Weakly exact categories and their relations

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Abstract. This work is a first step in extending to the non-commutative case previous works concerned with the study of spectral sequences via universal models. We characterise categories of relations on weakly exact categories, i.e. γ-categories in the sense of Burgin; we also study their lattices of subobjects and various lattice-theoretical properties of such categories.

0. Introduction

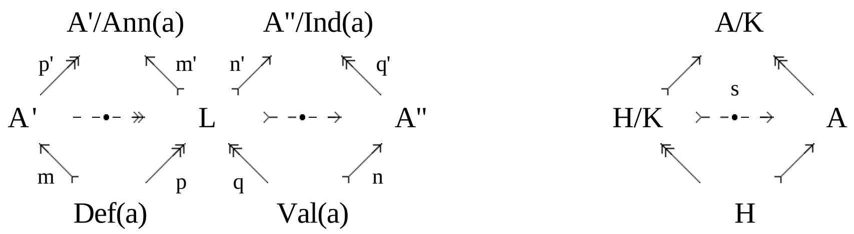
0.1. We want to extend to the non-commutative case a series of three papers [G10-12] on commutative homological algebra and universal models of spectral sequences. Here, we are mostly concerned with the extension of the first, [G10], which will be cited as Part I; the reference I.1, or I.1.2, or I.1.2.3 applies respectively to its Section 1, or Subsection 1.2, or item (3) in the latter.

The categorical frame we chose in the former series is **EX**, the 2-category of exact categories (in the sense of Puppe-Mitchell [Pu, Mi]), exact functors and natural transformations. We have shown that in such a frame the biuniversal model of "homological theories", as the filtered complex or the double complex, can be explicitly described: the biclassifying exact category can be "drawn" in the (discrete or real) plane, yielding a graphic tool (a sort of algebraic crossword chasing) for studying the associated spectral sequences; in the case of the discrete filtered complex we recover the Zeeman diagram [ZE, HW].

As a crucial, distinctive fact (see I.0.1), all the above theories are *distributive*, in the sense that their biclassifying exact category has distributive lattices of subobjects, while – generally – the lattices of subobjects of an exact category are just modular.

Notice that *such results cannot be achieved in* **AB**, the 2-category of abelian categories: the biclassifying abelian category of the above theories is not distributive (as any non-trivial abelian category); also, it is much more complicated than the exact one and probably cannot be given simple representations. Note also that the lack of an additive structure in **EX** will make easier the present extension of results to non-commutative algebra.

Homological algebra is – essentially – a calculus of subquotients. Therefore, an important tool to prove our results is the calculus of relations, which reduce subquotients in **E** to *subobjects* in Rel**E**. More precisely, for a p-exact category **E**, a relation $a\colon A' \nrightarrow A''$ has a *W-factorisation* $a = nq^{\#}pm^{\#}$ by four arrows of **E**, determined up to isomorphism, as in the lower part of the left diagram below

(1) 

it also has a dual *W*-factorisation* $a = q'^{\#}n'm'^{\#}p'$, as in the upper part; the fact that the two factorisations yield the same relation is ensured by the two squares being pullbacks (or, equivalently, pushouts). The dotted diagonals give the (essentially unique) factorisation of a in epirelation / monorelation; thus, a monorelation s with values in A amounts to a *subquotient* H/K of A in **E** (a quotient of a subobject, or equivalently a subobject of a quotient), as shown in the right diagram above, again a bicartesian square.

The construction of Rel**E** is realised by equivalence classes of *W-shaped diagrams* in **E**, up to three central isomorphisms (or, dually, by equivalence classes of *W*-shaped diagrams*): cf. Calenko [C1, 2], Brinkmann-Puppe [BP], or a brief description here (4.3). If **E** has finite products, than it is abelian and relations can be equivalently constructed as equivalence classes of spans (or cospans).

It is important to note that a subquotient $s: H/K \rightarrow A$ can also be represented by a *projection* $e: A \rightarrow A$, i.e. a symmetric idempotent endorelation $(e = ee = e^{\#})$

(2) $e = (A \leftarrowtail H \twoheadrightarrow H/K \twoheadleftarrow H \rightarrowtail A)$ $\qquad (e = ee = e^{\#})$,

with the advantage that e *is uniquely determined*: two monorelations s, t (with values in A) are equivalent if and only if they have the *same* associated projection: $ss^{\#} = tt^{\#}: A \rightarrow A$; further, we can simulate the numerator (H) and the denominator (K) of a subquotient by two *restrictions* (projections ≤ 1), which again are *strictly determined*

(3) $\underline{n}(e) = (A \leftarrowtail H \rightarrowtail A)$, $\qquad \underline{d}(e) = (A \leftarrowtail K \rightarrowtail A)$,

and *strictly preserved* by "good" functors, the ones which preserve involution and order.

This is why we preferred to replace the *pseudo*-complete 2-category **EX** with the *strictly* 2-complete category **RE** of *RE-categories*, involutive ordered categories generalising the categories of relations over exact categories. Thus, in **RE**, *strict* 2-universal problems can be solved, simplifying our work; a weak adjunction between **EX** and **RE** yields the transfer of results. (Note, however, that one *can* obtain directly in **EX** such results, applying suitable theorems on the existence of *bi*universal models, as in [BG].)

