Theory of Categories 1. Notes

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0. Introduction

Category Theory yields a general frame for studying mathematical structures and their *universal constructions*. Important mathematical tools can often be described as *adjoint functors* of obvious procedures. Historically, this theory was born within Algebraic Topology and Homological Algebra, around 1945.

These notes contain definition and statements, with few motivations and no proof (except some hints). Proofs can be found in the references at the end.

We shall formulate category theory within a particular Set Theory, called NBG (von Neumann - Bernays - Gödel), where one has *sets* and *classes*, and the *class of all sets* makes sense. This approach is followed in [Mt] and - essentially - also in [AHS]; a brief exposition of NBG can be found in the Appendix of [Ke].

Another setting widely used in category theory is ordinary set theory together with a Grothendieck *universe*. This approach is followed in [Ma, Bo].

CHAPTER 1. Categories

1.1. Definition. A *category* $\mathcal{C}$ consists of the following data:

(a) a *class* $\mathrm{Ob}\mathcal{C}$, whose elements are called *objects* of $\mathcal{C}$,

(b) for every pair X, Y of objects, a *set* $\mathcal{C}(X, Y)$ whose elements are called *morphisms* (or *maps*, or *arrows*) of $\mathcal{C}$ from X to Y and denoted as $f: X \to Y$,

(c) for every triple X, Y, Z of objects of $\mathcal{C}$, a mapping

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), \qquad (f, g) \mapsto gf,$$

called *composition*; notice that this *partial* composition law acts on pairs of *consecutive* morphisms, $f: X \to Y$ and $g: Y \to Z$.

These data must satisfy the following axioms.

(1) *Associativity*. Given three consecutive arrows, $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$, one has: $h(gf) = (hg)f$.

(2) *Identities*. Given an object X, there exists an *endo*map $e: X \to X$ which acts as an identity whenever composition makes sense; in other words if $f: Y \to X$ and $g: X \to Z$, one has: $ef = f$ and $ge = g$. One shows, in the usual way, that e is determined by X; it is called the *identity* of X and written as 1_X or $\mathrm{id}(X)$.

Remark. We will generally assume that the following condition is also satisfied:

(3) *Separation.* For X, X', Y, Y' objects of $\mathcal{C}$

if $\mathcal{C}(X, Y) \cap \mathcal{C}(X', Y') \neq \emptyset$ then $X = X'$ and $Y = Y'$.

Therefore, a map $f: X \to Y$ has a well-determined *domain* $\mathrm{Dom}(f) = X$ and *codomain* $\mathrm{Cod}(f) = Y$. Concretely, when constructing a category, one can forget about this condition, since one can always satisfy it by *redefining* a morphism $\hat{f}: X \to Y$ as a triple $(X, Y; f)$ where f is a morphism from X to Y in the original sense (possibly not satisfying the Separation axiom).

1.2. Categories of structured sets. We are mainly interested in categories of 'structured sets', where the morphisms are mappings which preserve the structure (in some specified sense) and the composition law is the ordinary composition of mappings. We shall use the following notation.

Set: the category of sets and mappings between sets;

Top: the category of topological spaces and continuous mappings;

Hsd: the category of Hausdorff topological spaces and continuous mappings (between them);

Ab: the category of abelian groups and their homomorphisms;

Gp: the category of groups and their homomorphisms;

Mon: the category of monoids (i.e. unitary semigroups) and their (unitary) homomorphisms;

Rng: the category of unitary rings and their (unitary) homomorphisms;

R-**Mod**: the category of modules on the unitary commutative ring R and their homomorphisms;

Ban: the category of (real or complex) Banach spaces and their linear continuous mappings;

$\mathbf{Ban}_1$: the category of (real or complex) Banach spaces and their linear mappings with norm ≤ 1;

Mtr: the category of metric spaces and Lipschitz mappings;

$\mathbf{Mtr}_1$: the category of metric spaces and weak contractions.

R-modules are always assumed to be unitary. If the commutative ring R is a field, a module on R is also called a vector space, and R-**Mod** can be written as R-**Vct**.

1.3. Small categories. A category $\mathcal{C}$ is said to be *small* if its class of objects is a set. Then the class $\mathrm{Mor}\mathcal{C}$ of all its morphisms is also a set.

(a) Every set X can be viewed as a small category $\underline{X}$, where the objects are the elements of X, the only morphisms are identities $\mathrm{id}(x)$ and the composition law only says that $\mathrm{id}(x).\mathrm{id}(x) = \mathrm{id}(x)$, for all $x \in X$. Such categories are called *discrete*. For instance, the cardinal $2 = \{0, 1\}$ is a finite, discrete category with two arrows, $\mathrm{id}(0)$ and $\mathrm{id}(1)$.

(b) Let X be a *preordered* set, which means that it is equipped with a preorder relation $x \prec x'$ (reflexive and transitive). Then X can be viewed as a small category $\underline{X}$, where the objects are the elements of X; the set $\underline{X}(x, x')$ contains precisely one arrow if $x \prec x'$ (which can be written as $(x, x'): x \to x'$), and no arrow otherwise. The composition is (necessarily) $(x', x'').(x, x') = (x, x'')$, and $\mathrm{id}(x) = (x, x)$. In particular, each finite ordinal defines a category, which will be written $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ Thus, $\mathbf{0}$ is the empty category, $\mathbf{1}$ is the discrete category on one object and $\mathbf{2}$ has precisely one non-identity arrow: $0 \to 1$.

(c) Let M be a monoid. Then M can be viewed as a small category $\underline{M}$, where there is one object (say $*$) and $\underline{M}(*, *) = M$. Composition is the multiplication in M; the identity is the algebraic unit.

(d) Let R be a unitary ring. One can define a small category **Mat**(R) whose objects are the natural numbers, whose arrows $n \to m$ are the matrices $m{\times}n$ with coefficients in R, and whose composition BA is matrix multiplication (for $A: p \to n$, $B: n \to m$ and $BA: p \to m$). The identity $I_n = id(n)$ is the $n{\times}n$ unit matrix. Notice that for every pair (m, n) there is a null matrix O_{mn}, *including* when $m = 0$ (no rows) or $n = 0$ (no columns); this will become clearer at the light of further developments, where such matrices correspond to null linear mappings (see 2.2, Exercise 1).

(e) Let X be a topological space. The *fundamental groupoid* $\Pi_1(X)$ is the small category whose objects are the elements of X and whose maps $[a]: x \to x'$ are equivalence classes of paths in X, from x to x', up to homotopy with fixed endpoints. Composition is by concatenation of consecutive paths. The general definition of a groupoid is given below (1.4).

1.4. Isomorphisms, monomorphism, epimorphisms. Let $\mathcal{C}$ be a category. A morphism $f: X \to Y$ is said to be *invertible*, or an *isomorphism*, if it has an inverse, i.e. a morphism $g: Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. Then, g is uniquely determined; it is called the *inverse* of f and written as f^{-1}.

Obvious verifications prove that:

(a) the identity of any object X is invertible, with $(1_X)^{-1} = 1_X$;

(b) the inverse of an isomorphism f is invertible, with $(f^{-1})^{-1} = f$;

(c) the composite of two consecutive isomorphisms f, g is invertible, with $(gf)^{-1} = f^{-1}g^{-1}$.

Thus, the *isomorphism relation* $X \cong Y$ between objects of $\mathcal{C}$ (meaning that there exists an isomorphism $X \to Y$) is an equivalence relation.

A morphism $f: X \to Y$ is said to be a *monomorphism*, or *mono*, if it satisfies the following cancellation property: for every pair of maps $u, v: X' \to X$ such that $fu = fv$, one has $u = v$.

A morphism $f: X \to Y$ is said to be an *epimorphism*, or *epi*, if it satisfies the 'other' cancellation property: for every pair of maps $u, v: Y \to Y'$ such that $uf = vf$, one has $u = v$.

Every isomorphism is mono and epi. A category is said to be *balanced* if the converse holds: every morphism which is mono and epi is invertible. A *groupoid* is a category where every map is invertible; e.g. the fundamental groupoid of a space (see 1.3(e)).

Proposition. Let $f: X \to Y$ and $g: Y \to Z$ be two consecutive maps in a category. Then:

- if f and g are both mono, gf is also mono; if gf is mono, then f is also;
- if f and g are both epi, gf is also epi; if gf is epi, then g is also.

Exercises. 1. Characterise the isomorphisms of the categories listed in 1.2 and 1.3.

2. Characterise the monomorphisms and epimorphisms of **Set**, **Top**, **Hsd**, **Ab**, R-**Mod**, **Ban**. Notice that epimorphisms in **Hsd** need not be surjective mappings. Which of these categories are balanced?

3. Prove that there exists a non-surjective morphism in **Mon** which is epi. (There is no elementary characterisation of epimorphisms in this category, nor in **Rng**.)

4. Epimorphism in **Gp** coincide with the surjective homomorphisms. The proof is not easy, see [Ma].

1.5. Retracts, split monos and epis. Suppose we have, in a category $\mathcal{C}$, two maps $i: A \to X$ and $p: X \to A$ such that $pi = idA$. Then i is a monomorphism (called a *section*, or a *split* *monomorphism*), p is an epimorphism (called a *retraction*, or a *split epimorphism*) and one says that A is a *retract* of X.

Proposition. Let $f: X \to Y$ and $g: Y \to Z$ be consecutive maps in a category. Then:

(a) if f and g are both split mono, gf is also; if gf is a split mono, f is also;

(a*) if f and g are both split epi, gf is also; if gf is a split epi, g is also;

(b) if f is a split mono and an epi, then it is invertible;

(b*) if f is a split epi and a mono, then it is invertible.

