Weak subobjects and the epi-monic completion of a category

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Abstract. The notion of *weak subobject*, or *variation*, was introduced in [Gr4] as an extension of the notion of subobject, adapted to homotopy categories or triangulated categories, and well linked with their *weak* limits. We study here some formal properties of this notion. The variations in $\mathbf{X}$ can be identified with the (distinguished) subobjects in the *Freyd completion* Fr$\mathbf{X}$, the free category with epi-monic factorisation system over $\mathbf{X}$, which extends the Freyd embedding of the stable homotopy category of spaces in an abelian category [Fr2]. If $\mathbf{X}$ has products and weak equalisers, as Ho**Top** and various other homotopy categories, Fr$\mathbf{X}$ is complete; similarly, if $\mathbf{X}$ has zero-object, weak kernels and weak cokernels, as the homotopy category of pointed spaces, then Fr$\mathbf{X}$ is a homological category [Gr1]; finally, if $\mathbf{X}$ is triangulated, Fr$\mathbf{X}$ is abelian and the embedding $\mathbf{X} \to$ Fr$\mathbf{X}$ is the universal homological functor on $\mathbf{X}$, as in the original case [Fr2]. These facts have consequences on the ordered sets of variations.

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Introduction

A *variation*, or *weak subobject*, of an object A in the category $\mathbf{X}$ is an equivalence class of morphisms with values in A, where $x \sim_A y$ if there exist maps u, v such that x = yu, y = xv [Gr4]; among them, the *monic variations* (having some representative which is a monomorphism) can be identified with subobjects. The variations of A form an ordered set Var(A), possibly large, which is a lattice under weak assumptions on $\mathbf{X}$ (1.1). Dually, a *covariation*, or *weak quotient*, of the object A is an equivalence-class of morphisms from A, extending the notion of a quotient.

Variations are well connected with *weak* limits (these are defined by the existence part of the usual universal property), much in the same way as subobjects are connected with limits; thus, they are of particular interest in homotopy categories and triangulated categories, which generally have ordinary products but only weak equalisers. Nevertheless, the study of weak subobjects in *ordinary* categories, like abelian groups or groups, is interesting in itself and relevant to classify variations in homotopy categories of spaces, by means of homology and homotopy functors.

Various classifications are given in [Gr4]. The choice of the ground-category is crucial to obtain results of interest. E.g., *finitely generated* abelian (co)variations always yield *countable* lattices, whereas any prime order group **Z**/p has *at least a continuum* of abelian variations and a *proper class* of abelian covariations. In the homotopy category of topological spaces, Ho**Top** = **Top**/$\simeq$, we get a distributive lattice $\mathrm{Var}_{\simeq}(A) = \mathrm{Fib}(A)$ of *types of fibrations* over the space A (1.3), which is hard to classify even in the simplest cases; restricting to CW-spaces, the *cw-variations* of the circle $\mathbf{S}^1$ are classified by the standard fibrations $n: \mathbf{S}^1 \to \mathbf{S}^1$ $(n > 0)$, together with the universal covering $\mathbf{R} \to \mathbf{S}^1$, *consistently with what one might expect as "homotopy subobjects" of the circle* (1.4).

Here, after a brief review of these results (Section 1), we show in Section 2 that *variations can be viewed as* (distinguished) *subobjects in a new category*, by a universal construction. In fact, any category $\mathbf{X}$ can be embedded in the *Freyd completion* Fr$\mathbf{X}$, or *epi-monic completion*, a quotient of the category of morphisms $\mathbf{X}^2$, by the "same" procedure used by Freyd to embed the stable homotopy category of spaces in an abelian category [Fr2]; in our general case, the result is the *free category with epi-monic factorisation structure* over $\mathbf{X}$ (2.3). The weak subobjects of X in $\mathbf{X}$ correspond to the distinguished subobjects of X in Fr$\mathbf{X}$. If $\mathbf{X}$ has ordinary products and *weak equalisers*, Fr$\mathbf{X}$ has all limits; moreover, counterimages of variations in $\mathbf{X}$ correspond to counterimages of distinguished subobjects in Fr$\mathbf{X}$. Dual results hold for covariations, sums and weak coequalisers in $\mathbf{X}$, quotients and colimits in Fr$\mathbf{X}$. The *regularity* of Fr$\mathbf{X}$ is considered in 2.6c.

Finally, the exactness properties of the completion are considered in Section 3. If $\mathbf{X}$ has zero object, weak kernels and weak cokernels, as the homotopy category of pointed spaces, Fr$\mathbf{X}$ is a *homological category*, in the sense of [Gr1], and the *normal variations* of an object in $\mathbf{X}$ form a lattice. If moreover in $\mathbf{X}$ every map is a weak kernel and a weak cokernel, then Fr$\mathbf{X}$ is *exact* in the sense of Puppe [Pu1, Mi, FS] and the variations of $\mathbf{X}$ (all normal) form modular lattices. Adding also the existence in $\mathbf{X}$ of finite products (or sums), Fr$\mathbf{X}$ is abelian; in particular, this holds for every triangulated category $\mathbf{X}$, and the embedding $\mathbf{X} \to \mathrm{Fr}\mathbf{X}$ is then the universal homological functor on $\mathbf{X}$ (3.7), as in Freyd's original result.

Normal or *regular* variations have appeared in Eckmann - Hilton [EH] and Freyd [Fr4-5], under the equivalent form of "principal right ideals" of maps, to deal with weak kernels or weak equalisers. Recently, in connection with proof theory, Lawvere [La] has considered a "proof-theoretic power set $\mathcal{P}_{\mathbf{X}}(A)$", defined as the "poset-reflection of the slice category $\mathbf{X}/A$", which amounts to Var(A). A different approach to "subobjects" in homotopy categories can be found in Kieboom [Ki]. Fr$\mathbf{X}$ is related with the *regular* and *Barr-exact completions* of a category with limits or weak limits, studied in Carboni [Ca] and Carboni - Vitale [CV]. Finally, let us recall that the pseudo algebras for the 2-monad $\mathbf{X} \mapsto \mathbf{X}^2$ are known to correspond to the factorisation systems over $\mathbf{X}$ (Coppey [Co]; Korostenski - Tholen [KT]); similar relations link the induced 2-monad $\mathbf{X} \mapsto \mathrm{Fr}\mathbf{X}$ with the *epi-monic* factorisation systems over $\mathbf{X}$ (2.3).

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1. Review of variations

This review of some results of [Gr4] is meant to motivate the interest of weak subobjects. Technically, only the main definitions are necessary for the sequel. $\mathbf{X}$ is a fixed category.

1.1. Variations and covariations. A *variation* $[x]_A$, or *weak subobject*, of the object A in $\mathbf{X}$ denotes a class of morphisms with values in A, equivalent with respect to mutual factorisation

(1) $x \sim_A y$ iff there exist u, v such that $x = yu, \ y = xv$

$$
(2) \qquad \begin{array}{ccc} X & \xrightarrow{x} & A \\ {\scriptstyle u}\downarrow\uparrow{\scriptstyle v} & & \| \\ Y & \xrightarrow[y]{} & A \end{array}
$$

In other words, x and y generate the same principal right ideal of maps with values in A (or, also, are connected by morphisms $x \to y \to x$ in the slice-category $\mathbf{X}/A$ of *objects over* A). By a standard abuse of notation, as for subobjects, a variation $[x]_A$ will generally be denoted by any of its representatives x. The *domain* of a variation $x: X \to A$ is only determined up to a pair of arrows $u: X \to Y$, $v: Y \to X$ such that $x.vu = x$ (x *sees* its domain as a retract of Y; and symmetrically, $x' = xv: Y \to A$ sees its domain as a retract of X).

The variations of A form a (possibly large) ordered set $\mathrm{Var}(A)$, with $x \leq y$ iff x factors through y. The *identity variation* 1_A is the maximum. If $\mathbf{X}$ has an initial object, $\mathrm{Var}(A)$ has also a minimum $0_A: \perp \to A$. *Weak pullbacks* give meets of covariations; sums give joins

(1) $\vee(x_i: X_i \to A) = x: \Sigma_i X_i \to A$

(but weak sums would be sufficient). Recall that a *weak (co)limit* is defined by the existence part of the usual universal property (see also 1.3 and 2.8).