More complete motivations can be found in the Introduction of Part I.

0.2. We want now to extend this study of "homological theories" to the non-commutative case, so that it can be applied to such categories as **Gp** (groups) or **Rng** (associative rings, without unit assumption). The weaker notion of exactness which is suitable for our purposes is given by γ-categories, a non selfdual notion introduced and studied by Burgin [Bu], together with the construction of relations by means of W-diagrams (while W*-diagrams work in the dual case, including the category of pointed sets). These γ-categories will also be called *weakly exact* or *w-exact*, for the sake of uniformity of terminology.

We begin here by extending Part I. Therefore we introduce and study *RW-categories*, by weakening the RE-axioms (I.4.1): a *projection* $e: A \nrightarrow A$, with $e = ee = e^{\#}$ (simulating a subquotient H/K of A), is no longer assumed to have a ^{c}-denominator $^c\underline{d}(e)$ (simulating the quotient A/K, *which would require* K *to be normal in* A) but only a denominator $\underline{d}(e)$ (simulating the subobject K normal in H). The categories of proper morphisms of "projection-complete" RW-categories appear to be precisely those categories whose connected components are γ-categories; conversely, RW-categories coincide with the "projection-full" involutive subcategories of the categories of relations over γ-categories.

This work is not a straightforward generalisation of Part I. The *normality* relation $x \triangleleft y$ between restrictions, which appears in the present case, makes various notions (e.g. RE-functors, their factorisations, RE-subcategories...) to have at least two extensions, of which the non-standard one, related with the reflection of normality, is often more interesting. Moreover we have to extend many results of related works [G6-8] which we used in Part I; in particular modular lattices and their exact category **Mlc** of modular connections [G8], simulating the covariant and contravariant images of subobjects in exact categories, are here generalised by *w-modular weak lattices* (wm-lattices; see 11.1, 2) and their w-exact category of *wm-connections*. These notions we consider well-adapted to study the lattices of subgroups and subrings, in the same way as modular lattices are the good notion for lattices of submodules.

0.3. In Sections 1-3 we introduce the 2-category **RW**: an RW-category **A** is provided with a regular involution $(-)^{\#}$ and a consistent order $\leq$, so that each projection $e: A \to A$ $(e = ee = e^{\#})$ has two associated *restrictions* (i.e. projections ≤ 1), the *numerator* and the *denominator*

(1) $\underline{n}(e): A \to A, \qquad\qquad \underline{d}(e): A \to A,$

and each object A has suitable null projections ω_A and Ω_A; RW-functors preserve all that (2.1). The canonical factorisation of RE-functors extends here to the *ordinary* factorisation (3.2) and to the less obvious but more interesting *closed* factorisation (3.6), via weak quotients and *closed* faithful RW-functors (reflecting normality between restrictions).

In Sections 4, 5 we prove the connections between RW-categories and γ-categories (w-exact categories) previously expounded. The RW-category w**Mlr** *of wm-lattices and wm-relations* is introduced in Section 6, together with the w-exact category w**Mlc** = Prp(w**Mlr**) formed by its proper morphisms, the *wm-connections*. In Section 7 we deal with the *transfer* functors

(2) $\mathrm{Rst}_{\mathbf{A}}: \mathbf{A} \to \mathrm{w}\mathbf{Mlr}, \qquad\qquad \mathrm{Sub}_{\mathbf{E}}: \mathbf{E} \to \mathrm{w}\mathbf{Mlc},$

of RW-categories and w-exact categories, their *lattice-theoretical properties* (e.g., *distributivity*) and a general treatment of *expansions* (7.5-11). Every wm-lattice *can be realised* as the w-lattice of subobjects of some object in a (fixed) w-exact category (w**Mlc**), and no other lattice-like notion can be suitable for w-exact categories.

The distributive and idempotent cases for RW-categories and w-exact categories are characterised in Section 8, and "universal representatives" of this type are given. In Section 9 a further lattice property is considered, corresponding to the stability of normal subobjects with respect to intersection in w-exact categories.

Last, Section 10 deals with the dual case of RW^0-categories and w*-exact ones; the self-dual case reduces to RE-categories and exact categories. Section 11 is an appendix containing the generalisation of modular and distributive lattices used throughout this work.

0.4. Conventions. We follow the same conventions as in Part I (I.0.6), which we briefly recall here. We generally use Mac Lane's terminology [Ma] for categories and Kelly - Street's [KS, Ke, S1, S2] for 2-categories. The set of subobjects ("chosen monos") or quotients ("chosen epis") of the object A in the category **C** is written $Sub_{\mathbf{C}}(A)$ or $Quo_{\mathbf{C}}(A)$, respectively.

0.5. RO-categories. The results of I.1-3 on involutive ordered categories need no adaptation to be used here; we only recall some basic terminology.

An *involutive* category $\mathbf{A} = (\mathbf{A}, ^\#)$ is a category provided with an *involution* $(–)^\#: \mathbf{A} \to \mathbf{A}$, i.e. a contravariant endofunctor identical on the objects and involutive, whose result on the morphism $a: A' \to A''$ will be written as $a^\#: A'' \to A'$. Actually we only consider *regular* involutions, satisfying

(1) $a = aa^\#a$, for every morphism a.