Exercise. Characterise split monomorphisms and split epimorphisms in **Set** and **Ab**.

Remark. There is no elementary characterisation of retracts in **Top**. Homology groups allow one to prove, for instance, that the n-sphere $\mathbf{S}^n$ is not a retract of the euclidean space $\mathbf{R}^{n+1}$.

1.6. Subcategories. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{C}'$ is defined by assigning:

(a) a subclass $\mathrm{Ob}\mathcal{C}' \subset \mathrm{Ob}\mathcal{C}$ (*weak* inclusion, of course), whose elements are called *objects of* $\mathcal{C}'$,

(b) for every pair of objects X, Y of $\mathcal{C}'$, a subset $\mathcal{C}'(X, Y) \subset \mathcal{C}(X, Y)$, whose elements are called *morphisms of* $\mathcal{C}'$, from X to Y,

so that the following conditions hold:

(1) for every pair of consecutive morphisms of $\mathcal{C}'$, their composite in $\mathcal{C}$ belongs to $\mathcal{C}'$,

(2) for every object of $\mathcal{C}'$, its identity in $\mathcal{C}$ belongs to $\mathcal{C}'$.

Then $\mathcal{C}'$, equipped with the induced composition law, is a category.

One says that $\mathcal{C}'$ is a *full* subcategory of $\mathcal{C}$ if, for every pair of objects X, Y of $\mathcal{C}'$, we have $\mathcal{C}'(X, Y) = \mathcal{C}(X, Y)$, so that $\mathcal{C}'$ is determined by assigning its subclass of objects. For instance, **Ab** is a full subcategory of **Gp**, which is a full subcategory of **Mon**; **Hsd** is a full subcategory of **Top**.

On the other hand, one says that $\mathcal{C}'$ is a *wide* subcategory of $\mathcal{C}$ if it contains all its objects. For instance, $\mathbf{Ban}_1$ is a wide subcategory of **Ban** and $\mathbf{Mtr}_1$ is a wide subcategory of **Mtr**.

1.7. Congruences and quotients of categories. A *congruence* $R = (R_{XY})$ in a category $\mathcal{C}$ consists of a family of equivalence relations R_{XY} in each set of morphisms $\mathcal{C}(X, Y)$; the family must be consistent with composition:

(1) if $f\, R_{XY}\, f'$ and $g\, R_{YZ}\, g'$, then $gf\, R_{XZ}\, g'f'$.

Then one defines the *quotient category* $\mathcal{D} = \mathcal{C} / R$: the objects are those of $\mathcal{C}$, and $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)/R_{XY}$; in other words, a morphism $[f]: X \to Y$ in $\mathcal{D}$ is an equivalence class of morphisms $X \to Y$ in $\mathcal{C}$. The composition is induced by that of $\mathcal{C}$, which is legitimate because of (1):

(2) $[g].[f] = [gf]$.

Exercises. 1. Prove that property (1) is equivalent to the conjunction of the following properties:

(3a) if $f\, R_{XY}\, f'$ and $g: Y \to Z$, then $gf\, R_{XZ}\, gf'$,

(3b) if $f: X \to Y$ and $g\, R_{YZ}\, g'$, then $gf\, R_{XZ}\, g'f$.

2. Prove that, in **Top**, the homotopy relation $f \simeq f'$ is a congruence of categories. The quotient category Ho**Top** = **Top**/$\simeq$ is called the *homotopy category of topological spaces*, and is important in Algebraic Topology. Prove that a continuous mapping $f: X \to Y$ is a homotopy equivalence if and only if its homotopy class $[f]$ is an isomorphism of Ho**Top**.

1.8. Product categories. If $\mathcal{C}$ and $\mathcal{D}$ are categories, one defines the *product category* $\mathcal{C}{\times}\mathcal{D}$. An object is a pair (X, Y) where X is in $\mathcal{C}$ and Y in $\mathcal{D}$. A morphism

(1) $(f, g): (X, Y) \to (X', Y'),$ $\qquad f \in \mathcal{C}(X, X'),\ \ g \in \mathcal{D}(Y, Y'),$

is a pair of morphisms in $\mathcal{C}$ and $\mathcal{D}$. The composition of (f, g) with $(f', g'): (X', Y') \to (X'', Y'')$ is component-wise

(2) $(f', g').(f, g) = (f'f, g'g).$

The axioms of categories are easily verified. More generally, one defines the product $\Pi_i\, \mathcal{C}_i$ of a family of categories indexed on a *set*.

1.9. Opposite category and duality. If $\mathcal{C}$ is a category, the *opposite* (or *dual*) category, written $\mathcal{C}^*$ or $\mathcal{C}^{op}$, has the same objects as $\mathcal{C}$ and 'reversed' arrows,

(1) $\mathcal{C}^*(X, Y) = \mathcal{C}(Y, X),$

with 'reversed composition' $g{*}f = fg$.

Every notion of category theory has a dual notion, which comes from the opposite category (or categories): thus, monomorphism and epimorphism are dual to each other, while isomorphism is a selfdual notion. Every statement of category theory has a dual one: for instance, in 1.5, the statements (a) and (a*) are dual, and it suffices to prove one of them; similarly, (b) and (b*) are dual.

CHAPTER 2. Functors and natural transformations

2.1. Covariant functors and isomorphism of categories. A (covariant) *functor* $F: \mathcal{C} \to \mathcal{D}$ consists of the following data:

(a) a mapping $F_0: \mathrm{Ob}\mathcal{C} \to \mathrm{Ob}\mathcal{D}$, whose action is generally written as $X \mapsto F(X)$,

(b) for every pair of objects X, X' in $\mathcal{C}$, a mapping $F_{XX'}: \mathcal{C}(X, X') \to \mathcal{D}(F(X), F(X'))$, whose action is generally written as $f \mapsto F(f)$,

so that composition and identities are preserved. In other words:

(1) if f, g are consecutive maps in $\mathcal{C}$, then $F(gf) = F(g).F(f)$,

(2) if X is in $\mathcal{C}$, then $F(\mathrm{id}X) = \mathrm{id}F(X)$.

Given a second functor $G: \mathcal{D} \to \mathcal{E}$, one defines in the obvious way the *composed* functor $GF: \mathcal{C} \to \mathcal{E}$. This composition is associative and has identities: the *identity functor* of each category

(2) $\mathrm{id}\mathcal{C}: \mathcal{C} \to \mathcal{C},$ $\qquad X \mapsto X, \quad f \mapsto f.$

An *isomorphism of categories* is a functor $F: \mathcal{C} \to \mathcal{D}$ which is invertible, i.e. admits an *inverse* $G: \mathcal{D} \to \mathcal{C}$. This means a functor such that $GF = \mathrm{id}\mathcal{C}$ and $FG = \mathrm{id}\mathcal{D}$.

Proposition (Characterisation of isomorphisms of categories). A functor $F: \mathcal{C} \to \mathcal{D}$ is an isomorphism if and only if all the mappings F_0 and $F_{XX'}$ considered above are bijective.

Being isomorphic categories is an equivalence relation $\mathcal{C} \cong \mathcal{D}$, by the usual argument (as in 1.4).

Exercise. Prove that **Ab** is isomorphic to the category **Z-Mod** of modules on the ring of integers.

Remarks. (a) Isomorphic categories are often perceived as 'the same thing'. For instance, the various equivalent ways of defining topological spaces give rise to isomorphic categories that are nearly never distinguished.

(b) Restricting to small categories (to avoid higher set-theoretic problems), there is a category **Cat** of small categories and their functors.

2.2. Forgetful and structural functors. (a) Forgetting structure, or part of it, yields various examples of functors between categories of structured sets, like the following obvious instances

(1) **Top** → **Set**, **Rng** → **Ab** → **Set**, **Ban** → **R-Vct** → **Ab**, **Ban** → **Mtr** → **Hsd**.

These are called *forgetful* functors, and often denoted by the letter U, which refers to the *underlying* set, or *underlying* abelian group, and so on.

(b) A subcategory $\mathcal{C}'$ of $\mathcal{C}$ yields an *inclusion* functor $\mathcal{C}' \to \mathcal{C}$, which we also write as $\mathcal{C}' \subset \mathcal{C}$. For instance, **Hsd** ⊂ **Top** and **Ab** ⊂ **Gp** ⊂ **Mon**. Notice that these functors *forget properties* rather than structure (being Hausdorff, etc.).

(c) A congruence R in a category $\mathcal{C}$ yields an obvious *projection* functor $P: \mathcal{C} \to \mathcal{C}/R$, which is the identity on objects and sends a morphism f to its equivalence class [f]. For instance, **Top** → Ho**Top** = **Top**/$\simeq$.

(d) A product category $\mathcal{C}\times\mathcal{D}$ has two obvious projection functors $P_1: \mathcal{C}\times\mathcal{D} \to \mathcal{C}$, $P_2: \mathcal{C}\times\mathcal{D} \to \mathcal{D}$.

(e) By definition, a *functor in two variables* is simply an ordinary functor $F: \mathcal{C}\times\mathcal{D} \to \mathcal{E}$ defined on the product of two categories. Fixing an object X_0 in $\mathcal{C}$, we have a functor $F(X_0, -): \mathcal{D} \to \mathcal{E}$; and symmetrically.

Exercises. 1. For a commutative unitary ring R, define a functor F: **Mat**(R) → R-**Mod** which sends the natural number n to the free R-module R^n.

2. Show that a functor **2** → $\mathcal{C}$ amounts to a map in $\mathcal{C}$, while a functor **2**×**2** → $\mathcal{C}$ amounts to a commutative square in $\mathcal{C}$.