A variation $x: X \to A$ is said to be *epi* if it has a representative which is so in $\mathbf{X}$, or equivalently if all of them are so. It is equivalent to the identity iff it is a retraction; a split epi onto A should thus be viewed as giving the same *information with values in* A as 1_A, with redundant duplication. A variation is said to be *a subobject* if it has some monic representative $m: M \to A$ (all the other representatives are then given by the split extensions of M, and include all monics equivalent to m). The ordered set $\mathrm{Sub}(A)$ of subobjects of A is thus embedded in $\mathrm{Var}(A)$.

Transformations of weak subobjects, induced by morphisms (direct and inverse images), or by adjoint functors, or by product decompositions of objects, are considered in [Gr4]; in particular, counterimages of variations are obtained via weak pullbacks.

Dually, the *covariations*, or *weak quotients*, of A form an ordered set $\mathrm{Cov}(A)$, containing its quotients.

1.2. Examples. In **Set**, every epi splits, by the axiom of choice, and the unique epi variation of a set A is its identity: *variations and subobjects coincide*. The covariations of a non-empty set coincide with its quotients; but $\emptyset$ has two covariations, the identity and $0^{\emptyset}: \emptyset \to \{*\}$.

Similarly, in any category with epi-monic factorisations where all epis split, weak subobjects and subobjects coincide. This property, and its dual as well, hold in the category $\mathbf{Set}^{\top}$ of pointed sets, or in any category of vector spaces (over a fixed field), or also in the category of relations over any (well-powered) abelian category. In all these cases, the sets $\mathrm{Var}(A)$ and $\mathrm{Cov}(A)$ are small.

Consider now the category **Ab** of abelian groups and its full subcategory $\mathbf{Ab}_{fg}$ of finitely generated objects (fg-abelian groups, for short). We have the lattice $\mathrm{Var}(A)$ of all *abelian variations* of A, and – if A is finitely generated – the sublattice $\mathrm{Var}_{fg}(A)$ of *fg-variations* of A (having representatives in $\mathbf{Ab}_{fg}$). By the structure theorem of fg-abelian groups, $\mathrm{Var}_{fg}(A)$ and $\mathrm{Cov}_{fg}(A)$ are always *countable*.

Since every subgroup of a free abelian group is free, it is easy to show that the abelian variations of a free abelian group F coincide with its subobjects (and are finitely generated whenever F is so). In particular, the weak subobjects of the group of integers $\mathbf{Z}$ form a noetherian distributive lattice, and can be represented by its "positive" endomorphisms

$$(1)\quad x_n: \mathbf{Z} \to \mathbf{Z}, \qquad x_n(a) = n.a \qquad (n \geq 0)$$

$$x_m \leq x_n \quad \text{iff} \quad m\mathbf{Z} \subset n\mathbf{Z}, \quad \text{iff} \quad n \text{ divides } m.$$

The prime-order group $\mathbf{Z}/p$ has two subobjects and a totally ordered set of fg-variations, anti-isomorphic to the ordinal $\omega+2$, which can be represented by the natural homomorphisms x_n (including the natural projection $x_\infty: \mathbf{Z} \to \mathbf{Z}/p$)

$$(2)\quad x_n: \mathbf{Z}/p^n \to \mathbf{Z}/p, \qquad x_n(\bar{1}) = \bar{1} \qquad (0 \leq n \leq \infty)$$

$$0 = x_0 < x_\infty < \ldots < x_3 < x_2 < x_1 = 1.$$

But $\mathbf{Z}/p$ has *at least a continuum* of *non-finitely generated* variations ([Gr4], 1.5). Similarly, $\mathbf{Z}/p$ has a totally ordered set of fg-abelian covariations, anti-isomorphic to $\omega+1$, and a *proper class* of abelian covariations ([Gr4], 1.6). The fg-variations of any cyclic group and of $\mathbf{Z}/p\oplus\mathbf{Z}/p$ are classified in [Gr4] (Section 4).

Also in the category $\mathbf{Gp}$ of groups, *the weak subobjects of a free group coincide with its subobjects*, by the Nielsen-Schreier theorem (any subgroup of a free group is free). But here a subgroup of a free group of finite rank may have countable rank ([Ku], § 36). Thus, the set $\mathrm{Var}_{fg}(G) \subset \mathrm{Var}(G)$ of fg-variations of an fg-group G is at most a continuum.

Consider now the full embedding $\mathbf{Ab} \subset \mathbf{Gp}$. For an abelian group A, the set of abelian variations $\mathrm{Var}(A)$ is embedded in the set $\mathrm{Var}_{\mathbf{Gp}}(A)$ of its *group-variations*. Every group-variation $y: G \to A$ has an obvious *abelian closure* ${}^{ab}y: ab(G) \to A$, which is the least abelian variation of A following y; the latter is abelian iff it is equivalent to ${}^{ab}y$; $\mathrm{Var}(A) \subset \mathrm{Var}_{\mathbf{Gp}}(A)$ is a retract.

The group-variations of $\mathbf{Z}$, which is also free as a group, coincide with its subobjects and are all abelian. On the other hand, $\mathbf{Z}/2$ has also non-abelian fg-variations ([Gr4], 1.7).

1.3. Homotopy variations. Consider a quotient category $\mathbf{X}/\simeq$, modulo a congruence $f \simeq g$ (an equivalence relation between parallel morphisms, consistent with composition), which may be viewed as a sort of homotopy relation, since our main examples will be of this type.

A $\simeq$*-variation* of A in $\mathbf{X}$ is just a variation in the quotient $\mathbf{X}/\simeq$. But it is simpler to take its representatives in $\mathbf{X}$, as morphisms $x: \bullet \to A$ modulo the equivalence relation: $x \simeq_A y$ iff there are u, v such that $x \simeq yu$, $y \simeq xv$. The ordered set $\mathrm{Var}_{\simeq}(A)$ is thus a quotient of $\mathrm{Var}(A)$, often more manageable and more interesting. Similarly for covariations.

For a space X, we consider thus the ordered set $\mathrm{Var}_{\simeq}(X)$ of its *homotopy variations*, in the *homotopy category* $\mathrm{Ho}\mathbf{Top} = \mathbf{Top}/\simeq$ of topological spaces (coinciding with the category of fractions of $\mathbf{Top}$ which inverts homotopy equivalences). This ordered set, invariant up to homotopy type, is a lattice. In fact, $\mathbf{Top}$ has (small) sums, consistent with homotopies, and homotopy pullbacks [Ma], whence the quotient $\mathbf{Top}/\simeq$ has sums and weak pullbacks. Moreover, each homotopy variation *can be represented by a fibration*, because every map in **Top** factors as a homotopy equivalence followed by a fibration; we can thus view $\mathrm{Var}_{\simeq}(X) = \mathrm{Fib}(X)$ as the lattice of *types of fibrations* over X. Each homology functor H_n: **Top** $\to$ **Ab** can be used to represent homotopy variations as abelian variations and, in particular, distinguish them.

A homotopy class $\varphi = [f]\colon X \to Y$ acts on such lattices by direct and inverse images, giving a covariant connection (an adjunction between ordered sets)

(2) $\varphi_* : \mathrm{Fib}(X) \rightleftarrows \mathrm{Fib}(Y) : \varphi^*$

$\varphi_*[x] = [fx]$, $\qquad \varphi^*[y] =$ class of a weak pullback of y along f

$1 \leq \varphi^*\varphi_*$, $\qquad \varphi_*\varphi^*[y] = [y] \wedge \varphi_*[1]$.

Globally, we get a homotopy-invariant functor Fib defined over **Top**, with values in the category of (possibly large) lattices and "right-exact" connections [Gr4]. Dual facts hold for the lattice $\mathrm{Cov}_{\simeq}(X) = \mathrm{Cof}(X)$ of homotopy covariations, or *types of cofibrations* from X. Most of these results can be extended to various other "categories with homotopies", as pointed spaces, chain complexes, diagrams of spaces, spaces under a space (or over), topological monoids, etc. (see [Gr2, Gr3, GM]; and references therein for other approaches to abstract homotopy).