A *projection* $e: A \to A$ is a symmetric idempotent endomorphism: $e = ee = e^\#$; an equivalent condition is: $e = e^\#e$, or also $e = ee^\#$.

The projections of the object A form a set $Prj_{\mathbf{A}}(A)$, non closed with respect to composition: the product ef of two projections is an idempotent; it *is a projection if and only if* e *and* f *commute*; conversely, every idempotent e is the product of two projections $e = (ee^\#)(e^\#e)$; Prj(A) is canonically ordered by

(2) $e \prec f$ if $e = ef$ $(\Leftrightarrow \; e = fe \; \Leftrightarrow \; e = fef)$.

Every morphism $a: A' \to A''$ has a covariant and contravariant *transfer of projections* (I.1.3), by order preserving mappings

(3) $a_P: Prj(A') \to Prj(A'')$, $a_P(e) = aea^\#$,
$a^P: Prj(A'') \to Prj(A')$, $a^P(f) = a^\#fa = (a^\#)_P(f)$,

in a functorial way: $(ba)_P = b_Pa_P$, $(ba)^P = a^Pb^P$. The morphism a has two associated projections, simulating its coimage and its image

(4) $\underline{c}(a) = a^P(1) = a^\#a \in Prj(A')$, $\underline{i}(a) = a_P(1) = aa^\# \in Prj(A'')$,
a is mono $\Leftrightarrow$ $a^\#a = 1$, a is epi $\Leftrightarrow$ $aa^\# = 1$,

thus, *all monos are split, as well as all epis*; if a is mono and epi then it is an iso, with $a^{-1} = a^\#$.

Now, a *RO-category* (I.1.2) $\mathbf{A} = (\mathbf{A}, ^\#, \leq)$ is a category **A** provided with a regular involution and an order relation $\leq$ on parallel morphisms, consistent with composition and involution; furthermore we assume that **A** is *Prj-small*, i.e. all its projection-sets Prj(A) are small.

Then, the set Prj(A) has two order relations, $\prec$ and $\leq$, generally different. It is easy to see that these orders coincide on the subset Rst(A) of the *restrictions* $x: A \to A$ (defined by $x \leq 1$), which is a meet-semilattice with respect to composition. On the other hand, they are opposite on the subset Crs(A) of the *corestrictions* x' of A $(x' \geq 1)$: $x' \prec y'$ if and only if $x' \geq y'$.

Proper morphisms $a: A' \to A''$ (defined by the usual conditions $a^{\#}a \geq 1$, $aa^{\#} \leq 1$) and *null* morphisms ($aa'a = a$, for all $a': A'' \to A'$) are also studied in I.1. If u, v are proper and $u \leq v$, then $u = v$.

A *RO-functor* $F: \mathbf{A} \to \mathbf{B}$ is a functor between RO-categories preserving involution and order (hence also projections, restrictions, corestrictions and proper morphisms). A *RO-transformation* (I.2.3)

(5) $\varphi: F \to F': \mathbf{A} \to \mathbf{B}$,

assigns, to each **A**-object A, a *proper* morphism $\varphi A: F(A) \to F'(A)$ of **B**, so that a *lax*-naturality condition holds

(6) $\varphi A''.Fa \leq Ga.\varphi A'$, for every $a \in \mathbf{A}(A', A'')$,

which implies equality when a *is proper*.

This defines **RO**, the 2-category of RO-categories, RO-functors and RO-transformations, equipped with an obvious 2-functor

(7) $\text{Prp}: \mathbf{RO} \to \mathbf{CAT}$.

Last we recall (from I.3) that the following conditions on a RO-category are equivalent:

(a) it has epi-mono factorisations (necessarily unique),

(b) every idempotent e splits (factors $e = ts^{\#}$, where s and t are mono),

(c) every projection e splits (factors $e = ss^{\#}$, where s is a mono);

then, we say that our RO-category is *projection-complete*, or *factorising*.

Every RO-category **A** has an associated *projection completion* Fct(**A**), solving the obvious biuniversal problem (I.3.8): the objects of Fct(**A**) are the projections of **A**, while a morphism $(a; e, f): e \to f$ is a morphisms of **A** such that $a = fae$. The well-known idempotent completion, with idempotents as objects and similar morphisms, is equivalent to Fct(**A**).

1. RW-categories

We introduce here our extension of RE-categories (I.4), which will be shown in Section 5 to generalise also the categories of relations on γ-categories. Basic notions and results on R0-categories (I.1-3, recalled here in 0.5) are often used without reference.

1.1. Definition. An *RW-category* will be a triple $\mathbf{A} = (\mathbf{A}, {}^{\#}, \leq)$ satisfying:

(RW.0) **A** is a RO-category (0.5).

(RW.1) For every projection e there exists precisely one restriction $\underline{n}(e)$ such that $e \prec \underline{n}e \leq e$, called the *numerator* of e.

(RW.2) Every object A has a null restriction $\omega = \omega_A$ and a null corestriction $\Omega = \Omega_A$.

