. Same proof as in 3.5, using the Subdivision Theorem with coefficients in G (5.2).

- *Dimension* and *coefficients*. Compute directly $H_n(\{*\}; G)$. □

5.4. Theorem (Mayer-Vietoris for singular homology with coefficients in G)

In the same hypotheses of 2.4 there is an exact sequence, natural in the same sense

$$(1)\quad ... \to H_n(A; G) \xrightarrow{h_n} H_n(U; G)\oplus H_n(V; G) \xrightarrow{k_n} H_n(X; G) \xrightarrow{\Delta_n} H_{n-1}(A; G) ...$$

Hint. Same proof as in 2.4, using the Subdivision Theorem with coefficients in G (5.2). □

5.5. Exercises

- Compute the homology of $\mathbf{S}^n$ and $\mathbf{P}^2$, with coefficients in $\mathbf{Q}$ and in $\mathbf{Z}_m$.

- Study the projection $\mathbf{P}^2 \to \mathbf{S}^2$, viewing both as quotients of $\mathbf{I}^2$. Hint: use $H_2(-; \mathbf{Z}_2)$.

6. The functor Hom [*Homological Algebra, Multilinear Algebra*]

R is always a commutative ring with unit.

6.1. The functor Hom

- If A, B are R-modules, $\mathrm{Hom}_R(A, B)$ denotes the set of R-homomorphisms $A \to B$, with the *pointwise* structure of R-module

$$(1)\quad (h + h')(a) = h(a) + h'(a), \qquad (\lambda.h)(a) = \lambda.h(a) \qquad (a\in A,\ \lambda\in R).$$

- Hom_R is a *functor in two variables*, contravariant in the first and covariant in the second

(2) Hom_R: R-$\mathbf{Mod}^{op}$ × R-$\mathbf{Mod}$ → R-$\mathbf{Mod}$,

$\mathrm{Hom}_R(f, g): \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A', B'), \qquad h \mapsto ghf \qquad (f: A' \to A,\ g: B \to B')$,

(3) $\mathrm{Hom}_R(\mathrm{id}A, \mathrm{id}B) = \mathrm{id}(\mathrm{Hom}_R(A, B))$, $\mathrm{Hom}_R(ff', g'g) = \mathrm{Hom}_R(f', g')\circ\mathrm{Hom}_R(f, g)$.

This functor is *bilinear* (additive and homogeneous in each variable):

(4) $\mathrm{Hom}_R(f+f', g) = \mathrm{Hom}_R(f, g) + \mathrm{Hom}_R(f', g)$, $\mathrm{Hom}_R(\lambda f, g) = \lambda.\mathrm{Hom}_R(f, g)$,

$\mathrm{Hom}_R(f, g+g') = \mathrm{Hom}_R(f, g) + \mathrm{Hom}_R(f, g')$, $\mathrm{Hom}_R(f, \lambda g) = \lambda.\mathrm{Hom}_R(f, g)$.

6.2. Exercises (for abelian groups: $R = \mathbf{Z}$, $\mathrm{Hom}_{\mathbf{Z}} = \mathrm{Hom}$)

- $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, B) = {}_mB$ (the subgroup of elements $b\in B$ such that $mb = 0$).
- $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}) = 0$, $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Q}) = 0$, $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_n) \cong \mathbf{Z}_d$ ($d = \mathrm{g.c.d.}(m, n)$).
- If m, n are coprime, then $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_n) = 0$.
- More generally, if $mA = 0$ and in B $mb = 0$ implies $b = 0$, then $\mathrm{Hom}_{\mathbf{Z}}(A, B) = 0$.
- If T is a *torsion abelian group* and B is *torsion-free*, then $\mathrm{Hom}_{\mathbf{Z}}(A, B) = 0$.

6.3. Basic properties of the functors Hom

R is a commutative unital ring. The *properties of* Hom_R *in each variable must be distinguished.*

(A) The module $A^* = \mathrm{Hom}_R(A, R)$ is called the *dual* of A. There is a canonical isomorphism:

(1) $\mathrm{Hom}_R(R, B) \to B$, $h \mapsto h(1_R)$, $b \mapsto (\lambda \mapsto \lambda.b)$.