Finally, it is interesting to show that the lattice $\mathrm{Fib}(A) = \mathrm{Var}_{\simeq}(A)$ is *distributive*, and actually *binary meets distribute over small joins*. First, note that pullbacks in **Top** distribute over sums (see [CLW] for the notion of "extensive" categories): given a (small) topological sum with injections u_i: $X_i \subset X$ and a map $f\colon Z \to X$, the pullback-spaces $Z_i = f^{-1}(X_i)$ have topological sum Z. Second, the sum-injections u_i are fibrations (every homotopy in X, starting from a map with values in X_i, has image contained in the latter), whence the previous pullbacks are homotopy pullbacks in **Top** and weak pullbacks in Ho**Top**. Now, given a family of variations $x_i\colon X_i \to A$, their join $x\colon X \to A$ and a variation $y\colon Y \to A$, form the following commutative diagram

(2)
$$\begin{tikzcd}
X_i \arrow[r, "u_i"'] \arrow[rr, bend left, "x_i"] & X \arrow[r, "x"'] & A \\
Z_i \arrow[u, "f_i"] \arrow[r, "v_i"] \arrow[rr, bend right, "z_i"'] & Z \arrow[u, "f"] \arrow[r, "z"] & Y \arrow[u, "y"']
\end{tikzcd}$$

where the right-hand square is a weak pullback in Ho**Top**, whence $yz = y \wedge x$ in $\mathrm{Fib}(A)$, and the left-hand square too, constructed as above. Also the rectangle is a weak pullback, for every i; since (v_i) is the family of injections of a topological sum, the join of $y \wedge x_i = yz_i$ is $yz = y \wedge x$.

1.4. CW-variations. But is important to restrict the class of spaces we are considering, to obtain more homogeneous sets of variations, which one might hopefully classify. A first standard restriction is the category **CW** of CW-spaces (pointed spaces having the homotopy type of a connected CW-complex), with pointed maps; the variations of X in **CW**$/\simeq$ will be called *cw-variations*, and $\mathrm{Var}_{cw}(X)$ *is a sublattice of the lattice* $\mathrm{Var}_{\simeq}(X)$ of all homotopy variations of X. But $\mathrm{Var}_{cw}(X)$ may still be large (as follows from Freyd's results on the non-concreteness of homotopy categories [Fr4-5]), and further restrictions should be considered.

The group variations of $\mathbf{Z}$, coinciding with its abelian variations $x_n \colon \mathbf{Z} \to \mathbf{Z}$ (1.2.1), have corresponding cw-variations of the pointed circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$

(1) $y_n \colon \mathbf{S}^1 \to \mathbf{S}^1, \qquad y_n[\lambda] = [n\lambda] \qquad (n \geq 0)$

$y_m \leq y_n$ iff n divides m

which realise them via π_1. In fact, this sequence classifies all the cw-variations of the circle ([Gr4], 3.4). Note that, for $n > 0$, y_n is the covering map of $\mathbf{S}^1$ of degree n; the universal covering map $p \colon \mathbf{R} \to \mathbf{S}^1$ corresponds to the weak subobject y_0, also represented by $\{*\} \to \mathbf{S}^1$.

More generally, any cluster of circles $\Sigma_I \mathbf{S}^1$ has cw-variations determined by the group-variations of the free group $\Sigma_I \mathbf{Z}$, i.e. its subgroups ([Gr4], 3.4). Cw-variations of the sphere and the projective plane are also studied in [Gr4] (2.4-5).

2. Weak subobjects and the generalised Freyd completion

We prove that variations in the category $\mathbf{X}$ can be identified with distinguished subobjects in the *Freyd completion* Fr$\mathbf{X}$, which extends the Freyd embedding of the stable homotopy category into an abelian category [Fr1-3].

2.1. Factorisation systems. A *category with factorisation system* $\mathbf{C}$, or *fs-category* for short, is equipped with a pair (E, M) satisfying the usual axioms [FK, KT, CJKP, JT]:

(fs.1) E, M are subcategories of $\mathbf{C}$ containing all the isomorphisms,

(fs.2) every morphism u has a factorisation $u = u''.u'$ with $u' \in E$, $u'' \in M$,

(fs.3) (*orthogonality*) given a commutative square $mf = ge$, with $e \in E$, $m \in M$, there is a unique morphism u making the following diagram commute

(1)
$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, "u"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

The factorisation $u = u''.u'$, determined up to a unique central isomorphism, will be called the *structural*, or *fs-factorisation*, of u; its middle object will be written $\mathrm{Im}(u)$. An *fs-functor*, of course, is a functor between fs-categories which preserves their structure.

We also need to consider a *strict factorisation system* (E_0, M_0) over $\mathbf{C}$, satisfying:

(i) E_0, M_0 are subcategories of $\mathbf{C}$ containing all the identities,

(ii) every morphism u has a *strictly unique* factorisation $u = u''.u'$ with $u' \in E_0$, $u'' \in M_0$.

Then, $E_0 \cap M_0$ is the subcategory of identities; and there is a unique factorisation system (E, M) containing the former (or *spanned* by it), where $u = u''.u'$ is in E (resp. M) if u'' (resp. u') is iso. To prove this, the only non obvious point is the closure of E, or M, under composition. First, one proves that, if $e \in E_0$, i is iso and $ei = n.f$ is the unique (E_0, M_0)-factorisation, then n is an iso. Now, if u and v are consecutive in E, with u'', v'' iso, we have $vu = v''v'.u''u' = v''.(v'u'').u' = v''.(a''.a').u' = v''a''.a'u'$, with $a' \in E_0$, $a'' \in M_0$ and a'' iso; thus $vu \in E$.

A factorisation system is *epi-monic* if all E-maps are epi and M-maps are monics. Then, the morphisms of E and M will be called *fs-epis* (or distinguished epis) and *fs-monics*, respectively. There is a decomposition property for E-maps: if $e \in E$ and $e = vu$ then $v \in E$; dually for M.

2.2. The factorisation completion. Let $\mathbf{X}$ be any category and $\mathbf{X}^2$ its category of maps. An object of the latter is an $\mathbf{X}$-morphism $x\colon X' \to X''$, which we *may* write as $\hat{x}$ when viewed as an object of $\mathbf{X}^2$; a morphism $f = (f', f'')\colon \hat{x} \to \hat{y}$ is a commutative square of $\mathbf{X}$

$$\text{(1)} \qquad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X'' & \xrightarrow{f''} & Y'' \end{array}$$

and the composition is obvious. $\mathbf{X}^2$ has a canonical factorisation system (not epi-monic, generally), where the map $f = (f', f'')$ is in E (resp. in M) iff f' (resp. f'') is an isomorphism

$$\text{(2)} \qquad \begin{array}{ccccc} X' & = & X' & \xrightarrow{f'} & Y' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle y} \\ X'' & \xrightarrow{f''} & Y'' & = & Y'' \end{array}$$

but it is relevant to note that, inside this system, the morphism f has a unique *strict* factorisation (2), whose middle object is the diagonal $\bar{f} = f''x = yf'$ of our square (1). In other words, our system is spanned by a canonical strict system, where (f', f'') is in E_0 (resp. in M_0) iff f' (resp. f'') is an identity.

$\mathbf{X}$ is fully embedded in $\mathbf{X}^2$, identifying the object X with $\hat{1}_X$, and $f\colon X \to Y$ with $(f, f)\colon \hat{1}_X \to \hat{1}_Y$. Each object $\hat{x}$ can be viewed as the structural image of the corresponding morphism $x\colon X' \to X''$ of $\mathbf{X} \subset \mathbf{X}^2$, whose (strict) factorisation is

(3) $\quad x = (x, 1).(1, x)\colon X' \to \hat{x} \to X'' \qquad\qquad \mathrm{Im}(x) = \hat{x}$.