For any $e \in \text{Prj}(A)$ the *denominator* of e is defined to be the restriction

(1) $\underline{d}(e) = \underline{n}(e\omega_A e)$.

(RW.3) For all parallel projections e, f

(a) $e \prec f \;\Leftrightarrow\; (\underline{n}e \prec \underline{n}f$ and $\underline{d}e \succ \underline{d}f)$,

(b) $e \leq f \;\Leftrightarrow\; (\underline{n}e \prec \underline{n}f$ and $\underline{d}e \prec \underline{d}f)$.

The projections ω_A, Ω_A are uniquely determined, since the axiom (RW.2) is equivalent to the following one (in the presence of (RW.0)):

(RW.2') For every object A, the set of endomorphisms $\mathbf{A}(A, A)$ has a least element ω_A and a greatest one Ω_A, satisfying $\omega = \omega\Omega\omega$, $\Omega = \Omega\omega\Omega$.

By I.4.1, I.4.4, the RE-categories are precisely those RW-categories which admit ^{c}*-denominators* (opposite to numerators, with respect to the order $\leq$): for every projection e there exists precisely one corestriction ${}^c\underline{d}(e)$ such that $e \prec {}^c\underline{d}e \geq e$.

1.2. Normality. Henceforth $\mathbf{A}$ is an RW-category and A, A', A'' are objects of $\mathbf{A}$.

Every projection e of A is determined by the pair $(\underline{n}e, \underline{d}e)$ of its numerator and denominator (RW.3); by applying (RW.3b) to the inequality $e\omega e \leq e$, we get: $\underline{d}e \leq \underline{n}e$.

Now, if $x, y \in \mathrm{Rst}(A)$, we write $y \lhd x$ (y is *normal* in x) whenever there exists a projection e such that $\underline{n}e = x$ and $\underline{d}e = y$; then $y \leq x$ while e is determined and will be written as x/y.

Thus, for $x \in \mathrm{Rst}(A)$ and $e \in \mathrm{Prj}(A)$

(1) $x = x/\omega$, $\qquad \omega \lhd x$, $\qquad 1 = 1/\omega$, $\qquad \omega \lhd 1$,

(2) $x\Omega x = x/x$, $\qquad x \lhd x$, $\qquad \omega = \omega/\omega$, $\qquad \Omega = 1/1$,

(3) $e \in \mathrm{Rst}(A) \;\Leftrightarrow\; \underline{d}e = \omega \;\Leftrightarrow\; e = \underline{n}e$,

(4) $e \in \mathrm{Crs}(A) \;\Leftrightarrow\; \underline{n}e = 1 \;\Leftrightarrow\; e = 1/\underline{d}e$,

(5) $e \in \mathrm{Nul}(\mathbf{A}) \;\Leftrightarrow\; \underline{n}e = \underline{d}e$.

Actually, for (5), if e is null then $\underline{d}e = \underline{n}(e\omega e) = \underline{n}e$; conversely, if $\underline{n}e = \underline{d}e$, then e and $e\omega e$ have the same numerator and denominator, hence coincide and e is null.

Last it follows from (5) and (RW.3b) that the meet-semilattice $\mathrm{Rst}(A)$ is isomorphic to the set $(\mathrm{Npr}(A), \leq)$ of null projections of A, via

(6) $x \mapsto x/x$, $\qquad\qquad e \mapsto \underline{n}e$.

1.3. Transfer of restrictions. For $a \in \mathbf{A}(A', A'')$ let us define the covariant and contravariant *transfer of restrictions* along a, by two mappings (deduced from the transfer of projections, 0.5.3)

(1) $a_R\colon \mathrm{Rst}(A') \to \mathrm{Rst}(A'')$, $\qquad a_R(x) = \underline{n}(a_P(x)) = \underline{n}(axa^\#)$,

$a^R\colon \mathrm{Rst}(A'') \to \mathrm{Rst}(A')$, $\qquad a^R(y) = \underline{n}(a^P(y)) = \underline{n}(a^\# ya) = (a^\#)_R(y)$,

which will be seen later to form a "wm-relation between wm-lattices" (7.1). Now

(2) $a_P(x/y) = a_R(x)/a_R(y)$, $\qquad$ for $y \lhd x$ in $\mathrm{Rst}(A')$,

$a_P(x) = a_R(x)/a_R(\omega)$, $\qquad$ for $x \in \mathrm{Rst}(A')$.

Indeed, as the transfer of projections a_P preserves the orders $\leq$ and $\prec$, by (RW.3)

(3) $a_P(x/y) \prec a_P(x)$, $\qquad a_P(x/y) \geq a_P(x)$,

$a_P(x/y) \succ a_P(y/y)$, $\qquad a_P(x/y) \geq a_P(y/y)$,

therefore, again by (RW.3)

(4) $\underline{n}(a_P(x/y)) = \underline{n}(a_P(x)) = a_R(x)$,

(5) $\underline{d}(a_P(x/y)) = \underline{d}(a_P(y/y)) = \underline{n}(a_P(y/y)) = a_R(y)$.

In particular we have

(6) $\underline{c}(a) = a^{\#}a = a^P(1) = a^R(1)/a^R(\omega)$, $\qquad \underline{i}(a) = aa^{\#} = a_P(1) = a_R(1)/a_R(\omega)$.