3. Define a functor **Set** × **Set** → **Set** which sends a pair of sets (X, Y) to their cartesian product X×Y.

4. The construction of the fundamental groupoid $\Pi_1(X)$ leads to a functor Π_1: **Top** → **Cat**. Since this functor is invariant up to homotopy, there is an induced functor Ho**Top** → **Cat**.

2.3. Faithful and full functors. For a functor $F: \mathcal{C} \to \mathcal{D}$, let us consider the mappings (of sets):

(1) $F_{XX'}: \mathcal{C}(X, X') \to \mathcal{D}(F(X), F(X'))$, $\quad f \mapsto F(f)$.

F is said to be *faithful* if all these mappings, for X, X' objects of $\mathcal{C}$, are injective; F is said to be *full* if all these mappings are surjective. An isomorphism of categories is full and faithful.

The inclusion $\mathcal{C}' \to \mathcal{C}$ of a subcategory is always a faithful functor; it is full if and only if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ (see examples in 1.6).

The forgetful functors listed in 2.2.1 are faithful, not full. The following functor is not faithful

(1) Ob: **Cat** $\to$ **Set**, $\quad\quad \mathcal{C} \mapsto \mathrm{Ob}\mathcal{C}, \quad F \mapsto F_0$

(It sends a small category to its set of objects and a functor F to the mapping F_0.)

Definition. A *concrete* category will be a category $\mathcal{C}$ equipped with a faithful functor $U: \mathcal{C} \to$ **Set**, called its *forgetful functor*. Notice that U *reflects monos and epis* (see the Proposition below), but need not preserve them.

Remarks. 1. All the categories of structured sets of 1.2 can be made concrete with the obvious 'underlying set' functor. However, $\mathbf{Ban}_1$ has a more important forgetful functor, the *unit ball* (see 2.9)

(2) B_1: $\mathbf{Ban}_1 \to$ **Set**, $\quad\quad B_1(X) = \{x \in X \mid \|x\| \leq 1\}$.

2. The functor **Mat**(R) $\to$ R-**Mod** of Section 2.2 is faithful, and can be used to make **Mat**(R) into a concrete category. It is often better to replace the vague notion of 'category of structured sets' with the precise, more general notion of concrete category.

3. *Not every category can be made concrete*, but there are no elementary examples of this fact. P. Freyd has proved that the homotopy category Ho**Top** cannot be made concrete (1.7).

Proposition (Preservation and reflection properties of functors). (a) Every functor preserves commutative diagrams, isomorphisms, retracts, split monos and split epis. (b) A faithful functor *reflects* monos and epis (i.e. if F(f) is mono or epi, then f is also) and commutative diagrams. (c) A full and faithful functor reflects isomorphisms, split monos and split epis.

Remarks. As a consequence of point (b) above, in a concrete category every morphism whose underlying mapping is injective (resp. surjective) is mono (resp. epi). We already know that the converse need not be true (cf. 1.4). As an application of point (a), the usual way of proving that a topological subspace $A \subset X$ is not a retract (in **Top**) is to find a functor F: **Top** $\to$ **Ab** (e.g. a homology functor H_n) such that the associated homomorphism $F(A) \to F(X)$ is not a split mono in **Ab**.

2.4. Natural transformations. Given two functors $F, G: \mathcal{C} \to \mathcal{D}$, a natural transformation $\varphi: F \to G: \mathcal{C} \to \mathcal{D}$ consists of the following data:

(a) for each object X of $\mathcal{C}$, a morphism $\varphi X: FX \to GX$ in $\mathcal{D}$ (called the *component* of φ on X, and also written as φ_X),

so that, for every arrow $f: X \to X'$ in $\mathcal{C}$, we have a commutative square in $\mathcal{D}$:

(1)
$$\begin{array}{ccc} FX & \xrightarrow{\varphi X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FX' & \xrightarrow[\varphi X']{} & GX' \end{array} \qquad\qquad \varphi X'.F(f) = G(f).\varphi X \qquad\qquad (\textit{naturality condition}).$$

In particular, a functor $F: \mathcal{C} \to \mathcal{D}$ always has a natural transformation $\mathrm{id}F: F \to F$, with components $(\mathrm{id}F)X = \mathrm{id}(FX)$.

Exercise. Characterise the natural transformations $\mathrm{id}\mathcal{C} \to \mathrm{id}\mathcal{C}$, where $\mathcal{C}$ is **Set**, **Ab** or R-**Mod**.

2.5. Vertical composition and categories of diagrams. Suppose we have two *vertically consecutive* natural transformations φ, ψ

$$\text{(1)} \qquad \mathcal{C} \; \underset{H}{\overset{F}{\substack{\longrightarrow \\ \downarrow\varphi \\ \longrightarrow \\ \downarrow\psi \\ \longrightarrow}}} \; \mathcal{D} \qquad\qquad \begin{array}{l} \varphi\colon F \to G\colon \mathcal{C} \to \mathcal{D}, \\[2ex] \psi\colon G \to H\colon \mathcal{C} \to \mathcal{D}, \end{array}$$

Their *vertical composition* $\psi\varphi\colon G \to H\colon \mathcal{C} \to \mathcal{D}$ is simply obtained by composing the components of φ and ψ:

(2) $(\psi\varphi)X = \psi X.\varphi X\colon\ FX \to GX \to HX.$

Again, this composition is associative and has identities, given by the identities of functors (2.4).

A natural transformation $\varphi\colon F \to G$ is *invertible* if it admits an inverse $\psi\colon G \to F$ for vertical composition ($\psi\varphi = \mathrm{id}F$, $\varphi\psi = \mathrm{id}G$). Then φ is also called a *natural isomorphism*, or an *isomorphism of functors*; we say that F, G are *isomorphic functors*, and we write $F \cong G$. The latter is, again, an equivalence relation.

Proposition (Characterisation of isomorphisms of functors). The natural transformation $\varphi\colon F \to G\colon \mathcal{C} \to \mathcal{D}$ is invertible if and only if all its components are isomorphisms of $\mathcal{D}$. Then the inverse has components $\psi X = (\varphi X)^{-1}$, for all objects of $\mathcal{C}$.

Let $\mathbf{I}$ be a small category and $I = \mathrm{Ob}\mathbf{I}$ its set of objects. A functor $X\colon \mathbf{I} \to \mathcal{C}$ is also called a *diagram in* $\mathcal{C}$ *of shape* $\mathbf{I}$. It will often be written in 'index notation'

(3) $X\colon \mathbf{I} \to \mathcal{C}, \qquad i \mapsto X_i, \qquad a \mapsto (X_a\colon X_i \to X_j) \qquad$ (for $i \in I$ and $a\colon i \to j$ in $\mathbf{I}$).

One writes $\mathcal{C}^{\mathbf{I}}$ the category whose objects are the functors $\mathbf{I} \to \mathcal{C}$ and whose morphisms are the natural transformations $\varphi\colon X \to Y\colon \mathbf{I} \to \mathcal{C}$, with vertical composition.

Examples. For the discrete category 2 (1.3), $\mathcal{C}^2$ is isomorphic to the product category $\mathcal{C}\times\mathcal{C}$. For the ordinal category $\mathbf{2}$, $\mathcal{C}^{\mathbf{2}}$ is the category of morphisms of $\mathcal{C}$ and commutative squares. As to $\mathcal{C}^{\mathbf{2}\times\mathbf{2}}$, we already know (from 2.2) that a diagram of this type is a commutative square of $\mathcal{C}$

$$\text{(4)} \qquad \begin{array}{ccc} X_{00} & \longrightarrow & X_{10} \\ \downarrow & \searrow & \downarrow \\ X_{01} & \longrightarrow & X_{11} \end{array}$$

while a morphism $\varphi\colon X \to Y\colon \mathbf{2}\times\mathbf{2} \to \mathcal{C}$ amounts to a commutative cube in $\mathcal{C}$.

Exercise. Prove that assigning a natural transformation $\varphi\colon F \to G\colon \mathcal{C} \to \mathcal{D}$ is equivalent to giving a functor $\mathcal{C} \to \mathcal{D}^{\mathbf{2}}$, or also to giving a functor $\mathcal{C}\times\mathbf{2} \to \mathcal{D}$.

2.6. Whisker composition and horizontal composition. In the following situation:

(1) $$\mathcal{C}' \xrightarrow{H} \mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}}{\scriptstyle \downarrow\varphi}\ \mathcal{D} \xrightarrow{K} \mathcal{D}'$$

one defines the *whisker composition* $K\varphi H\colon KFH \to KGH\colon \mathcal{C}' \to \mathcal{D}'$, with components

(2) $(K\varphi H)X' = K(\varphi(HX'))\colon KFH(X') \to KGH(X')$ (for every object X' in $\mathcal{C}'$).

This 'ternary' composition law is associative and has identities, in the appropriate sense:

(3) $K'(K\varphi H)H' = (K'K)\, \varphi\, (HH')$ (*associativity*),

$1_{\mathcal{D}}\, \varphi 1_{\mathcal{C}} = \varphi$, $K.\mathrm{id}(F).H = \mathrm{id}(KFH)$. (*identities*).

Moreover, we have a *reduced interchange property*:

(4) $$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}}{\scriptstyle \downarrow\varphi}\ \mathcal{D} \underset{G'}{\overset{F'}{\rightrightarrows}}{\scriptstyle \downarrow\psi}\ \mathcal{E} \qquad \psi G.F'\varphi = G'\varphi.\psi F.$$

This allows one to define the *horizontal composition* of two natural transformations φ, ψ which are *horizontally consecutive*, as in diagram (4)

(5) $\psi * \varphi = \psi G.F'\varphi = G'\varphi.\psi F\colon F'F \to G'G\colon \mathcal{C} \to \mathcal{E}$.