One deduces easily that the category with factorisation system $\mathbf{X}^2$ is the *factorisation completion*, or the *free fs-category* on $\mathbf{X}$: every functor $F\colon \mathbf{X} \to \mathbf{C}$ with values in an fs-category has an fs-extension $G\colon \mathbf{X}^2 \to \mathbf{C}$, determined up to a unique natural isomorphism

(4) $\quad G(\hat{x}) = \mathrm{Im}_{\mathbf{C}}(Fx)$

$$\text{(5)} \qquad \begin{array}{ccccc} FX' & \xrightarrow{(Fx)'} & G(\hat{x}) & \xrightarrow{(Fx)''} & FX'' \\ {\scriptstyle Ff'}\downarrow & & \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle Ff''} \\ FY' & \xrightarrow{(Fy)'} & G(\hat{y}) & \xrightarrow{(Fy)''} & FY'' \end{array}$$

(the uniqueness of Gf follows from the orthogonality axiom).

2.3. The Freyd completion. Now, the *Freyd completion*, or *epi-monic completion*, Fr**X** is a quotient $\mathbf{X}^2/R$ of the category of morphisms of **X**: two parallel morphisms $f = (f', f"): x \to y$ and $g = (g', g"): x \to y$ of $\mathbf{X}^2$ are R-equivalent whenever their diagonals $\overline{f}$, $\overline{g}$ coincide

$$\text{(1)} \qquad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, "\overline{f}"] & Y' \arrow[d, "y"] \\ X'' \arrow[r, "f''"'] & Y'' \end{tikzcd}$$

The morphism of Fr**X** represented by f will be written as [f] or [f', f"]. Plainly, if f' is epi (resp. f" is monic) in **X**, so is [f] in Fr**X**.

Consider the previous *strict* factorisation in $\mathbf{X}^2$ (2.2.2) and note that its middle object is precisely $\overline{f}$, as well as the diagonals of both morphisms. Thus, our strict factorisation system in $\mathbf{X}^2$ induces a similar system in Fr**X**, which is now epi-monic: a *canonical epi* (resp. monic) in Fr**X** is a morphism which can be represented by a square whose upper (resp. lower) arrow is an identity, and every morphism [f] has a strictly unique *canonical factorisation* as a canonical epi followed by a canonical monic (2.2.2). By 2.1, this strict system spans an epi-monic factorisation system for Fr**X**, in the usual sense: the distinguished epis (or fs-epis, denoted by $\twoheadrightarrow$) are those maps [f] whose factorisation presents an iso at the right-hand; dually for fs-monics (denoted by $\rightarrowtail$).

The quotient induces a full embedding $\mathbf{X} \to \mathrm{Fr}\mathbf{X}$, which identifies X with $1_X: X \to X$, and $f: X \to Y$ with $[f, f]: 1_X \to 1_Y$. Fr**X** *is the free category with epi-monic factorisation system over the category* **X**: every functor $F: \mathbf{X} \to \mathbf{C}$ with values in an epi-monic fs-category has an essentially unique fs-extension $G: \mathrm{Fr}\mathbf{X} \to \mathbf{C}$. The construction of G is the same as above (2.2.4-5); now, G[f] is well defined, independently of the representative f, because E- and M-maps of **C** are respectively epi and monic, and the diagonal of the rectangle 2.2.5 is determined by [f].

It is interesting to recall that the **Cat**-endofunctor $\mathbf{X} \mapsto \mathbf{X}^2$ has an obvious 2-monad structure (with "diagonal" multiplication) whose pseudo algebras $\mathbf{X}^2 \to \mathbf{X}$ correspond to the factorisation systems over **X** ([Co, KT]). Similarly, as suggested by F.W. Lawvere, one can show that the pseudo algebras for the induced **Cat**-monad $\mathbf{X} \mapsto \mathrm{Fr}\mathbf{X}$ correspond to the *epi-monic* factorisation systems over **X**.

2.4. Subobjects. The ordered set $\mathrm{Sub}_{\mathrm{Fr}\mathbf{X}}(X)$ of *subobjects* of X in Fr**X** (pertaining to the fs-structure, i.e. determined by *fs-monics*, or equivalently by the canonical ones) can be identified with the ordered set of variations of X in **X**

$$\text{(1)} \qquad \begin{tikzcd} M \arrow[r, "m"] \arrow[d, "m"'] & X \arrow[d, "1"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad\qquad \mathrm{Sub}_{\mathrm{Fr}\mathbf{X}}(X) = \mathrm{Var}_{\mathbf{X}}(X).$$

More generally, the ordered set $\mathrm{Sub}_{\mathrm{Fr}\mathbf{X}}(\hat{x})$ can be identified with the set of **X**-variations of X" lesser than x. Indeed, consider two morphisms $m: M \to X'$, $n: N \to X'$ and the associated fs-monics; $[m, 1] \leq [n, 1]$ means that there exist a map $[f', f"]: xm \to xn$ forming a commutative triangle $[n, 1].[f', f"] = [m, 1]$

$$(2)\qquad \begin{array}{ccccc} M & \xrightarrow{f'} & N & \xrightarrow{n} & X' \\ {\scriptstyle xm}\downarrow & & \downarrow{\scriptstyle xn} & & \downarrow{\scriptstyle x} \\ X'' & \xrightarrow{f''} & X'' & = & X'' \end{array} \quad = \quad \begin{array}{ccc} M & \xrightarrow{m} & X' \\ {\scriptstyle xm}\downarrow & \searrow^{xm} & \downarrow{\scriptstyle x} \\ X'' & = & X'' \end{array}$$

This is equivalent to the existence of f', f" in **X** making the following diagram commute

$$(3)\qquad \begin{array}{ccc} M & \xrightarrow{f'} & N \\ {\scriptstyle xm}\downarrow & \searrow^{xm} & \downarrow{\scriptstyle xn} \\ X'' & \xrightarrow[f'']{} & X'' \end{array}$$

but f" can always be replaced with $1_{X''}$, and all this reduces to $xm \leq_{X''} xn$.

Every object UX is *fs-projective* in Fr**X** (satisfies the usual lifting property with respect to fs-epis) and, dually, fs-injective. Moreover, each object $\hat{x}$ can be viewed as the image of the morphism $x: X' \to X''$ of $\mathbf{X} \subset \mathrm{Fr}\mathbf{X}$, whose canonical factorisation is

$$(4)\quad x = [x, 1].[1, x]: X' \twoheadrightarrow \hat{x} \rightarrowtail X'' \qquad\qquad \mathrm{Im}(x) = \hat{x}$$

so that $\hat{x}$ is a quotient of an fs-projective (namely, X') and a subobject of an fs-injective (X"): Fr**X** *has sufficient projectives and injectives, as an fs-category, belonging to the same class* U(Ob**X**) (a sort of "Frobenius condition", according to [Fr1-2]). As an easy consequence, the fs-projectives of Fr**X** coincide with the retracts of such objects UX; the fs-injectives as well.

2.5. Limits. We show now that: if **X** has (finite) ordinary products and weak equalisers, then Fr**X** has all (finite) limits. These hypotheses apply to homotopy categories like Ho**Top**, which do have ordinary products and weak equalisers (see 2.8) but lack ordinary equalisers (a fact related to the notion of *flexible* limits in bicategories, in the sense of [BKPS]). Of course, limits and weak limits are understood to be *small*.

a) Products in **X** give products in Fr**X**, in the obvious way (inherited from $\mathbf{X}^2$)

$$(1)\qquad \begin{array}{ccc} \Pi X'_i & \xrightarrow{p_i} & X'_i \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x_i} \\ \Pi X''_i & \xrightarrow{q_i} & X''_i \end{array} \qquad\qquad x = \Pi x_i$$

and $U: \mathbf{X} \to \mathrm{Fr}\mathbf{X}$ preserves the existing products. Note that the cancellation property of the family of projections $[p_i, q_i]$ comes from the similar property of the family (q_i) in **X**; generally, we cannot replace $\Pi X''_i$ with a *weak* product.

b) Weak equalisers in **X** produce ordinary equalisers in Fr**X**

$$(2)\qquad \begin{array}{ccccc} E & \xrightarrow{e} & X' & \overset{f'}{\underset{g'}{\rightrightarrows}} & Y' \\ {\scriptstyle xe}\downarrow & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X'' & = & X'' & \overset{f''}{\underset{g''}{\rightrightarrows}} & Y'' \end{array}$$

given a pair $[f], [g]: x \to y$, if e is a *weak equaliser of the pair* (yf', yg'), the fs-monic $[e, 1_{X''}]$ is the equaliser of the pair.