The transfer of restrictions is functorial, like the one of projections (0.5)

(7) $(ba)_R(x) = b_R(a_R(x))$, $\qquad (ba)^R(y) = a^R(b^R(y))$,

as it follows from (2)

(8) $(ba)_P(x) = b_P(a_P(x)) = b_P(a_R(x)/a_R(\omega)) = b_R(a_R(x))/b_R(a_R(\omega))$.

1.4. Definition and annihilator. For a morphism $a \colon A' \to A''$ we shall consider the following restrictions of its domain and codomain, called *definition, annihilator, values* and *indetermination* of a (simulating the analogous subobjects, which exist when **A** is the category of relations of some w-exact category: see 4.6 and 5.6)

(1) $\underline{def}(a) = \underline{n}(a^{\#}a) = a^R(1) \in \mathrm{Rst}(A')$, $\qquad \underline{ann}(a) = \underline{d}(a^{\#}a) = a^R(\omega) \in \mathrm{Rst}(A')$,

$\underline{val}(a) = \underline{n}(aa^{\#}) = a_R(1) \in \mathrm{Rst}(A'')$, $\qquad \underline{ind}(a) = \underline{d}(aa^{\#}) = a_R(\omega) \in \mathrm{Rst}(A'')$,

so that

(2) $\underline{ann}(a) \triangleleft \underline{def}(a)$, $\qquad \underline{ind}(a) \triangleleft \underline{val}(a)$,

(3) $\underline{def}(a) = \underline{val}(a^{\#})$, $\qquad \underline{ann}(a) = \underline{ind}(a^{\#})$,

(4) a is mono $\Leftrightarrow$ ($\underline{def}(a) = 1$ and $\underline{ann}(a) = \omega$),

a is epi $\Leftrightarrow$ ($\underline{val}(a) = 1$ and $\underline{ind}(a) = \omega$),

(5) a is proper $\Leftrightarrow$ ($\underline{def}(a) = 1$ and $\underline{ind}(a) = \omega$),

(6) a is null $\Leftrightarrow$ $\underline{def}(a) = \underline{ann}(a)$ $\Leftrightarrow$ $\underline{val}(a) = \underline{ind}(a)$.

Moreover, for a projection e

(7) $\underline{n}e = \underline{def}(e) = \underline{val}(e)$, $\qquad \underline{d}e = \underline{ann}(e) = \underline{ind}(e)$.

1.5. The operation &. The set $\mathrm{Prj}(A)$ will be provided with the binary operation &

(1) $e\&f = efe = e_P(f)$,

a sort of generalised meet, which will be calculated in the next subsection. The operation is idempotent, with identity 1_A, generally *neither associative nor commutative*: we shall prove that & is associative (for all objects) if and only if **A** is *w-distributive* (7.5; 8.1). We have

(a) $e \prec f \Leftrightarrow e = f\&e$,

(b) e and f commute $\Leftrightarrow$ e&f = f&e $\Leftrightarrow$ e&f $\prec$ f&e.

Indeed, (a) follows from (0.5.2); for (b), if the third property holds

(2) ef = ef.ef.ef = efe.fef = efe,

hence ef is a projection and e, f commute.

1.6. Theorem (Calculus of projections). In the RW-category **A**, for all projections e, f $\in$ Prj(A)

(a) e&f = efe = $e_P(f)$ = $e^P(f)$ = $(xz \vee y)/(xt \vee y)$ (for e = x/y, f = z/t);

(b) e and f commute if and only if $\underline{n}e \succ \underline{d}f$ and $\underline{n}f \succ \underline{d}e$; in this case

(1) ef = fe = e&f = f&e = $(\underline{n}e.\underline{n}f)/(\underline{d}e \vee \underline{d}f)$;

(c) for all y' $\in$ Rst(A), y' = $\underline{d}e$ $\Leftrightarrow$ (y' $\leq \underline{n}e$ and y'e = ey' $\in$ Nul**A**).

(Recall that Rst(A) is a meet-semilattice for composition (0.5); joins need not exist (see 1.7). We also note that in Part I the analogous, more particular, result for RE-categories (I.6.9) was deduced from the theory of exact categories; the present direct approach is more satisfactory.)

Proof. First we prove the following part of (b)

(b') if the restriction x is $\geq$ t, then x commutes with f = z/t.

Indeed, by 1.3.2 and (RW.3)

(2) xfx = $x_P(f)$ = $x_R(z)/x_R(t)$ = xz/xt = xz/t;

xz = zxz $\leq$ fxf; fxf $\leq$ f,

Thus $\underline{n}$(xfx) = xz $\leq$ $\underline{n}$(fxf) and $\underline{d}$(xfx) = $\underline{d}$f $\geq$ $\underline{d}$(fxf); by (RW.3a), x&f $\prec$ f&x; by 1.5b, x and f commute.

Now we prove (a). By 1.3.2

(3) efe = $e_P(f)$ = $e_R(z)/e_R(t)$,

therefore we need only to verify that

(4) $\underline{n}$(eze) = $xz \vee y$.