Proposition. The horizontal composition of natural transformations is associative, has identities (consisting of the identity transformations of identity functors) and satisfies the *middle-four interchange property* with vertical composition:

(1) $$\mathcal{C} \ \overset{F}{\underset{H}{\Rrightarrow}}{\scriptstyle \downarrow\varphi,\ \downarrow\psi}\ \mathcal{D} \ \overset{F'}{\underset{H'}{\Rrightarrow}}{\scriptstyle \downarrow\sigma,\ \downarrow\tau}\ \mathcal{E} \qquad (\tau\sigma) * (\psi\varphi) = (\tau * \varphi)(\sigma * \varphi).$$

Remark. Restricting to small categories, we have enriched the category **Cat** of small categories and functors (2.1) with 'higher morphisms', the natural transformations, having two composition laws, vertical and horizontal, which satisfy various algebraic equations (for associativity, identities, interchange). All this can be expressed by saying that **Cat** forms a *2-category* (see 5.4); it is the beginning of 'higher-dimensional category theory' - which is presently one of the main fields of research in Category Theory.

2.7. Equivalence of categories and skeleta. An *equivalence of categories* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ which is invertible up to isomorphism of functors, i.e. there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}\mathcal{C}$ and $FG \cong \mathrm{id}\mathcal{D}$.

An *adjoint equivalence of categories* is a four-tuple $(F, G, \eta, \varepsilon)$ where:

(1) $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are functors,

$\eta\colon \mathrm{id}\mathcal{C} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}\mathcal{D}$ are isomorphisms of functors,

$F\eta = (\varepsilon F)^{-1}\colon F \to FGF$, $\eta G = (G\varepsilon)^{-1}\colon G \to GFG$.

Theorem (Characterisation of the equivalence of categories). The following conditions on a functor $F: \mathcal{C} \to \mathcal{D}$ are equivalent:

(a) F is an equivalence of categories,

(b) F can be completed to an adjoint equivalence of categories $(F, G, \eta, \varepsilon)$,

(c) F is faithful, full and *essentially surjective on objects*.

The last condition means that: for every object Y of $\mathcal{D}$ there exists some object X in $\mathcal{C}$ such that F(X) is isomorphic to Y in $\mathcal{D}$. The proof of the equivalence is rather long and requires the axiom of choice, for classes.

Remark. One says that two categories $\mathcal{C}, \mathcal{D}$ are *equivalent*, and one writes $\mathcal{C} \simeq \mathcal{D}$, if there exists an equivalence as above. From property (c) it follows easily that this is indeed an equivalence relation. Being isomorphic categories, *written* $\mathcal{C} \cong \mathcal{D}$, is a stronger fact.

Exercise. Prove that, for any commutative unitary ring R, the functor F: **Mat**(R) → R-**Mod** (see 2.2, Ex. 1) induces an equivalence of categories

(1) F: **Mat**(R) → $\mathcal{D}$,

where $\mathcal{D}$ is the full subcategory of R-**Mod** determined by the finite-dimensional free R-modules.

Definition. A category $\mathcal{C}$ is said to be *skeletal* if any two isomorphic objects of $\mathcal{C}$ coincide. The *skeleton* of a category $\mathcal{C}$ is, by definition, a skeletal category $\mathcal{C}_0$ equivalent to $\mathcal{C}$. One can prove its existence by choosing (with the axiom of choice for classes) precisely one object in every isomorphism class of objects of $\mathcal{C}$ and letting $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ determined by the chosen objects. Then the inclusion $\mathcal{C}_0 \subset \mathcal{C}$ is an equivalence of categories, by the previous characterisation theorem.

Proposition. (a) Two skeletal categories are equivalent if and only if they are isomorphic.

(b) If $\mathcal{C}, \mathcal{D}$ have skeleta $\mathcal{C}_0, \mathcal{D}_0$ (respectively), then $\mathcal{C}, \mathcal{D}$ are equivalent if and only if $\mathcal{C}_0, \mathcal{D}_0$ are isomorphic.

Remark. Skeleta are not really important. But they make clear that an equivalence of categories is, loosely speaking, an 'isomorphism up to multiplication of isomorphic objects'.

Exercises. 1. Without using the axiom of choice, construct a skeleton of the category f**Set** of finite sets and mappings.

2. Prove that, if K is a commutative field, then **Mat**(K) is a skeleton of the category of finite dimensional vector spaces on K.

2.8. Contravariant functors. A *contravariant functor* $F: \mathcal{C} \dashrightarrow \mathcal{D}$ (notice the dot-marked arrow) consists of the following data:

(a) a mapping $F_0: \mathrm{Ob}\mathcal{C} \to \mathrm{Ob}\mathcal{D}$, generally written as $X \mapsto F(X)$,

(b) for every pair of objects X, X' in $\mathcal{C}$, a mapping $F_{XX'}: \mathcal{C}(X, X') \to \mathcal{D}(F(X'), F(X))$, generally written as $f \mapsto F(f)$,

so that composition is 'reversed' and identities are preserved. In other words:

(1) if f, g are consecutive maps in $\mathcal{C}$, then $F(gf) = F(f).F(g)$,

(2) if X is in $\mathcal{C}$, then $F(\mathrm{id}X) = \mathrm{id}F(X)$.

F can be viewed as a covariant functor $F'\colon \mathcal{C}^{op} \to \mathcal{D}$, which allows us to reduce contravariant functors to the covariant ones.

Exercises. 1. Define a *covariant* functor $\mathcal{P}_*\colon \mathbf{Set} \to \mathbf{Set}$ which sends every set X to its set of parts $\mathcal{P}X$ and a *contravariant* functor $\mathcal{P}^*\colon \mathbf{Set} \nrightarrow \mathbf{Set}$ which acts in the same way on objects.

2. Define, for an arbitrary category $\mathcal{C}$, a functor

(3) $\mathrm{Mor}\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$, $\qquad (X, Y) \mapsto \mathcal{C}(X, Y)$,

and prove that, for $\mathcal{C} = \mathbf{Set}$, the functor $\mathcal{P}^*$ considered above is isomorphic to the functor $\mathrm{Mor}(-, 2)\colon \mathcal{C}^{op} \to \mathbf{Set}$ (where $2 = \{0, 1\}$).

2.9. Representable functors. Let X_0 be an object of the category $\mathcal{C}$. The functor

(1) $\mathrm{Mor}(X_0, -)\colon \mathcal{C} \to \mathbf{Set}$,

is said to be *represented* by the object X_0. More generally, a functor $F\colon \mathcal{C} \to \mathbf{Set}$ is said to be *representable* if it is isomorphic to the functor (1), for a suitable object X_0 in $\mathcal{C}$. One proves that this object is determined by F up to isomorphism.

Exercises. Prove that the standard forgetful functor $U\colon \mathcal{C} \to \mathbf{Set}$ is representable, when $\mathcal{C}$ is one of the following categories: **Set**, **Top**, **Hsd**, **Ab**, **Gp**, **Mon**, R-**Mod**, **Ban**. Prove the same for the unit-ball functor $B_1\colon \mathbf{Ban}_1 \to \mathbf{Set}$. The last exercise of 2.8 amounts to saying that $\mathcal{P}^*\colon \mathbf{Set}^{op} \to \mathbf{Set}$ is representable. Prove that $\mathcal{P}_*\colon \mathbf{Set} \to \mathbf{Set}$ is *not* representable.

Yoneda Lemma. Let $F, G\colon \mathcal{C} \to \mathbf{Set}$ be two functors, with $F = \mathrm{Mor}(X_0, -)$. Then the canonical mapping

(1) $y\colon \mathrm{Nat}(F, G) \to G(X_0)$, $\qquad \varphi \mapsto (\varphi X_0)(\mathrm{id}X_0) \in G(X_0)$,

from the set of natural transformations $\varphi\colon F \to G$ to the set $G(X_0)$ is a bijection. This result can be extended to a representable functor $F \cong \mathrm{Mor}(X_0, -)$, making use of this isomorphism.

Hint. One constructs the inverse mapping as follows:

(2) $y'\colon G(X_0) \to \mathrm{Nat}(F, G)$, $\qquad y'(x)(X)\colon \mathrm{Mor}(X_0, X) \to GX, \ \ f \mapsto (Gf)(x)$. □

CHAPTER 3. Limits and colimits

3.1. Products and terminal object. In a category $\mathcal{C}$, the *product* of a family $(X_i)_{i\in I}$ of objects, indexed on a *set* I, is defined as an object X equipped with a family of morphisms $p_i\colon X \to X_i$ ($i \in I$), called *projections*, which satisfy the following *universal property*:

- for every object Y and every family of morphisms $f_i\colon Y \to X_i$, there exists a unique morphism $f\colon Y \to X$ such that, for all $i \in I$, $p_i f = f_i$.

The solution need not exist. But, if it exists, it is determined up to a unique *coherent* isomorphism, in the sense that if Y is also a product of the family $(X_i)_{i\in I}$ with projections $q_i\colon Y \to X_i$, then the morphism $f: X \to Y$ which commutes with all projections (i.e. $q_i f = p_i$, for all indices i) is an isomorphism. The product of the family (X_i) is denoted as $\Pi_i X_i$.

Hint. Let $g: Y \to X$ be the unique morphism such that $p_i g = q_i$, for all i, and prove that f and g are inverse.

Definition. We say that a category $\mathcal{C}$ *has products* (resp. *has finite products*) if every family of objects indexed *on a set* (resp. on a finite set) has a product in $\mathcal{C}$.