Moreover, the embedding U *takes a weak equaliser* $e: E \to X$ (of a pair $f, g: X \to Y$) *to a cone* $Ue: UE \to UX$ *which factors through the equaliser* $[e, 1]$ *by a distinguished epi* $[1, e]$

(3)
$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, equal] & E \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, equal] & Y \arrow[d, equal] \\ E \arrow[r, "e"'] & X \arrow[r, equal] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g''"'] & Y \end{tikzcd}$$

c) Our initial statement on the completeness of Fr**X** is proved. But it is interesting and easy to give an explicit construction of arbitrary limits (assuming **X** has products and weak equalisers). Let be given a diagram $D = ((\hat{x}_i), ([f_u]))$: $\mathbf{S} \to \mathrm{Fr}\mathbf{X}$ (a functor defined over a small category, with $i \in \mathrm{Ob}\mathbf{S}$ and $u \in \mathrm{Mor}\mathbf{S}$). Choose a weak limit L, in **X**, of the diagram formed of all the arrows $x_i: X'_i \to X''_i$ and all the arrows $\bar{f}_u: X'_i \to X''_j$; L comes equipped with arrows $a_i: L \to X'_i$ such that $\bar{f}_u.a_i = x_j a_j: L \to X''_j$ (weakly universally); take a product $\Pi X''_i$ with projections q_i

(4)
$$\begin{tikzcd} L \arrow[r, "a_i"] \arrow[d, "a"'] & X'_i \arrow[r, "f'_u"] \arrow[d, "x_i"] & X'_j \arrow[d, "x_j"] \\ \Pi X''_i \arrow[r, "q_i"] & X''_i \arrow[r, "f''_u"] & X''_j \end{tikzcd}$$

and let $a = \langle x_i a_i \rangle: L \to \Pi X''_i$. Then the cone $[a_i, q_i]: \hat{a} \to \hat{x}_i$ is the limit of D in Fr**X**.

d) In particular, if **X** has products, U satisfies a property with respect to all the existing weak limits of **X**, extending the one already considered for weak equalisers:

(5)
$$\begin{tikzcd} L \arrow[r, equal] \arrow[d, equal] & L \arrow[r, "a_i"] \arrow[d, "a"'] & X_i \arrow[d, equal] \\ L \arrow[r, "a"'] & \Pi X_i \arrow[r, "q_i"] & X_i \end{tikzcd}$$

(∗) U takes any weak limit (L, a_i) of a diagram $X = ((X_i), (f_u))$: $\mathbf{S} \to \mathbf{X}$ to a cone which is connected to the limit-cone of UX in Fr**X** by a distinguished epi.

e) As a marginal remark, one can note that, if in **X** weak equalisers exist and *every map is a weak equaliser of some pair*, the fs-monics of Fr**X** coincide with the regular monics and are categorically determined. Practically, this assumption on **X** is mostly of interest for triangulated categories and stable homotopy, when Fr**X** is even abelian (3.7).

2.6. Theorem: Completeness properties of Fr**X.** Let **X** be a category, $F: \mathbf{X} \to \mathbf{B}$ a functor with values in an epi-monic fs-category, $G: \mathrm{Fr}\mathbf{X} \to \mathbf{B}$ its fs-extension. In the following, one can also restrict everything to the *finite* case: finite products, finite limits, finite weak limits...

a) If **X** has products, also Fr**X** has them and U preserves them. Moreover, F preserves them iff G does (Fr**X** is the free epi-monic fs-category with products over **X**, as a category with products).

b) If $\mathbf{X}$ has products and weak equalisers, Fr$\mathbf{X}$ is complete and fs-monics, fs-epis are stable under pullbacks. The ordered sets of variations in $\mathbf{X}$ have small meets. U preserves products and satisfies the property ($*$) on weak limits (2.5). Moreover, the following conditions are equivalent

(1) F preserves products and satisfies ($*$) on equalisers

(2) F preserves products and satisfies ($*$) on weak limits

(3) G preserves all limits.

The property (*) in (1) can be equivalently replaced with the following (and similarly for (2)):

($**$) every diagram $X = ((X_i), (f_u))\colon \mathbf{S} \to \mathbf{X}$ has a weak limit (L, a_i) taken by U to a cone which is connected to the limit-cone of UX in Fr$\mathbf{X}$ by a distinguished epi.

c) If $\mathbf{X}$ has finite products and weak equalisers, and every map is a weak coequaliser, then Fr$\mathbf{X}$ is a regular category and all distinguished epis are regular.

Proof. After the previous construction of (co)limits in Fr$\mathbf{X}$, we only need to check the stability property of the fs-factorisation under pullbacks, which is done below (2.7). The fact that ($**$) implies ($*$) is easy: we know that $Ua_i = p_i.e$ with $e \in E$, where $(\bar{L}, p_i)$ is the limit of UX in Fr$\mathbf{X}$

(4)
$$\begin{tikzcd}
UL \arrow[dr, "e"] \arrow[drr, "Ua_i"] \arrow[dd, "Uu"'] & & \\
& \bar{L} \arrow[r, dashed, "p_i"] & UX_i \\
UM \arrow[ur, "f"] \arrow[urr, "Ub_i"'] & &
\end{tikzcd}$$

if (M, b_i) is also a weak limit of X, take $u\colon L \to M$ such that $a_i = b_i u$ (all i) and let $Ub_i = p_i.f$; cancelling the limit cone (p_i), we have that $e = f.Uu$, whence also $f \in E$.

Note that the property ($*$) is not closed under composition, and does not lead - naturally - to a category of categories with weak limits; to express the universal property b) as an adjunction would require artificial constructs, probably of scarce interest.

2.7. Counterimages. In particular, we are interested in the construction of pullbacks in Fr$\mathbf{X}$, from weak pullbacks and ordinary products $A \times B$ in $\mathbf{X}$.

Given $[f]\colon x \to z$ and $[g]\colon y \to z$ in the left-hand diagram below

(1)
$$\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dd, dashed, "x" near start] & & Z' \arrow[dd, "z"] \\
P \arrow[ur, "h'"] \arrow[rr, "k'" near start] \arrow[dd, "a"'] & & Y' \arrow[ur, "g'"] \arrow[dd, "y" near start] & \\
& X'' \arrow[rr, dashed, "f''" near start] & & Z'' \\
X''\times Y'' \arrow[ur, "h''"] \arrow[rr, "k''"] & & Y'' \arrow[ur, "g''"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X' \arrow[rr, "f'"] \arrow[dd, dashed, "x" near start] & & Z' \arrow[dd, "z"] \\
P \arrow[ur, "h'"] \arrow[rr, "k'" near start] \arrow[dd, "xh'"'] & & Y' \arrow[ur, "g'"] \arrow[dd, "y" near start] & \\
& X'' \arrow[rr, dashed, "f''" near start] & & Z'' \\
X'' \arrow[ur, "1"] \arrow[rr, "f''"] & & Z'' \arrow[ur, "1"'] &
\end{tikzcd}$$

let h", k" be the projections of the product $X''\times Y''$, let (P, h', k') be a weak pullback of (zf', zg') and $a = \langle xh', yk'\rangle\colon P \to X''\times Y''$. The morphisms $[h]\colon a \to x$ and $[k]\colon a \to y$ are our solution.

In Fr**X**, *fs-epis and fs-monics are stable under pullbacks*; the first property is obvious (if $g' = 1$, one can take $h' = 1$); the second is shown by another construction of the pullback, in the particular case $g'' = 1$, given in the right-hand diagram above.

The stability of fs-monics also shows that the identification $\mathrm{Var}_{\mathbf{X}}(X) = \mathrm{Sub}_{\mathrm{Fr}\mathbf{X}}(X)$ (2.4) is consistent with counterimages, given by weak pullbacks of variations in **X** ([Gr4], 3.1) and pullbacks of fs-monics in Fr**X**. Take $f: X \to Z$ in **X** ($x = 1_X$, $z = 1_Z$) and $g' = y \in \mathrm{Var}_{\mathbf{X}}(Z)$; the pullback of the fs-monic $[g', 1]: \hat{y} \to Z$ is realised as above, so that the **X**-counterimage $h' = f^*(y) \in \mathrm{Var}_{\mathbf{X}}(X)$ corresponds to the Fr**X**-counterimage $[h', 1] = [f]^*[y, 1] \in \mathrm{Sub}_{\mathrm{Fr}\mathbf{X}}(X)$.