Actually

(5) xz = xzx $\leq$ eze, y $\leq$ y/y.y/y = (y/y)z(y/y) $\leq$ eze,

hence, by (RW.3b), xz $\leq$ $\underline{n}$(eze) and y $\leq$ $\underline{n}$(eze). Take now some restriction r greater than xz and y in Rst(A); by (b'), r commutes with e = x/y and

(6) r.(eze) = r.ex.ze = er.xz.e = e.xz.e = eze.

Thus eze $\prec$ r and $\underline{n}$(eze) $\leq$ r, which achieves (4).

It is now easy to deduce (b): e and f commute iff e&f = f&e (1.5b), iff

(7) $xz \vee y$ = $xz \vee t$; $xt \vee y$ = $zy \vee t$,

if and only if x $\succ$ t and z $\succ$ y.

Last, for (c), $\underline{d}e$ commutes with e by (b); conversely, if y' satisfies the conditions in (c)

(8) $y' = y'.\underline{n}e = \underline{n}(y'e) = \underline{d}(y'e) = \omega \vee \underline{d}e = \underline{d}e$. □

1.7. Theorem (Normomodularity). For all objects A of the RW-category **A**, the set $\mathrm{Rst}(A)$ is a w-modular w-lattice (11.1, 2) with respect to its relations $\leq$ and $\lhd$.

Proof. We already know that $\mathrm{Rst}(A)$ is a meet-semilattice with respect to $\leq$ and product, with minimum ω and maximum 1; also the axiom (wl.1) in 11.1 is known to hold (1.2.1-2). Take $x, y, z, t \in \mathrm{Rst}(A)$.

Assume that $y \lhd x$. For (wl.2), it is sufficient to note that (by 1.6)

(2) $z \,\&\, (x/y) = zx/zy$,

hence $zy \lhd zx$. If moreover $t \lhd z \leq x$

(3) $(x/y) \,\&\, (z/t) = (xz \vee y)/(xt \vee y) = (z \vee y)/(t \vee y)$,

which proves both (wl.3), by taking $z = t$, and (wl.4).

Now, for the first normomodularity condition (wm.1), suppose that $y \lhd x$ and $t \leq xz$. Consider the projections

(4) $e = z \,\&\, ((x/y) \,\&\, t) = z\,(x/y)\,t\,(x/y)\,z$,

(5) $f = (z \,\&\, (x/y)) \,\&\, t = z\,(x/y)\,ztz\,(x/y)\,z$,

which are equal because $t \leq z$; calculating their numerators, by 1.6, we get our goal

(6) $\underline{n}e = z \wedge \underline{n}((x/y) \,\&\, t) = z \wedge (xt \vee y) = z \wedge (t \vee y)$,

(7) $\underline{n}f = (\underline{n}(z \,\&\, (x/y)) \wedge t) \vee \underline{d}(z \,\&\, (x/y)) = (zx \wedge t) \vee zy = t \vee zy$.

Last (wm.2) is proved in a similar way. Suppose that $y \lhd x$, $t \leq x$, $y \leq z$ and consider the projections

(8) $e = z \,\&\, ((x/y) \,\&\, t) = z(x/y)t(x/y)z$, $\qquad f = (x/y) \,\&\, zt = (x/y)zt(x/y)$,

which coincide since z commutes both with x/y (1.6) and t. Also here it suffices to calculate their numerators

(9) $\underline{n}e = z \wedge \underline{n}((x/y) \,\&\, t) = z \wedge (xt \vee y) = z \wedge (t \vee y)$,

$\underline{n}f = xzt \vee y = zt \vee y$. □

1.8. Normal RW-categories. The *normal restrictions* $x \lhd 1$ of A form a join-semilattice $\mathrm{Nrm}(\mathrm{Rst}(A))$ (11.4), which is anti-isomorphic to the ordered set $(\mathrm{Crs}(A), \prec)$, hence isomorphic to $(\mathrm{Crs}(A), \leq)$, via

(1) $\mathrm{NrmRst}(A) \to \mathrm{Crs}(A)$, $\qquad x \mapsto 1/x$,

(2) $\mathrm{Crs}(A) \to \mathrm{NrmRst}(A)$, $\qquad e \mapsto \underline{d}e$.

We say that the RW-category **A** is *normal* (resp. *subnormal*) when all its wm-lattices $\mathrm{Rst}(A)$ are so, i.e. when every restriction x of each object A is normal (resp. subnormal).

It is easy to see that **A** *is normal if and only if it is an RE-category* (I.4.1). Actually, if **A** is normal, it satisfies (RE.1b): each projection $e = x/y$ has denominator $y \lhd 1$, hence by (RW.3) there is exactly one corestriction x' of **A** such that $e \leq x'$ and $e \prec x'$, i.e. $x' = 1/y$; thus **A** is an RE-category. The converse is trivial.

We shall consider in 9.1 certain RW-categories in which normal restrictions are meet-stable, so that all sets NrmRst(A) are modular lattices.