(B) There are canonical isomorphisms:

(2) $\prod_{i\in I} \mathrm{Hom}_R(A, B_i) \to \mathrm{Hom}_R(A, \prod_{j\in J} B_j)$, $(h_j)_{j\in J} \mapsto h$, $h(a) = (h_j(a))_{j\in J}$,

(3) $\prod_{i\in I} \mathrm{Hom}_R(A_i, B) \to \mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B)$, $(h_i)_{i\in I} \mapsto h$, $h((a_i)_{i\in I}) = \sum_{i\in I} h_i(a_i)$,

(C) Corollary. There are canonical isomorphisms:

(4) $\mathrm{Hom}_R(A, R^J) \cong A^J = \prod_{j\in J} A$, $\mathrm{Hom}_R(R^{(I)}, B) \cong B^I = \prod_{i\in I} B$,

$\mathrm{Hom}_R(A, R^n) \cong A^n$, $\mathrm{Hom}_R(R^m, B) \cong B^m$, $\mathrm{Hom}_R(R^m, R^n) \cong R^{m.n}$.

(D) Exponential law. There is a canonical isomorphism:

(5) $\mathrm{Hom}_R(A\otimes B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$, $h \mapsto h'$, $h'(a): b \mapsto h(a\otimes b)$.

6.4. Exactness properties of the functors Hom

(A) The (covariant) functor $\mathrm{Hom}_R(X, -)$ is *left-exact*: it transforms an exact sequence (1) into an exact sequence (2) (equivalently: *it preserves kernels*)

(1) $0 \to A \to B \to C$

(2) $0 \to \mathrm{Hom}_R(X, A) \to \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, C)$.

(B) The (contravariant) functor $\mathrm{Hom}_R(-, Y)$ transforms an exact sequence (3) into an exact sequence (4) (equivalently: *it transforms cokernels into kernels*)

(3) $A \to B \to C \to 0$

(4) $0 \to \mathrm{Hom}_R(C, Y) \to \mathrm{Hom}_R(B, Y) \to \mathrm{Hom}_R(A, Y)$.

- Exercise. For $R = \mathbf{Z}$, deduce $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_n) \cong \mathbf{Z}_d$ from (B).

- Exercise. For $R = \mathbf{Z}$, show that $\mathrm{Hom}_{\mathbf{Z}}(-, \mathbf{Z}_n)$ is not exact.

(C) For every module X, the functors $\mathrm{Hom}_R(X, -)$ and $\mathrm{Hom}_R(-, X)$ preserve all *split* exact sequences (because these functors are additive).

(D) For $R = \mathbf{Z}$ and every abelian group X, the functors $\mathrm{Hom}_{\mathbf{Z}}(X, -)$ and $\mathrm{Hom}_{\mathbf{Z}}(-, X)$ preserves all exact sequences of *free abelian groups*.

7. Relative singular cohomology with coefficients in a group

G is an abelian group. We use the contravariant functor $\mathrm{Hom}(-, G) = \mathrm{Hom}_{\mathbf{Z}}(-, G)$.

7.1. Cochain complexes

- A *cochain complex* $A = ((A^n), (d^n))$ of abelian groups is a sequence

$$(1) \qquad 0 \longrightarrow A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \ldots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \ldots$$

with $d^{n+1}.d^n = 0$. A *morphism* $\varphi\colon A \to B$ of cochain complexes is a sequence of homomorphisms $\varphi^n\colon A^n \to B^n$ commuting with differentials: $d^n.\varphi^n = \varphi^{n+1}.d^n$. They form the category $C^*\mathbf{Ab}$ of cochain complexes of abelian groups.

- The n-cohomology functor of chain complexes:

(2) $H^n\colon C^*\mathbf{Ab} \to \mathbf{Ab}$ $\qquad (n \geq 0)$,

$H^n(A) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$, $\qquad H^n(\varphi)[\zeta] = [\varphi^n(\zeta)]$ $\qquad (d^n(\zeta) = 0)$.

7.2. Main definitions

- The *contravariant* functor $\mathrm{Hom}(-, G)\colon \mathbf{Ab}^{op} \to \mathbf{Ab}$ transforms *chain* complexes into *cochain complexes*

(1) $\mathrm{Hom}(-, G)\colon (C_*\mathbf{Ab})^{op} \to C^*\mathbf{Ab}$,

$\mathrm{Hom}(A, G) = (\ldots \to \mathrm{Hom}(A_n, G) \to \mathrm{Hom}(A_{n+1}, G) \to \ldots)$, $\qquad d^n = \mathrm{Hom}(\partial_{n+1}, G)$

$\mathrm{Hom}(f, G)^n = \mathrm{Hom}(f_n, G)\colon \mathrm{Hom}(B_n, G) \to \mathrm{Hom}(A_n, G)$ $\qquad$ (for $f\colon A \to B$ in $C_*\mathbf{Ab}$).