Remark. A family $(X_i)_{i \in I}$ of objects of $\mathcal{C}$ amounts to a mapping $X: I \to \mathrm{Ob}\mathcal{C}$. Therefore, there is *one* empty family of such objects, the trivial mapping $\emptyset \to \mathrm{Ob}\mathcal{C}$. Its product means an object X (equipped with no projections) such that for every object Y (equipped with no maps) there is a unique morphism $f: Y \to X$ (satisfying no conditions). The solution is called the *terminal* object of $\mathcal{C}$; again, it need not exist, but is determined up to a unique isomorphism. It can be written as $\top$.

Exercises. 1. Prove that a category has finite products if and only if it has binary products $X_1 \times X_2$ and a terminal object.

2. Prove that the categories **Set**, **Top**, **Hsd**, **Ab**, **Gp**, **Mon**, R-**Mod** have products. Prove that **Ban**, **Ban**$_1$, **Mtr** and **Mtr**$_1$ have finite products.

Remark. **Ban**$_1$ also has infinite products, but their construction is less elementary.

3.2. Equalisers and regular subobjects. Let $f, g: X \to Y$ be two parallel maps. Their equaliser is a map $m: E \to X$ such that

(a) $fm = gm$,

(b) every map $h: Z \to X$ such that $fh = gh$ factorises uniquely through m, i.e. there exists a unique map $w: Z \to E$ such that $mw = h$.

The solution, if it exists, is determined up to a unique isomorphism coherent with the data. The uniqueness part in (b) is equivalent to saying that m is a monomorphism.

A monomorphism $m: E \to X$ which is the equaliser of some pair of maps $f, g: X \to Y$ is said to be a *regular* monomorphism. A regular mono $m': E' \to X$ is *equivalent* to m if there exists an isomorphism $u: E \to E'$ such that $m = m'u$ (which means that m' is also an equaliser of the same pair). A *regular subobject* of X in $\mathcal{C}$ is an equivalence class $[m]$ of regular monomorphisms of codomain X, in this sense; or, better, a chosen representative of such a class. (More generally, one defines a *subobject* as a distinguished monomorphism, in a similar equivalence class.)

Exercise. Characterise the equalisers, regular monos and regular subobjects in **Set**, **Top**, **Hsd**, **Ab**, R-**Mod**, **Ban**$_1$, **Ban**, **Mtr** and **Mtr**$_1$.

3.3. Pullbacks. Let $f: X \to Z$, $g: Y \to Z$ be two morphisms with the same codomain. Their *pullback* is an object A equipped with two maps $u: A \to X$, $v: A \to Y$ such that:

(1)
$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

(a) $fu = gv$,

(b) for every triple (A', u', v') such that $fu' = gv'$, there exists a unique map $w: A' \to A$ such that $uw = u'$, $vw = v'$.

The solution, if it exists, is determined up to a unique isomorphism coherent with the data.

Theorem. If a category has binary products and equalisers, it also has pullbacks.

Exercise. Describe pullbacks in **Set**, **Top**, **Hsd**, **Ab**, R-**Mod**.

3.4. General limits. Let **I** be a small category and $X: \mathbf{I} \to \mathcal{C}$ a diagram, for which we use the index notation described in 2.5.

A *cone* for X is an object A of $\mathcal{C}$ equipped with a family of maps $(f_i: A \to X_i)_{i \in I}$ in $\mathcal{C}$ such that all the following triangles commute

$$(1) \qquad \begin{array}{ccc} A & \xrightarrow{f_i} & X_i \\ & {\scriptstyle f_j}\searrow & \downarrow {\scriptstyle X_a} \\ & & X_j \end{array} \qquad X_a.f_i = f_j \qquad (a: i \to j \text{ in } \mathbf{I}).$$

The *limit* of $X: \mathbf{I} \to \mathcal{C}$ is a universal cone $(L, (u_i: L \to X_i)_{i \in I})$. This means a cone of X such that every cone $(A, (f_i: A \to X_i)_{i \in I}$ 'factorises uniquely through the former': i.e. there is a unique map $f: A \to L$ such that, for all $i \in I$, $u_i f = f_i$.

Again, the solution need not exist. When it does, it is determined up to a unique coherent isomorphism. The object L is denoted as $\mathrm{Lim}(X)$.

One says that a functor $F: \mathcal{C} \to \mathcal{D}$ *preserves the limit* $(L, (u_i: L \to X_i)_{i \in I})$ of a functor $X: \mathbf{I} \to \mathcal{C}$ if the cone $(FL, (Fu_i: FL \to FX_i)_{i \in I})$ is the limit of the composed functor $FX: \mathbf{I} \to \mathcal{D}$. One says that F *preserves limits* if it preserves those limits *which exist* in $\mathcal{C}$. Analogously for the preservation of products, equalisers, finite limits, etc.

Exercises. 1. Prove that products, equalisers and pullbacks are limits, over convenient small categories.

2. Prove that the uniqueness of f in the universal property of the limit is equivalent to saying that the family $(u_i: L \to X_i)_{i \in I}$ is *jointly mono*. This means that, given two maps $f, g: X \to L$ such that, for all $i \in I$, $u_i f = u_i g$, one has $f = g$.

3. Prove that a representable functor $\mathcal{C} \to$ **Set** always preserves limits.

3.5. Complete categories. A category $\mathcal{C}$ is said to be *complete* (resp. *finitely complete*) if it has a limit for every functor $\mathbf{I} \to \mathcal{C}$ defined over a *small* category (resp. a *finite* category).

Theorem (Construction and preservation of limits). (a) A category $\mathcal{C}$ is complete (resp. finitely complete) if and only if it has equalisers and products (resp. finite products).

(b) If $\mathcal{C}$ is complete, a functor $F: \mathcal{C} \to \mathcal{D}$ preserves all limits (resp. all finite limits) if and only if it preserves equalisers and products (resp. finite products).

Examples. We conclude that **Set**, **Top**, **Hsd**, **Ab**, **Gp**, **Mon**, R-**Mod**, $\mathbf{Ban}_1$ are complete, while **Ban**, **Mtr** and $\mathbf{Mtr}_1$ are finitely complete.

Remark. Most forgetful functors between such categories preserve limits. The unit-ball functor $\mathbf{Ban}_1 \to \mathbf{Set}$ preserves all limits, while the underlying-set functor of $\mathbf{Ban}_1$ just preserves finite limits (as the construction of infinite products in $\mathbf{Ban}_1$ would show).

Exercise. Let $\mathbf{I}$ be a small category. Suppose that the category $\mathcal{C}$ has all $\mathbf{I}$-based limits, i.e. it has the limit of every functor $\mathbf{I} \to \mathcal{C}$. Then there is a functor which sends every diagram $X\colon \mathbf{I} \to \mathcal{C}$. to its limit

(1) $\mathrm{Lim}\colon \mathcal{C}^{\mathbf{I}} \to \mathcal{C}, \qquad\qquad X \mapsto \mathrm{Lim}(X).$

Theorem. *A small category which is complete is necessarily a preordered set* (and therefore a complete 'pre-lattice').

3.6. Sums. We begin now to dualise the previous notions, writing things in a less detailed way. The *sum*, or *coproduct* in $\mathcal{C}$ of a family $(X_i)_{i\in I}$ of objects, indexed on a set I, is an object X equipped with a family of morphisms $u_i\colon X_i \to X$ $(i \in I)$, called *injections*, which satisfy the following *universal property*:

- for every object Y and every family of morphisms $f_i\colon X_i \to Y$, there exists a unique morphism $f\colon X \to Y$ such that, for all $i \in I$, $fu_i = f_i$.

Again, if the solution exists, it is determined up to a unique coherent isomorphism. The sum of the family (X_i) is denoted as $\Sigma_i X_i$.

The sum of the empty family is the initial object $\perp$: this means that, for every object X there is precisely one map $\perp \to X$.

Exercise. Prove that the categories **Set**, **Top**, **Hsd**, **Ab**, **Gp**, **Mon**, R-**Mod** have sums. Prove that **Ban**, $\mathbf{Ban}_1$ have finite sums. (Again, $\mathbf{Ban}_1$ has arbitrary sums, but their construction is less elementary.)

3.7. Coequalisers, regular quotients and pushouts. Let $f, g\colon X \to Y$ be two parallel maps. Their coequaliser is a map $p\colon Y \to C$ such that

(a) $pf = pg$,

(b) every map $h\colon Y \to Z$ such that $hf = hg$ factorises uniquely through p, i.e. there exists a unique map $w\colon C \to Z$ such that $wp = h$.

The uniqueness part is equivalent to say that p is epi. An epimorphism $p\colon Y \to C$ is said to be *regular* if it is the coequaliser of some pair of maps $f, g\colon X \to Y$; it is *equivalent* to another regular epi $p'\colon Y \to C'$ if there exists an isomorphism $u\colon C \to C'$ such that $p' = up$. A *regular quotient* of Y in $\mathcal{C}$ is an equivalence class $[p]$ of regular epis of domain Y, in this sense.

Let $f\colon X \to Y$, $g\colon X \to Z$ be two morphisms with the same domain. Their *pushout* is an object A equipped with two maps $u\colon Y \to A$, $v\colon Z \to A$ such that:

(1)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow[v]{} & A \end{array}$$

(a) $uf = vg$,

(b) for every triple (A', u', v') such that $u'f = v'g$, there exists a unique map $w: A \to A'$ such that $wu = u'$, $wv = v'$.

Theorem. If a category has binary sums and coequalisers, it also has pushouts.