1.9. Theorem (The projection completion). The projection complete RO-category $\mathbf{B} = \mathrm{Fct}(\mathbf{A})$ (0.5) on the RW-category **A** is an RW-category. Moreover, if $e \in \mathrm{Prj}(A)$ and

(1) $f' = (f; e, e) \in \mathrm{Prj}_{\mathbf{B}}(A)$, $\qquad f = x/y \prec e$,

then

(2) $\omega_e = (e\omega_A e; e, e)$, $\qquad \Omega_e = (e\Omega_A e; e, e)$,

(3) $\underline{n}f' = (ex; e, e) = (\underline{n}f/\underline{d}e; e, e)$, $\qquad \underline{d}f' = (ey; e, e) = (\underline{d}f/\underline{d}e; e, e)$.

Proof. We already know that **B** is a projection complete RO-category. Clearly it satisfies (RW.2) with null restrictions and corestrictions as specified in (2). As to (RW.1), by (1)

(4) $\underline{d}e \leq y \leq x \leq \underline{n}e$,

so that both x and y commute with e (1.6b). The restriction $x' = (xe; e, e)$ of e is a numerator for f' since

(5) $x' \leq f'$ $\quad (xe = xex \leq fef = f)$, $\qquad x'.f' = (xef; e, e) = (f; e, e) = f'$.

Conversely, if $x'' \in \mathrm{Rst}(e)$ and

(6) $x'' = (g; e, e)$, $\qquad x'' \leq f' \prec x''$,

we have

(7) $g \prec e$, $\qquad g \leq e$, $\qquad g \leq f \prec g$,

so that, by (RW.3) in **A**

(8) $\underline{n}g = \underline{n}f$ and $\underline{d}g = \underline{d}e$;

thus g is determined, as $g = \underline{n}f/\underline{d}e$. Last

(9) $\underline{d}f' = \underline{n}(f'\omega_e f') = \underline{n}(fe\omega ef; e, e) = \underline{n}(f\omega f; e, e) = (\underline{n}(f\omega f)/\underline{d}e; e, e) = (\underline{d}f/\underline{d}e; e, e)$.

It follows easily that **B** satisfies (RW.3). □

1.10. Lemma (Characterisation of order). Let $a, b \in \mathbf{A}(A, A')$ and

(1) $x = \underline{\mathrm{def}}(a)$, $\qquad z = \underline{\mathrm{def}}(b) \in \mathrm{Rst}(A)$, $\qquad x' = \underline{\mathrm{val}}(a) \in \mathrm{Rst}(A')$.

Then $a \leq b$ if and only if

(2) $a = az$ and $a.b^{\#}b.a^{\#}a = x'bx$.

In particular

(3) $(a \leq b,\ \underline{c}a = \underline{c}b,\ \underline{i}a = \underline{i}b) \Rightarrow a = b$.

Proof. If (2) holds, then

(4) $a = az = axz = azx = a.\underline{n}(b^{\#}b).\underline{n}(a^{\#}a) \leq a.b^{\#}b.a^{\#}a = x'bx \leq b$.

Conversely, if $a \leq b$, clearly $a = az$ and

(5) $a.b^{\#}b.a^{\#}a = x' (a.b^{\#}b.a^{\#}a) x \leq x' (b.b^{\#}b.b^{\#}b) x = x'bx$,

(6) $a.b^{\#}b.a^{\#}a \geq aa^{\#}.b.a^{\#}a \geq \underline{n}(aa^{\#}).b.\underline{n}(a^{\#}a) = x'bx$. □

1.11. Order and restrictions. A first consequence of the above lemma is that the order $\leq$ in **A** *is determined by the restrictions* of **A**. More precisely, let $\mathbf{A}_1 = (\mathbf{A}, {}^{\#}, \leq_1)$ and $\mathbf{A}_2 = (\mathbf{A}, {}^{\#}, \leq_2)$ be two RW-structures on the same involutive category $(\mathbf{A}, {}^{\#})$, and assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same restrictions. Since the numerator $\underline{n}e$ of the projection e of $\mathbf{A}_i$ is the smallest (for $\prec$) restriction x such that xe = e, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same numerators; by the above lemma the orders $\leq_1$ and $\leq_2$ coincide.

2. The complete 2-category RW

A and **B** are always RW-categories.

2.1. RW-functors and transformations. An *RW-functor* will be an RO-functor F: **A** → **B** between RW-categories which preserves null morphisms (or, equivalently, all projections ω; or also, all projections Ω). Hence it preserves projections, restrictions and their meet, corestrictions, numerators, denominators, as well as the normality relation $\lhd$.

It also preserves $\lhd$-unions (11.1) of restrictions: indeed, if $y \lhd x$ and $z \lhd x$, consider the following projection e (applying the calculus of projections 1.6)

(1) $e = (x/y)z(x/y) = (z \vee y)/y$,

(2) $F(z \vee y) = F(\underline{n}e) = \underline{n}(Fe) = \underline{n}((Fx/Fy).Fz.(Fx/Fy)) = F(z) \vee F(y)$.

An RW-transformation

(3) α: F → G: **A** → **B**,

is an RO-transformation (0.5) between RW-functors; recall that such transformations are just *lax*-natural (with respect to the 2-categorical structure given by the order $\leq$), yet "natural on proper morphisms".

These functors and transformations define **RW**, a sub-2-category of **RO**.

It is easy to see that an RW-functor is an isomorphism (resp. an equivalence) in **RW** if and only if it is bijective on objects and morphisms (resp. faithful, representative and full).