2.8. Spaces and homotopy. Let now $\mathbf{X} = \mathrm{Ho}\mathbf{Top} = \mathbf{Top}/\simeq$, the homotopy category of spaces.

Ho**Top** has small products and weak limits. In fact, **Top** has small products, satisfying the obvious 2-dimensional property with respect to homotopies (any family of homotopies $\alpha_i: f_i \to g_i: A \to X_i$ has a unique lifting $\alpha: f \to g: A \to \Pi X_i$). Moreover, it has *homotopy equalisers*, making a pair of parallel maps homotopic, in a universal way (which induce weak equalisers in Ho**Top**)

(1) $\quad E \xrightarrow{e} X \overset{f,\, g}{\rightrightarrows} Y \qquad\qquad \varepsilon: fe \to ge: E \to Y$

(2) $\quad E = \{(x, \eta) \in X{\times}PY \mid \eta\in(0) = f(x),\ \eta\in(1) = g(x)\}, \qquad e(x, \eta) = x, \qquad \varepsilon(x, \eta; t) = \eta(t)$

where the path-space $PY = Y^{[0, 1]}$ has the compact-open topology.

Similarly, Ho**Top** has small sums and weak colimits, because **Top** has small sums, consistent with homotopies, and *homotopy coequalisers*

(3) $\quad X \overset{f,\, g}{\rightrightarrows} Y \xrightarrow{c} C \qquad\qquad \kappa: cf \to cg: X \to C$

where C is a quotient of $(I{\times}X)+Y$, identifying $[0, x] = [fx]$ and $[1, x] = [gx]$, for $x\in X$.

It follows that the Freyd completion FrHo**Top** is complete and cocomplete. Every homotopy invariant functor F: **Top** $\to$ **B** with values in an epi-monic fs-category has a unique extension to an fs-functor G: FrHo**Top** $\to$ **B**. If F preserves (finite) products and satisfies the condition $(*)$ over weak equalisers (2.5) then G preserves (finite) limits; and dually.

In particular, for every space S, the S-homotopy functor $\pi_S = [S, -]$: **Top** $\to$ **Set** has a unique extension to a limit-preserving fs-functor π'_S: FrHo**Top** $\to$ **Set**; since **Set** has unique epi-monic factorisations, fs-functor means here to take fs-epis to epis and fs-monics to monics. Similarly, the S-cohomotopy functor $\pi^S = [-, S]$: **Top** $\to$ $\mathbf{Set}^{\mathrm{op}}$ has a unique extension to a colimit-preserving fs-functor π'^S: FrHo**Top** $\to$ $\mathbf{Set}^{\mathrm{op}}$.

3. Weak subobjects and exactness properties of the Freyd completion

If **X** has a zero-object, weak kernels and weak cokernels, as the homotopy category of pointed spaces $\mathrm{Ho}\mathbf{Top}^{\top}$, then Fr**X** is a *homological category* in the sense of [Gr1]. Further hypotheses (3.6) make Fr**X** Puppe exact, or abelian as in the original Freyd's result [Fr2]; the latter case occurs, in particular, if **X** is triangulated (3.7). All this has consequences on the variations in **X**.

3.1. Exactness. In this section, we consider "exactness properties" for *pointed* categories (i.e., having zero object) and pointed functors (preserving the latter).

At the "weak level" (of **X**) we just need considering *weak kernels* and *weak cokernel*s, and the well-known notion of *triangulated category* (see Puppe [Pu2, 3], Verdier [Ve], Hartshorne [Ha], Neeman [Ne]). At the "strict level" (of Fr**X**) we need the notion of *Puppe exact category* (see Puppe [Pu1], Mitchell [Mi], Freyd - Scedrov [FS]) and of *homological category*, a generalisation of the former introduced in [Gr1]; both notions are briefly reviewed below. We refer to [Gr1] for examples and a synopsis of homological algebra in homological categories; to references therein for the same subject in Puppe exact categories (including the diagrammatical techniques of *distributive homological algebra*, relevant for the study of spectral sequences and with no counterpart in the abelian context).

Let **A** be a pointed category with kernels and cokernels. Then every morphism f has a *normal factorisation* $f = mgp$, where $p = \mathrm{cok}(\ker(f))$ is its *normal coimage* (the cokernel of the kernel), $m = \ker(\mathrm{cok}(f))$ *its normal image*, and g is the unique map completing the factorisation

(1)
$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\ker(f)"] & A \arrow[rr, "f"] \arrow[d, "p"'] & & B \arrow[r, "\mathrm{cok}(f)"] & \mathrm{Cok}(f) \\
& \mathrm{Cok}(\ker(f)) \arrow[r, "g"] & \mathrm{Ker}(\mathrm{cok}(f)) \arrow[ur, "m"'] & &
\end{tikzcd}$$

and we say that f is an *exact morphism* if g is iso.

The category **A** (pointed, with kernels and cokernels) is said to be *Puppe exact* if all its morphisms are exact. Then, every monic is a *normal monic* (i.e., a kernel of some arrow) and every epi is a *normal epi*. Every morphism has a unique factorisation by a normal epi and a normal monic; and, conversely, this condition ensures that a pointed category is Puppe exact. For every object A, kernels and cokernels yield an anti-isomorphism between the lattices Sub(A) and Quo(A) of subobjects and quotients of A; such lattices are always *modular*. A category is abelian iff it is Puppe exact and has finite products (or finite sums). A functor between Puppe exact categories is said to be *exact* if it preserves kernels and cokernels (whence also zero objects); this amounts to preserving exact sequences, or also the short exact ones. An exact functor between abelian categories automatically preserves finite limits and colimits, and the sum of maps.

More generally, a pointed category **A** is *homological* [Gr1] if it satisfies the following axioms:

(ex1) (*semiexactness*) kernels and cokernels exist;

(ex2) normal monics and normal epis are closed under composition;

(ex3) (*subquotient axiom*, or *homology axiom*) given a normal subobject $m: M \to A$ and a normal quotient of $q: A \to Q$, with $m \geq \ker(q)$, the composite qm is an exact morphism (whose factorisation determines a *subquotient* of A).

Let us also note that in a pointed epi-monic fs-category **C**, any normal monic is necessarily a distinguished monic; in particular this holds for $(0 \to A) = \ker(1_A)$. And dually for normal epis.

3.2. Kernels and cokernels. Assume now that the pointed category **X** has weak kernels and weak cokernels. We shall prove (in 3.2-5) that Fr**X** is a (pointed) homological category.

To begin with, the zero object of **X** is still so in Fr**X**. Moreover, every morphism [f] in Fr**X** has kernel and cokernel, whose *canonical representatives* can be constructed choosing, in **X**, a weak kernel m and a weak cokernel p of the diagonal $\overline{f}$

$$(1)\qquad \begin{array}{ccccccc} M & \xrightarrow{m} & X' & \xrightarrow{f'} & Y' & = & Y' \\ {\scriptstyle xm}\downarrow & & {\scriptstyle x}\downarrow & \searrow{\scriptstyle \overline{f}} & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle py} \\ X'' & = & X'' & \xrightarrow[f'']{} & Y'' & \xrightarrow[p]{} & P \end{array} \qquad \begin{array}{c} m \in \mathrm{wker}(\overline{f}) \\ \\ p \in \mathrm{wcok}(\overline{f}) \end{array}$$

(as in Section 2, the existence part of the universal properties comes from the analogue in **X**; the uniqueness part from the fact that [m, 1] is monic in Fr**X**, and [1, p] is epi). We shall write

(2) $[m, 1] = \ker[f]\colon xm \rightarrowtail x, \qquad\qquad [1, p] = \mathrm{cok}[f]\colon y \twoheadrightarrow py.$

We have thus proved that Fr**X** satisfies (ex1), i.e. is (pointed) *semiexact* [Gr1]. The normal factorisation [f] = nim[f].[g].ncm[f] considered above (3.1) is given by the following three maps

$$(3)\qquad \begin{array}{ccccccc} X' & = & X' & \xrightarrow{g'} & I & \xrightarrow{i} & Y' \\ {\scriptstyle x}\downarrow & & {\scriptstyle cx}\downarrow & & \downarrow{\scriptstyle yi} & & \downarrow{\scriptstyle y} \\ X'' & \xrightarrow[c]{} & C & \xrightarrow[g'']{} & Y'' & = & Y'' \end{array}$$

(4) $c \in \mathrm{wcok}(xm), \qquad i \in \mathrm{wker}(py); \qquad\qquad g'i = f', \qquad g''c = f''$

(the existence of g' and g" being provided by the weak universal properties of i and c, since $py.f' = p\,\overline{f} = 0$ and $f''.xm = \overline{f}m = 0$).