Exercise. Characterise the coequalisers, regular epis, regular quotients in **Set**, **Top**, **Ab**, R-**Mod**. Describe pushouts in the same categories.

3.8. General colimits and cocomplete categories. Let **I** be a small category and $X: \mathbf{I} \to \mathcal{C}$ a diagram, in the notation of 2.5.

A *cocone* for X is an object A of $\mathcal{C}$ equipped with a family of maps $(f_i: X_i \to A)_{i \in I}$ in $\mathcal{C}$ such that all the following triangles commute

$$(1) \qquad \begin{array}{ccc} X_i & \xrightarrow{f_i} & A \\ {\scriptstyle X_a}\downarrow & \nearrow_{f_j} & \\ X_j & & \end{array} \qquad f_j.X_a = f_i \qquad \text{(for all arrows } a: i \to j \text{ in } \mathbf{I}).$$

The *colimit* of $X: \mathbf{I} \to \mathcal{C}$ is a universal cocone $(L, (u_i: X_i \to L)_{i \in I})$. This means a cocone of X such that every cocone $(A, (f_i: X_i \to A)_{i \in I}$ 'factorises uniquely through the former': i.e. there is a unique map $f: L \to A$ such that, for all $i \in I$, $fu_i = f_i$. The object L, determined up to a unique coherent isomorphism, is denoted $\mathrm{Colim}(X)$.

A category $\mathcal{C}$ is *cocomplete* (resp. *finitely cocomplete*) if it has a colimit for every functor $\mathbf{I} \to \mathcal{C}$ defined over a small category (resp. a finite category). If, for a fixed small category **I**, the category $\mathcal{C}$ has all **I**-based colimits, there is a functor

$$(2) \quad \mathrm{Colim}: \mathcal{C}^{\mathbf{I}} \to \mathcal{C}, \qquad X \mapsto \mathrm{Colim}(X).$$

Theorem (Construction and preservation of colimits). (a) A category $\mathcal{C}$ is cocomplete (resp. finitely cocomplete) if and only if it has coequalisers and sums (resp. finite sums).

(b) If $\mathcal{C}$ is cocomplete, a functor $F: \mathcal{C} \to \mathcal{D}$ preserves all colimits (resp. all finite colimits) if and only if it preserves coequalisers and sums (resp. finite sums).

3.9. Universal arrows, limits and colimits. We have a functor $U: \mathcal{A} \to \mathcal{C}$ and an object X of $\mathcal{C}$.

(a) A *universal arrow from the object* X *to the functor* U is a pair $(A, \eta: X \to UA)$ consisting of an object A of $\mathcal{A}$ and arrow η of $\mathcal{C}$ which is universal, in the sense that every similar pair $(B, f: X \to UB)$ factorises uniquely through (A, η): there exists a unique $g: A \to B$ in $\mathcal{A}$ such that the following triangle commutes in $\mathcal{C}$

$$(1) \qquad \begin{array}{ccc} X & \xrightarrow{\eta} & UA \\ & {}_f\searrow & \downarrow{\scriptstyle Ug} \\ & & UB \end{array} \qquad Ug.\eta = f.$$

(a*) Dually, a *universal arrow from the functor* U *to the object* X is a pair $(A, \varepsilon: UA \to X)$ consisting of an object A of $\mathcal{A}$ and arrow ε of $\mathcal{C}$ such that every similar pair $(B, f: UB \to X)$

factorises uniquely through (A, ε): there exists a unique $g: B \to A$ in $\mathcal{A}$ such that the following triangle commutes in $\mathcal{C}$

$$\text{(2)} \qquad \begin{array}{ccc} UA & \xrightarrow{\varepsilon} & X \\ {\scriptstyle Ug}\uparrow & \nearrow_{f} & \\ UB & & \end{array} \qquad\qquad \varepsilon.Ug = f.$$

Exercise. Construct the universal arrows from a set X to the forgetful functor **Mon** $\to$ **Set**; from a set X to the forgetful functor **Ab** $\to$ **Set**; from a group G to the inclusion functor **Ab** $\to$ **Gp**.

We show now that limits and colimits can be viewed as universal arrows. Let $\mathbf{I}$ be a small category and $\mathcal{C}$ a category. Consider the category $\mathcal{C}^{\mathbf{I}}$ of diagrams $\mathbf{I} \to \mathcal{C}$ and their natural transformations (2.5). Consider the *diagonal functor*

(3) $D: \mathcal{C} \to \mathcal{C}^{\mathbf{I}}$, $\qquad (DA)i = A, \ (DA)a = \mathrm{id}A \qquad (i \in I, \ a \text{ in } \mathbf{I})$

which sends an object A to the constant functor at A, and a morphism $f: A \to B$ to the natural transformation $Df: DA \to DB: \mathbf{I} \to \mathcal{C}$ whose components are constant at f.

Exercise. Prove that the limit of X in $\mathcal{C}$ is the same as a universal arrow $(L, \varepsilon: DL \to X)$ from the functor D to the object X of $\mathcal{C}^{\mathbf{I}}$. Dually, the colimit of X in $\mathcal{C}$ is the same as a universal arrow $(L, \eta: X \to DL)$ from the object X of $\mathcal{C}^{\mathbf{I}}$ to the functor D.

CHAPTER 4. Adjoint functors

4.1. Theorem and definitions of adjunction. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Adjoint functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$, with F *left adjoint* to G (notation: $F \dashv G$) can be equivalently presented in four main forms.

(a) We assign two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ together with a family of bijections

$$\varphi_{XY}: \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY) \qquad\qquad (X \text{ in } \mathcal{C}, \ Y \text{ in } \mathcal{D}),$$

which is natural in X, Y. More formally, the family (φ_{XY}) is an invertible natural transformation

$$\varphi: \mathcal{D}(F(-), .) \to \mathcal{C}(- , G(.)): \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}.$$

(b) We assign a functor $G: \mathcal{D} \to \mathcal{C}$ and, for every object X in $\mathcal{C}$, a universal arrow

$(F_0X, \eta X: X \to GF_0X)$ $\qquad$ from the object X to the functor G (3.9).

(b*) We assign a functor $F: \mathcal{C} \to \mathcal{D}$ and, for every object Y in $\mathcal{D}$, a universal arrow

$(G_0Y, \varepsilon Y: FG_0Y \to Y)$ $\qquad$ from the functor F to the object Y (3.9).

(c) We assign two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$, together with two natural transformations

$\eta: \mathrm{id}\mathcal{C} \to GF$ (the *unit*), $\qquad$ $\varepsilon: FG \to \mathrm{id}\mathcal{D}$ (the *counit*),

which satisfy the *triangular identities*: $\varepsilon F.F\eta = \mathrm{id}F$, $G\varepsilon.\eta G = \mathrm{id}G$

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F \qquad\qquad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$
$$\underset{\mathrm{id}F}{\longrightarrow} \qquad\qquad\qquad\qquad \underset{\mathrm{id}G}{\longrightarrow}$$

Given (a), one defines

$\eta X = \varphi_{X,FX}(1_{FX})\colon X \to GFX, \qquad \varepsilon Y = (\varphi_{GY,Y})^{-1}(1_{GY})\colon FGY \to Y.$

Given (b) or (c) and a map $v\colon FX \to Y$ in $\mathcal{D}$, one defines

$\varphi_{XY}(v) = G(v).\eta X\colon X \to GFX \to GY.$

Moreover, in case (b), one defines F on maps in the unique way which makes the family (ηX) a natural transformation $\eta\colon \mathrm{id}\mathcal{C} \to GF$.

4.2. Remarks. The previous forms have different features. Form (a) is the classical definition of an adjunction, and is at the origin of the name (compare with adjoint maps of Hilbert spaces). Form (b) is used when one starts from an 'easily defined' functor and wants to construct its left adjoint; form (b*) is dual to the previous one, and used in a dual way. Form (c) is the most adequate for the formal theory of adjunctions (and makes sense *in* an abstract 2-category).

An adjoint equivalence (2.7) is the same as an adjunction where the unit and counit are invertible; this follows immediately from form (c).

4.3. Main properties of adjoint functors

Theorem (Uniqueness). Given a functor, its left adjoint (if it exists) is uniquely determined up to isomorphism.

Theorem (Composing adjoint functors). Given two consecutive adjunctions

(1) $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :G, \qquad \eta\colon 1 \to GF, \quad \varepsilon\colon FG \to 1,$

$H\colon \mathcal{D} \rightleftarrows \mathcal{E} :K, \qquad \rho\colon 1 \to KH, \quad \sigma\colon HK \to 1,$

there is a composed adjunction from the first to the third category:

(2) $HF\colon \mathcal{C} \rightleftarrows \mathcal{E} :GK,$

$G\rho F.\eta\colon\ 1 \to GF \to GK.HF, \qquad \sigma.H\varepsilon K\colon HF.GK \to HK \to 1.$

Theorem (Adjoints and limits). A left adjoint preserves (the existing) colimits, a right adjoint preserves (the existing) limits.

4.4. Exercises on adjunctions. 1. Construct the following adjoint functors.

(a) The left adjoint and the right adjoint to the forgetful functor U: **Top** $\to$ **Set**.

(b) The left adjoint to the forgetful functors from **Ab**, **Gp**, **Mon**, R-**Mod** to **Set**. Prove also that such functors do *not* have a right adjoint, showing that each of them does not preserve some colimit.

(c) The left adjoint to the embedding **Ab** $\to$ **Gp**. Prove also that the right adjoint does not exist.

2. Prove that the following constructions can be obtained as adjoints to obvious functors.

(d) Completion of a metric space.

(e) Stone-Cech compactification of a topological space.