- The *singular cochain complex* of a space, *with coefficients in* G

(2) $C^*(-; G)\colon \mathbf{Top}^{op} \to C^*\mathbf{Ab}$, $\qquad C^*(X; G) = \mathrm{Hom}(C_*(X), G)$,

$C^n(X; G) = \mathrm{Hom}(C_n(X), G) \cong \{\lambda\colon \square_n X \to G \mid \lambda(a) = 0 \text{ when } a \in \mathrm{Deg}_n X\}$,

$(d\lambda)(a) = \lambda(\partial a)$,

$f^{\#n}(\mu) = (\mu \circ (\square f)_n)$, $\qquad f^{\#n}(\mu)(a) = \mu(fa)$ $\qquad$ (for $f\colon X \to Y$).

- Note: $C^n(X; G) \cong \prod_a G$ $(a \in \square_n X \setminus \mathrm{Deg}_n X)$.

- The *singular cochain complex* of pair of spaces, *with coefficients in* G

(3) $C^*(-; G)\colon (\mathbf{Top}_2)^{op} \to C^*\mathbf{Ab}$, $\qquad C^*(X, A; G) = \mathrm{Hom}(C_*(X, A), G)$,

$C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G) \cong \{\lambda\colon \square_n X \to G \mid \lambda(a) = 0 \text{ for } a \in (\square_n A) \cup (\mathrm{Deg}_n X)\}$,

$(d\lambda)(a) = \lambda(\partial a)$.

- *Singular Cohomology* of a pair of spaces, *with coefficients in* G

(4) $H^n(-; G)\colon (\mathbf{Top_2})^{op} \to \mathbf{Ab}$ $\qquad H^n(X, A; G) = H^n(C^*(X, A; G))$,

$H^n(f) = f^{*n}\colon H^n(Y, B; G) \to H^n(X, A; G)$, $\qquad f^{*n}[\mu] = [f^{\#n}(\mu)]$ $\qquad (\mu \in C^n(Y, B; G))$.

- For $G = \mathbf{Z}$, one writes: $C^*(X, A) = C^*(X, A; \mathbf{Z})$.

7.3. Theorem (Subdivision for cohomology with coefficients in G)

In the hypotheses of 2.3, the canonical morphism $C^*(X; G) \to \mathrm{Hom}(C_*(X; \mathcal{U}), G)$ induces isomorphism in cohomology, in every degree.

Hint. The proof is similar to the one for homology with coefficients in G (5.2) □

7.4. Theorem (Relative Singular Cohomology with coefficients in G and E-S axioms)

Relative Singular Cohomology with coefficients in G is a *cohomology* theory with coefficients in G (in the sense of Eilenberg-Steenrod).

Hint. The axioms for cohomology are dual to the ones for homology. The proof is similar to 5.3. □

7.5. Theorem (Mayer-Vietoris for singular cohomology with coefficients in G)

In the same hypotheses of 2.4 there is an exact sequence, contravariantly natural

$$(1) \quad \dots \leftarrow H^n(A; G) \xleftarrow{h^n} H^n(U; G)\oplus H^n(V; G) \xleftarrow{k^n} H^n(X; G) \xleftarrow{\Delta^{n-1}} H^{n-1}(A; G) \dots$$

Hint. As in 2.4. □

TEXTS

- *Algebraic Topology*

J. Vick, Homology Theory, Academic Press 1973.

W. Massey, Singular Homology Theory, Springer 1980.

W. Massey, Algebraic Topology, an Introduction, Harcourt 1967.

W. Massey, A basic course in algebraic topology, Springer 1991

S. Eilenberg - N. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press 1952.

A. Dold, Lectures on algebraic topology, Springer 1972.

E. Spanier, Algebraic topology, McGraw-Hill 1966.

R. Brown, Topology, Ellis Horwood 1988.

S.T. Hu, Homotopy theory, Academic Press 1959.

A. Hatcher, Algebraic Topology, 2002. `www.math.cornell.edu/~hatcher/`

- *Homological Algebra*

H. Cartan - S. Eilenberg, Homological algebra, Princeton Univ. Press 1956.

S. Mac Lane, Homology, Springer 1963.

C.A. Weibel, An introduction to homological algebra, Cambridge Univ. Press 1994.

- *Category Theory*

S. Mac Lane, Categories for the working mathematician, Springer 1971.

J. Adámek - H. Herrlich - G. Strecker, Abstract and concrete categories, Wiley Interscience Publ., 1990.

F. Borceux, Handbook of categorical algebra. 1-2-3, Cambridge University Press, Cambridge, 1994.