Since ncm[f] and nim[f] are a *distinguished* epi and monic (3.1), respectively, the fs-factorisation of [g] yields the one of [f]. The morphism [f] is exact when this [g] is iso; then, the normal factorisation of [f] "coincides" with its fs-factorisation.

3.3. Short exact sequences. We prove now that every commutative diagram in **X** with

$$(1)\qquad \begin{array}{ccccc} M & \xrightarrow{m} & X' & = & X' \\ {\scriptstyle xm}\downarrow & & {\scriptstyle x}\downarrow & \searrow{\scriptstyle p} & \downarrow{\scriptstyle px} \\ X'' & = & X'' & \longrightarrow & P \end{array} \qquad \begin{array}{c} m \in \mathrm{wker}(px) \\ \\ p \in \mathrm{wcok}(xm) \end{array}$$

yields a short exact sequence in Fr**X**, and actually the generic one (any such can be obtained in this way, up to isomorphism)

(2) $\bullet \overset{[m,\,1]}{\rightarrowtail} x \overset{[1,\,p]}{\twoheadrightarrow} \bullet \qquad\qquad [m, 1] = \ker[1, p], \qquad [1, p] = \mathrm{cok}[m, 1].$

In fact, take a normal subobject of $x\colon X' \to X''$, $[m, 1] = \ker[f', f'']$, with $m \in \mathrm{wker}(f''x)$. Take now its cokernel [1, p], with $p \in \mathrm{wcok}(xm)$. General properties of semiexact categories would ensure that [m, 1] = ker[1, p]; but, directly, we can say more: the pair (m, 1) is actually a *canonical representative for this kernel*, i.e. $m \in \mathrm{wker}(px)$, noting that f" factors through p, so that any morphism v which annihilates px also annihilates f"x and factors through m

(3)
$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[dr, dashed] & X' \arrow[r, "x"] & X'' \arrow[r, "f''"] \arrow[d, "p"] & \bullet \\
& \bullet \arrow[u, "v"] \arrow[r, "0"] & P \arrow[ur, "u"'] &
\end{tikzcd}$$

In particular, a normal subobject of $x: X' \to X''$ is always determined by an arrow $\mu = [m, 1]$, where $m: M \to X'$ is a weak kernel in $\mathbf{X}$ of px, for some $p: X'' \to \bullet$. As in 2.4, two weak kernels $m: M \to X'$, $n: N \to X'$ give equivalent normal monics $[m, 1] \sim [n, 1]$ (determine the same normal subobject of x) iff xm and xn provide the same variation of X''.

Just because Fr$\mathbf{X}$ is semiexact, the normal subobjects of any object $x: X' \to X''$ form a (possibly large) *lattice* Nsb(x) [Gr1]. The normal quotients of x form a second lattice Nqt(x), *anti*-isomorphic to the former via kernel-cokernel duality. In particular, for $x = 1_X$, we get the lattice of *normal variations* of X in $\mathbf{X}$, determined by weak kernels $m: M \to X$ up to mutual factorisation.

3.4. The axiom (ex2). We prove now that normal monics in Fr$\mathbf{X}$ are closed under composition; by duality, the same holds for normal epis.

Consider the consecutive normal monics $[n, 1]$ and $[m, 1]$; by 3.3, we know that they are linked to their cokernels $[1, p]$ and $[1, q]$ by the following relations

(1)
$$\begin{tikzcd}
& & & M \arrow[dr, "m"] \arrow[ddd, "a"] & \\
N \arrow[r, "n"] \arrow[d, "xmn"'] & M \arrow[r, "m"'] \arrow[d, "xm"'] \arrow[urr, "1"] & X' \arrow[rr, equal] \arrow[d, "x"'] & & X' \arrow[d, "px"] \\
X'' \arrow[r, equal] & X'' \arrow[r, equal] & X'' \arrow[rr, "p"] \arrow[dr, "q"'] & & P \\
& & & Q \arrow[ur, dashed, "u"'] &
\end{tikzcd}$$
$(a = qxm)$

(2) $m \in \mathrm{wker}(px)$, $p \in \mathrm{wcok}(xm)$; $n \in \mathrm{wker}(q.xm)$, $q \in \mathrm{wcok}(xm.n)$.

Thus, p vanishes over xmn and factors through q, as $p = uq$. It follows easily that $[mn, 1] = \ker\,[q, 1_{X'}]$

(3)
$$\begin{tikzcd}
N \arrow[r, "mn"] \arrow[d, "xmn"'] & X' \arrow[r, equal] \arrow[d, "x"'] & X' \arrow[d, "qx"] \\
X'' \arrow[r, equal] & X'' \arrow[r, "q"'] & Q
\end{tikzcd}$$
$mn \in \mathrm{wker}(qx)$

(If $qx.v = 0$, also $px.v = 0$, whence $v = mv'$; now $qxm.v' = qx.v = 0$, and v' factors through n, which means that v factors through mn.)

3.5. The subquotient axiom (ex3). We finish proving that Fr$\mathbf{X}$ is homological. Given a normal subobject and a normal quotient of $x: X' \to X''$, satisfying the following relation

(1) $[m, 1]: m \rightarrowtail x$, $[1, q]: x \twoheadrightarrow q$, $[m, 1] \geq [n, 1] = \ker[1, q]$

we have to show that the composite $[1, q].[m, 1]$ is exact.

Let [1, p] = cok [m, 1] ≤ [1, q]. By hypothesis, there exist morphisms f, g such that xm.f = xn, and g.qx = px. It suffices thus to consider the following commutative diagram

(2)
$$\begin{tikzcd}
 & & & M \arrow[dr, "m"] \arrow[ddd, "a"] & & \\
N \arrow[r, "f"] \arrow[d, "xn"'] & M \arrow[r, "m"'] \arrow[d, "xm"'] \arrow[urr, "1"] & X' \arrow[rr, equal] \arrow[d, "x"'] & & X' \arrow[r, equal] \arrow[d, "qx"] & X' \arrow[d, "px"] \\
X'' \arrow[r, equal] & X'' \arrow[r, equal] \arrow[drr, "q"'] & X'' \arrow[rr, "q"] & & Q \arrow[r, "g"] & P \\
 & & & Q \arrow[ur, "1"'] & &
\end{tikzcd} \qquad (a \ = \ qxm)$$

where $[1_M, q]$ is a normal epi (since $q \in wcok(xn) = wcok(xm.f)$) and $[m, 1_Q]$ a normal monic (since $m \in wker(px) = wcok(g.qx)$).