(f) Tensor product of modules.

(g) Rings of fractions.

(h) Limits and colimits in a category.

3. Show that the direct-sum functor $- \oplus - :$ **Ab** × **Ab** → **Ab** is, at the same time, *left and right adjoint* to the diagonal functor D: **Ab** → **Ab** × **Ab**. The same holds, more generally, for R-**Mod**.

4.5. Galois connections. A *Galois connection* is essentially an adjunction between ordered sets, viewed as categories. Given a pair X, Y of ordered sets, it can also be presented in a *contravariant form*, in the following equivalent ways.

(a) We assign two *decreasing* mappings $f: X \to Y$ and $g: Y \to X$ such that:

$$y \leq f(x) \text{ in } Y \iff x \leq g(y) \text{ in } X.$$

(b) We assign a *decreasing* mapping $g: Y \to X$ such that, for every $x \in X$, there exists:

$$f(x) = \min\{y \in Y \mid x \leq g(y)\}.$$

(c) We assign two *decreasing* mappings $f: X \to Y$ and $g: Y \to X$ such that $gf \geq \mathrm{id}X$ and $fg \geq \mathrm{id}Y$.

Remarks. This contravariant form is *symmetric*: there is no *left* and *right* part. An element of X is said to be *closed* in the connection if $x = gf(x)$, or equivalently if $x \in g(Y)$.

Exercises. 1. Prove that $f = fgf$ and $g = gfg$. Prove that the connection restricts to a bijection between the closed elements of X and those of Y.

2. For a commutative unitary ring R, define a Galois connection between the following two sets of parts, ordered by inclusion:

$\mathcal{P}(R^n)$, $\qquad\qquad$ $\mathcal{P}(R[X_1, X_2,..., X_n])$.

Plainly, every 'closed' subset of the second set is an ideal of polynomials, but the converse is not true. If R is an algebraically closed commutative field, the 'closed' subsets of polynomials coincide with the radical ideals (*Nullstellensatz*).

4.6. Reflective subcategories. A subcategory $\mathcal{C}' \subset \mathcal{C}$ is said to be *reflective* (notice: not 'reflexive') if the inclusion functor has a left adjoint, and *coreflective* if it has a right adjoint.

Exercises. Prove that **Ab** is reflective in **Gp**. Prove that **Hsd** is reflective in **Top**. Prove that the full subcategory of **Ab** formed by torsion abelian groups is coreflective in **Ab**.

4.7. Theorem (Faithful and full adjoints). Suppose we have an adjunction $F \dashv G$, with counit $\varepsilon: FG \to 1$.

(a) G is faithful if and only if all the components εY of the counit are epi;

(b) G is full if and only if all the components εY of the counit are split mono;

(c) G is full and faithful if and only if the counit is invertible.

4.8. The adjoint Functor Theorem

Theorem (P. Freyd). Let $G: \mathcal{D} \to \mathcal{C}$ be a functor defined on a complete category. Then G has a left adjoint if and only if it preserves all limits and:

(*Solution Set Condition*) for every X in $\mathcal{C}$ there exists a *solution set*, i.e. a *set* of objects $S(X)$ in $\mathcal{D}$ such that every morphism $f: X \to GY$ (with Y in $\mathcal{D}$) factorises as

$$(1) \qquad \begin{array}{ccc} X & \xrightarrow{f_0} & GY_0 \\ & \searrow^{f} & \downarrow Gg \\ & & GY \end{array} \qquad\qquad Gg.f_0 = f,$$

for convenient $Y_0 \in S(X)$, f_0 in $\mathcal{C}$ and g in $\mathcal{D}$.

Exercises. Prove in this way the existence of the left adjoint to the forgetful functor **Gp** $\to$ **Set** and to the embedding **Hsd** $\to$ **Top**.

4.9. A digression on mathematical structures and categories. When studying a mathematical structure with the help of category theory, it is crucial to choose the 'right' kind of structure and the 'right' kind of morphisms, so that the result is sufficiently general and 'natural' to have good properties (with respect to the goals of our study) - even if we are interested in more particular situations.

For instance, the category **Top** of topological spaces and continuous mappings is a classical framework for studying topology. Among its good properties there is the fact that all (co)products and (co)equalisers exist, and are computed *as in* **Set**, then equipped with a suitable topology determined by the structural maps. (More generally, this is true of all limits and colimits, and is a consequence of the fact that the forgetful functor **Top** $\to$ **Set** has a left *and* a right adjoint, corresponding to discrete and chaotic topologies). Hausdorff spaces are certainly important, but it is often better to view them in **Top**, as their category is less well behaved: coequalisers exist, but are not computed as in **Set**, i.e. preserved by the forgetful functor to **Set**.

(Many category theorists would agree with Mac Lane [Ma], saying that even **Top** is not sufficiently good, because it is not a cartesian closed category, and prefer - for instance - the category of compactly generated spaces; however, a reader interested in Algebraic Topology can be satisfied with the fact that the standard interval and its powers are exponentiable in **Top**; see 5.2.)

Similarly, if we are interested in ordered sets, it is often better to view them in the category of *pre*ordered sets and (weakly) increasing mappings, where (co)products and (co)equalisers not only exist, but again are computed *as in* **Set**, with a suitable preorder determined by the structural maps.

Another point to be kept in mind is that the isomorphisms of the category (i.e. its invertible arrows) should indeed 'preserve' the structure we are interested in, or we risk of studying something different from our purpose. As a trivial example, the category **T** of topological spaces and *all* mappings between them has practically nothing to do with topology: an isomorphism of **T** is any bijection between topological spaces. Indeed, **T** is *equivalent to the category of sets*, and is a 'deformed' way of looking at the latter. Less trivially, the category **M** of metric spaces and continuous mappings misses crucial properties of metric spaces, since its invertible morphisms do not preserve completeness. In fact, **M** is equivalent to the category of *metrisable topological spaces* and continuous mappings, and should be viewed in this way. A 'reasonable' category of metric spaces should be based on *Lipschitz* maps, as **Mtr** or $\mathbf{Mtr}_1$.

Excluding particular cases is, generally speaking, a bad option. **Ab** and **Gp** are both important, but the category of *non-commutative groups* seems to be of no importance; certainly, it has practically none of the good categorical properties of **Ab** and **Gp**. Of course, one can always consider such groups within the category **Gp**, when useful.

A striking example of this kind is concerned with the category **Sgr** of semigroups and their homomorphisms. In the domain of 'Universal Algebra', a semigroup is assumed to be non-empty; now, this exclusion - which is never assumed in Category Theory - would destroy much of the good properties of **Sgr**, both from a categorical and a 'practical' point of view: for instance, subsemigroups would not be closed under intersection and counterimages; the subsemigroups of a given semigroup would no longer form a lattice; the category of semigroups would not be complete nor cocomplete.

CHAPTER 5. Complements

This is an outline of subjects which can be developed in seminars.

5.1. Monads and algebras. A *monad* in the category $\mathcal{C}$ is a triple (T, η, μ) where $T\colon \mathcal{C} \to \mathcal{C}$ is an endo-functor, while $\eta\colon 1 \to T$ and $\mu\colon T^2 \to T$ are natural transformations (called the *unit* and *multiplication* of the monad) which make the following diagrams commute:

(1)
$$\begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \diagdown\!\diagdown & \downarrow{\scriptstyle \mu} & /\!/ & \\ & & T & & \end{array} \qquad\qquad \begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

It is easy to verify that an adjunction

(2) $F\colon \mathcal{C} \rightleftarrows \mathcal{A} :U, \qquad\qquad \eta\colon 1 \to UF, \quad \varepsilon\colon FU \to 1,$

yields a monad (T, η, μ) on $\mathcal{C}$, where $T = UF\colon \mathcal{C} \to \mathcal{C}$, η is the unit of the adjunction and $\mu = U\varepsilon F\colon UF.UF \to UF$.

Given an arbitrary monad, as above, one defines the category $\mathcal{C}^T$ of T-*algebras* (or *Eilenberg-Moore algebras* for T): these are pairs $(X, a\colon TX \to X)$ consisting of an object X of $\mathcal{C}$ and a map a (the *algebraic structure*) satisfying two coherence axioms:

(2) $a.\eta X = 1_X, \qquad\qquad a.Ta = a.\mu X,$

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & TX \\ & \diagdown\!\diagdown & \downarrow{\scriptstyle a} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} T^2X & \xrightarrow{Ta} & TX \\ {\scriptstyle \mu X}\downarrow & & \downarrow{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array}$$

A morphism of T-algebras $f\colon (X, a) \to (Y, b)$ is a morphism $f\colon X \to Y$ of $\mathcal{C}$ which preserves the algebraic structures, in the sense that: $fa = b.Tf$.

There is an adjunction

(4) $F^T\colon \mathcal{C} \rightleftarrows \mathcal{C}^T : U^T$, $\qquad \eta^T = \eta\colon 1 \to U^TF^T, \qquad \varepsilon^T\colon F^TU^T \to 1$,

whose associated monad coincides with the given one.

A functor $U\colon \mathcal{A} \to \mathcal{C}$ is said to be *monadic*, or to make $\mathcal{A}$ *monadic over* $\mathcal{C}$, if it has a left adjoint $F\colon \mathcal{C} \to \mathcal{A}$ and moreover the following *comparison functor* from $\mathcal{A}$ to the category of algebras $\mathcal{C}^T$ of the monad associated to the adjunction

(5) $K\colon \mathcal{A} \to \mathcal{C}^T$, $\qquad K(A) = (UA, U\varepsilon A\colon UFUA \to UA)$,

is an equivalence of categories.

This *formalisation of the algebraic character of a category* is much wider than that given in Universal Algebra: for instance, the category of compact Hausdorff spaces is monadic over **Set** (see [Ma]).

Another important topic in this subject is *Kleisli algebras*: a category of free algebras on the given monad.

Exercise. Present the category **Ab** of abelian groups as a category of T-algebras over **Set**.

5.2. Monoidal categories. A *monoidal category* $(\mathcal{C}, \otimes, E)$ is a category equipped with a *tensor product*, which is a functor in two variables

(1) $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\qquad (A, B) \mapsto A \otimes B$.

The latter is assumed to be associative up to a natural isomorphism $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$, and the object E is assumed to be an identity, up to natural isomorphisms $E\otimes A \cong A \cong A\otimes E$. All these isomorphisms must form a *coherent system*, which allows one 'to forget them' and write $(A\otimes B)\otimes C = A\otimes(B\otimes C)$, $E\otimes A = A = A\otimes E$.

A *symmetric* monoidal category is further equipped with a symmetry isomorphism, coherent with the other ones:

(2) $s(X, Y)\colon X\otimes Y \to Y\otimes X$.

The latter cannot be omitted: notice that $s(X, X)\colon X\otimes X \to X\otimes X$ is not the identity, in general.

In a *symmetric* monoidal category $\mathcal{C}$, an object A is said to be *exponentiable* if the functor $-\otimes A\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint, often written as $(-)^A\colon \mathcal{C} \to \mathcal{C}$ or $\mathrm{Hom}(A, -)$ (and called then an *internal hom*). Since adjunctions compose, it follows easily that all its powers A^n are also exponentiable, with

(3) $\mathrm{Hom}(A^n, -) = (\mathrm{Hom}(A, -))^n$.

A symmetric monoidal category is said to be *closed* if all its objects are exponentiable. In the non-symmetric case, one should consider a left and a right Hom functor. (For instance, this is the case of the Kan tensor product of cubical sets.)

A category $\mathcal{C}$ with finite products has a symmetric monoidal structure given by the categorical product. This structure is called *cartesian*, and $\mathcal{C}$ is said to be *cartesian closed* if all its objects are exponentiable with respect to the cartesian product. **Cat** is cartesian closed, with the exponential defined in 2.5. The category **Ab** of abelian groups is symmetric monoidal closed, with respect to the usual tensor product and Hom functor.

Ab *is not cartesian closed*: for every abelian group $A \neq 0$, the product $-\times A$ does not preserve sums. **Top** *is not cartesian closed*: for a fixed Hausdorff space A, the product $-\times A$ preserves quotients, i.e. coequalisers, (if and) only if A is locally compact ([Mc], Thm. 2.1 and footnote (5)). However, *every locally compact Hausdorff space is exponentiable* in **Top**; in particular, the standard interval (and its powers) is exponentiable, a crucial fact in homotopy theory.

5.3. Additive categories. A *preadditive category* **C** is a category enriched on **Ab**. Explicitly, this means that every hom-set $\mathbf{C}(X, Y)$ is equipped with a structure of abelian group and that composition is bilinear. The zero element of $\mathbf{C}(X, Y)$ is written as $0_{XY}\colon X \to Y$.

Let **C** be preadditive. The following conditions on the object Z are equivalent:

(a) Z is terminal,

(b) Z is initial,

(c) $\mathbf{C}(X, X)$ is the null group,

(d) $\mathrm{id}Z = 0_{ZZ}$.

In this case Z is the zero object, written as 0.

In the same situation, given two objects X_1, X_2, their *biproduct* $X = X_1 \oplus X_2$ comes with *injections* $u_i\colon X_i \to X$ and *projections* $p_i\colon X \to X_i$ satisfying the following equivalent properties:

(i) (X, p_1, p_2) is the product of X_1, X_2 and the injections are $u_1 = (\mathrm{id}X_1, 0)$, $u_2 = (0, \mathrm{id}X_2)$;

(ii) (X, u_1, u_2) is the sum of X_1, X_2 and the projections are $p_1 = [\mathrm{id}X_1, 0]$, $p_2 = [0, \mathrm{id}X_2]$;

(iii) the following relations hold

$$\begin{array}{ccccc} X_1 & & & & X_2 \\ & \searrow^{u_1} & & {}^{u_2}\swarrow & \\ \| & & X & & \| \\ & \swarrow_{p_1} & & {}_{p_2}\searrow & \\ X_1 & & & & X_2 \end{array} \qquad\qquad \begin{array}{l} p_i u_i = \mathrm{id}X_i, \\ \\ u_1 p_1 + u_2 p_2 = \mathrm{id}X. \end{array} \tag{1}$$

Therefore, in a preadditive category, the existence of binary (or finite) products is equivalent to the existence of binary (or finite) sums, which are called biproducts and written as $\bigoplus_i X_i$.

An *additive category* is a preadditive category with finite biproducts. A preadditive category is finitely complete if and only if it is additive and has kernels.

Further topics to be developed are: additive functors; kernels and cokernels; abelian categories; exact sequences; left and right exact functors.

5.4. Two-dimensional categories. A *sesqui-category* is a category $\mathcal{C}$ equipped with:

(a) for each pair of parallel morphisms $f, g\colon X \to Y$, a set of *2-cells* $\mathcal{C}_2(f, g)$ whose elements are written as $\varphi\colon f \to g\colon X \to Y$ (or $\varphi\colon f \to g$), so that each map f has a *trivial* (or *degenerate*, or *identity*) *endocell* $\mathrm{id}(f)\colon f \to f$ (here also, φ must determine its domain and codomain);

(b) a *whisker composition*, or *reduced horizontal composition*, for maps and homotopies

(1) $$X' \xrightarrow{\ h\ } X \;\begin{array}{c} \xrightarrow{\ f\ } \\ \downarrow\varphi \\ \xrightarrow[\ g\ ]{} \end{array}\; Y \xrightarrow{\ k\ } Y' \qquad\qquad k\circ\varphi\circ h\colon\ kfh \to kgh\colon\ X' \to Y',$$

also written as $k\varphi h$;

(c) a *concatenation*, or *vertical composition* of 2-cells $\psi\cdot\varphi$

(2) $$X \;\begin{array}{c} \xrightarrow{\ f\ } \\ \downarrow\varphi \\ \longrightarrow \\ \downarrow\psi \\ \xrightarrow[\ h\ ]{} \end{array}\; Y \qquad\qquad \psi\cdot\varphi\colon f \to h\colon X \to Y.$$

These data must satisfy the following axioms:

(3) $k'\circ(k\circ\varphi\circ h)\circ h' \;=\; (k'k)\circ\varphi\circ(hh')$, $\qquad \chi\cdot(\psi\cdot\varphi) \;=\; (\chi\cdot\psi)\cdot\varphi$, $\qquad$ (*associativities*),

$1_Y\circ\varphi\circ 1_X \;=\; \varphi$, $\qquad k\circ id(f)\circ h \;=\; id(kfh)$ $\qquad \varphi\cdot id(f) \;=\; \varphi \;=\; id(g)\cdot\varphi$, $\qquad$ (*identities*),

$k\circ(\psi\cdot\varphi)\circ h \;=\; (k\circ\psi\circ h)\cdot(k\circ\varphi\circ h)$ $\qquad$ (*distributivity of the vertical composition*).

A *2-category* is a sesqui-category which also satisfies:

(4) $$X \;\begin{array}{c} \xrightarrow{\ f\ } \\ \downarrow\varphi \\ \xrightarrow[\ g\ ]{} \end{array}\; Y \;\begin{array}{c} \xrightarrow{\ f'\ } \\ \downarrow\psi \\ \xrightarrow[\ g'\ ]{} \end{array}\; Z \qquad\qquad \psi g\cdot f'\varphi \;=\; g'\varphi\cdot\psi f \qquad (\textit{reduced interchange}).$$

Then, one defines the *horizontal composition* of 2-cells φ, ψ which are *horizontally consecutive*, as in diagram (4)

(5) $\psi\circ\varphi \;=\; \psi g\cdot f'\varphi \;=\; g'\varphi\cdot\psi f\colon\ f'f \to g'g\colon X \to Z.$

One can prove that the horizontal composition of 2-cells is associative, has identities (consisting of the identity endocells $id(1_X)$ of identity arrows) and satisfies the *middle-four interchange property* with vertical composition:

(6) $$X \;\begin{array}{c} \xrightarrow{\ f\ } \\ \downarrow\varphi \\ \longrightarrow \\ \downarrow\psi \\ \xrightarrow[\ h\ ]{} \end{array}\; Y \;\begin{array}{c} \xrightarrow{\ f'\ } \\ \downarrow\sigma \\ \longrightarrow \\ \downarrow\tau \\ \xrightarrow[\ h'\ ]{} \end{array}\; Z \qquad\qquad (\tau\cdot\sigma)\circ(\psi\cdot\varphi) \;=\; (\tau\circ\psi)\cdot(\sigma\circ\varphi).$$

Examples and remarks. The prime example of such a structure is the 2-category **Cat** of small categories, functors and natural transformations.

The usual definition of a 2-category is based on the complete horizontal composition, rather than on the reduced one. But practically one generally works with the reduced horizontal composition; and there are important cases of sesqui-categories where the reduced interchange property does not hold (and one does not define a complete horizontal composition): for instance, the sesqui-category of chain complexes, chain morphisms and homotopies.

The category **Top**, equipped with the ordinary homotopies of continuous mappings, forms a 2-dimensional structure of a more general kind: it is *not* even a sesqui-category, since - for instance - the vertical composition of homotopies is not associative (but *associative up to higher homotopies*).

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