3.6. Theorem: Exactness properties of Fr**X.** Let **X** be a pointed category, F: **X** → **B** a zero-preserving functor with values in a pointed epi-monic fs-category with kernels and cokernels, G: Fr**X** → **B** its fs-extension.

a) Let **X** have weak kernels and weak cokernels. Then Fr**X** is a pointed homological (3.1) epi-monic fs-category; the normal variations (3.3) of X in **X** form a lattice, identified with the lattice of normal subobjects of X in Fr**X**. The functor G preserves kernels and cokernels iff

(∗∗) every **X**-morphism f: X → Y has a weak kernel k: K → X and a weak cokernel c: Y → C such that, in the following commutative diagram of **B**, u is an fs-epi and v an fs-monic

(4)
$$\begin{tikzcd}
FK \arrow[r, "Fk"] \arrow[d, "u"'] & FX \arrow[r, "Ff"] \arrow[d, equal] & FY \arrow[r, "Fc"] \arrow[d, equal] & FC \\
Ker(Ff) \arrow[r] & FX \arrow[r, "Ff"] & FY \arrow[r] & Cok(Ff) \arrow[u, "v"']
\end{tikzcd}$$

iff the same happens for *all* weak kernels and weak cokernels of f. In particular, this holds for F = U.

a') If, in the same hypotheses, **B** is Puppe exact, then G preserves kernels and cokernels iff every **X**-morphism f: X → Y has a weak kernel K → X and a weak cokernel Y → C such that the sequence FK → FX → FY → FC is exact in **B**.

b) Assume that **X** has weak kernels and weak cokernels, and moreover every morphism is a weak kernel and a weak cokernel. Then Fr**X** is Puppe exact; all variations in **X** are normal and form a modular lattice $Var(A) = Sub_{Fr\mathbf{X}}(A)$. If also **B** is Puppe exact, the functor G is exact if and only if the condition (∗∗) is satisfied for its left-hand part (concerning k and u), if and only if it is satisfied for its right-hand part (concerning c and v).

c) If **X** satisfies the hypotheses of b) and has finite products (or sums), then Fr**X** is abelian.

Proof. a) The first part has been proved above (3.2-5). The limit-colimit preserving properties are as in 2.6. a') is a trivial consequence. Since c) follows from b), as already recalled in 3.1, we only need to verify the latter.

b) First, we have to prove that an arbitrary morphism $f = [f', f'']$ factors as a normal epi followed by a normal monic. In fact, recall the distinguished factorisation $f = [f', 1].[1, f'']$ considered at the beginning (2.2.2). Under the new hypotheses, f'' is a weak cokernel in **X** and $[1, f'']$ is a cokernel in Fr**X**, while f' is a weak kernel and $[f', 1]$ a kernel. Thus, Fr**X** is Puppe exact, by the general theory recalled above (3.1). If also **B** is so, the functor G is exact iff it preserves short exact sequences, iff it preserves kernels and epimorphisms; but the last condition is always satisfied by G, because all the epis of Fr**X** and **B** are distinguished.

3.7. Universal homology theories (Freyd)**.** It follows easily that, if **X** is a triangulated category, then Fr**X** is abelian and U: **X** → Fr**X** is the universal homological functor over **X**. (See [Fr2], Lemma 4.1, for **X** the stable homotopy category of spaces.)

In fact, the hypotheses of 3.6c are satisfied. First, if (u, v, w) is a (distinguished) triangle, it is easy to show that v is a weak cokernel of u (and dually a weak kernel of w)

(1)
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d, "f"] & Z \arrow[r, "w"] \arrow[d, dashed] & \Sigma X \arrow[d] \\
0 \arrow[r] & A \arrow[r, "1"] & A \arrow[r] & 0
\end{tikzcd}$$

But any arrow can appear in a triangle, in any position, and the conclusion follows. Note that, in a triangle, any arrow is a weak kernel of the following, and a weak cokernel of the preceding one.

Moreover, the functor U: **X** → Fr**X** is homological (or the sequence $U_n = U\Sigma^{-n}$ is a homology theory), since it takes every triangle to an exact sequence; actually, it is the universal homological functor on **X** (by 3.6a'): for every homological functor H: **X** → **B** (with values in an abelian category, or more generally in a Puppe exact one) there is an essentially unique exact functor G: Fr**X** → **B** such that GU = H.

References

[BKPS] G.J. Bird - G.M. Kelly - A.J. Power - R.H. Street, *Flexible limits for 2-categories*, J. Pure Appl. Algebra **61** (1989) 1-27.

[Ca] A. Carboni, *Some free constructions in realizability and proof theory*, J. Pure Appl. Algebra **103** (1995), 117-148.

[CLW] A. Carboni - S. Lack - R.F.C. Walters, *Introduction to extensive and distributive categories*, J. Pure Appl. Algebra **84** (1993), 145-148.

[CV] A. Carboni - E. Vitale, *Regular and exact completions*, J. Pure Appl. Algebra, to appear.

[Co] L. Coppey, *Algèbres de décompositions et précategories*, Diagrammes **3** (1980, supplément).

[EH] B. Eckmann - P.J. Hilton, *Unions and intersections in homotopy theory*, Comm. Math. Helv. 38 (1963-64), 293-307.

[Fr1] P. Freyd, *Representations in abelian categories*, in: Proc. Conf. Categ. Alg., La Jolla 1965, pp. 95-120, Springer 1966.

[Fr2] P. Freyd, *Stable homotopy*, in: Proc. Conf. Categ. Algebra, La Jolla 1965, pp. 121-176, Springer 1966.

[Fr3] P. Freyd, *Stable homotopy II*, in: Proc. Symp. Pure Maths. XVII, pp. 161-183, Amer. Math. Soc. 1970.

[Fr4] P. Freyd, *On the concreteness of certain categories*, in: Symposia Mathematica, Vol. IV (INDAM, Roma 1968/69), pp. 431-456, Academic Press, London 1970.

[Fr5] P. Freyd, *Homotopy is not concrete*, in: The Steenrod algebra and its applications, Battelle Memorial Inst., Columbus, Ohio, 1970, pp. 25-34, Lecture Notes in Maths. vol. 168, Springer 1970.

[FK] P. Freyd - G.M. Kelly, *Categories of continuous functors*, J. Pure Appl. Algebra **2** (1972) 1-18.

[FS] P. Freyd - A.Scedrov, *Categories, allegories*, North-Holland Publishing Co., Amsterdam, 1990.

[Gr1] M. Grandis, *On the categorical foundations of homological and homotopical algebra*, Cahiers Top. Géom. Diff. Catég. **33** (1992), 135-175.

[Gr2] M. Grandis, *Homotopical algebra in homotopical categories*, Appl. Categ. Structures, **2** (1994), 351-406.

[Gr3] M. Grandis, *Categorically algebraic foundations for homotopical algebra,* Appl. Categ. Structures **5** (1997), 363-413.

[Gr4] M. Grandis, *Weak subobjects and weak limits in categories and homotopy categories*, Cahiers Top. Géom. Diff. Catég., to appear.

[GM] M. Grandis - J. MacDonald, *Homotopy structures for algebras over a monad,* Appl. Categ. Structures, to appear.

[Ha] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. vol. 20, Springer 1966.

[JT] G. Janelidze - W. Tholen, *Functorial factorisation, well-pointedness and separability*, to appear.

[Ki] R.W. Kieboom, *On stable homotopy monomorphisms*, Appl. Categ. Structures **5** (1997), 131-142.

[KT] M. Korostenski - W. Tholen, *Factorisation systems as Eilenberg-Moore algebras*, J. Pure Appl. Algebra **85** (1993), 57-72.

[Ku] A.G. Kurosh, *The theory of groups*, Chelsea Publ. Co., New York 1960.

[La] F.W. Lawvere, *Adjoints in and among bicategories,* in: "Logic and Algebra" (Pontignano, 1994), pp. 181-189, M. Dekker, New York 1996.

[Ma] M. Mather, *Pull-backs in homotopy theory*, Can. J. Math. **28** (1976), 225-263.

[Mi] B. Mitchell, *Theory of categories*, Academic Press, 1965.

[Ne] A. Neeman, *New axioms for triangulated categories*, J. of Algebra **139** (1991), 221-255.

[Pu1] D. Puppe, *Korrespondenzen in abelschen Kategorien*, Math. Ann. **148** (1962), 1-30.

[Pu2] D. Puppe, *On the formal structure of stable homotopy theory*, in: Colloquium on Algebraic Topology, pp. 65-71, Mat. Inst., Aarhus Univ. 1962.

[Pu3] D. Puppe, *Stabile Homotopietheorie I*, Math. Ann. **169** (1967), 243-274.

[Ve] J. L. Verdier, *Catégories dérivées*, in: Séminaire de Géométrie algébrique du Bois Marie SGA $4^1/_2$, Cohomologie étale, pp. 262-311, Lecture Notes in Math. vol. 569, Springer 1977.