The full embedding (I.3.8; 1.9)

(4) $\eta\mathbf{A}$: **A** → Fct**A**, A ↦ (A, 1), a ↦ (a; 1, 1),

yields a biuniversal arrow (I.0.6) from the object **A** to the 2-functor Fct: **RW** → F**RW**, the latter being the full sub-2-category of **RW** determined by projection complete RW-categories.

2.2. Basic reflection properties. We need to study various reflection properties for an RW-functor F: **A** → **B**; the trivial ones are exposed below, others will follow (e.g.: 2.3, 2.4, 3.1, 3.3).

Every projection (resp. restriction) e' ∈ **B**(FA, FA) such that e' = Fa for some a: A → A' can be written as Fe for some projection (resp. restriction) e: A → A: just take $e = \underline{c}a = a^{\#}a$ (resp. $e = \underline{def}(a) = \underline{n}(a^{\#}a)$).

Moreover, for x, y ∈ $Rst_{\mathbf{A}}(A)$ and Fy ≤ Fx in $Rst_{\mathbf{B}}(FA)$ there exist $x_0, y_0 \in Rst_{\mathbf{A}}(A)$ such that $y_0 \leq x_0$, $Fx_0 = Fx$, $Fy_0 = Fy$: just take $x_0 = x$, $y_0 = xy$. The analogous property for the relation ⊲ *need not hold*, but yields the following definition.

2.3. Closed RW-functors. The RW-functor F: **A** → **B** will be said to be *⊲-closed*, or *closed*, whenever

(a) for all x, y ∈ $Rst_{\mathbf{A}}(A)$ such that Fy ⊲ Fx, there exist $x_0, y_0 \in Rst_{\mathbf{A}}(A)$ such that $y_0 \lhd x_0$, $Fx_0 = Fx$, $Fy_0 = Fy$.

By 2.2, a faithful RW-functor is ⊲-closed if and only if it reflects the relation ⊲, when acting on parallel restrictions. Closed RW-functors are stable for composition. Conversely, if the composite $F = F_2F_1$ of two RW-functors is closed and the second (F_2) is faithful (more generally, Rst-faithful (3.3)) then the first functor (F_1) is closed too.

2.4. Lemma (Order reflection). Let F: **A** → **B** be an RW-functor and a, b ∈ **A**(A', A"). Then Fa ≤ Fb in **B** if and only if there exist c, c', c" ∈ **A**(A', A") such that

(1) $a \sim_F c \leq c' \sim_F c'' \leq b$,

where $a \sim_F b$ means that a and b are parallel in **A** and Fa = Fb.

Proof. The condition (1) is clearly sufficient. Conversely if Fa ≤ Fb, consider the restrictions x, z, x' of Lemma 1.10 and take

(2) $c = az$, $\qquad c' = a.b^{\#}b.a^{\#}a$, $\qquad c'' = x'bx$,

so that $c = az = azx \leq c'$ and $c'' \leq b$, while Fa = Fc and Fc' = Fc" by 1.10 applied to Fa and Fb in **B**. □

2.5. RW-subcategories. An *RW-subcategory* **A'** of **A** is an involutive subcategory such that

(a) for each object A of **A'** and each $e \in Prj_{\mathbf{A'}}(A)$ the projections ω_A, Ω_A and $\underline{n}e$ belong to **A'**.

Then **A'** will be equipped with the induced RW-structure, i.e. the only one making the inclusion F: **A'** → **A** an RW-functor.

We say that **A'** is *⊲-closed*, or closed, in **A** if the inclusion F is so, which is equivalent to each of the following conditions

(b) for all restrictions x, y of **A'**, if x ⊲ y in **A** then this holds in **A'**,

(b') every projection of **A**, whose numerator and denominator are in **A'**, belongs to **A'**.

Every full subcategory of **A** is a closed RW-subcategory. More generally, every Rst-full (resp. Prj-full (3.3)) involutive subcategory of **A** is an RW-subcategory (resp. a closed one). Last, any intersection of (closed) RW-subcategories is so. A closed RW-subcategory of an RE-category (i.e., a normal RW-subcategory (1.8)) is an RE-category.

2.6. Generation by subgraphs. Let Δ be a subgraph of **A**; the last remark above proves the existence of the *RW-subcategory* **A**' (resp. the *closed RW-subcategory* **A**") *of* **A** *spanned by* Δ. **A**" will also be called the *RW- closure* of Δ in **A**, and written as $\overline{\Delta} = \mathrm{cl}_{\mathbf{A}}(\Delta)$.

A' can be constructed as

(1) $\mathrm{Ob}\mathbf{A}' = \mathrm{Ob}\Delta$, $\qquad$ $\mathrm{Mor}\mathbf{A}' = \bigcup \Delta_n$,

where the sets $\Delta_n \subset \mathrm{Mor}\mathbf{A}$ $(n \geq 0)$ are inductively defined by:

(a) $\Delta_0 = (\mathrm{Mor}\Delta) \cup \{1_A, \omega_A, \Omega_A \mid A \in \mathrm{Ob}\Delta\}$,

(b_1) if $a \in \Delta_n$, then $a^\# \in \Delta_{n+1}$,

(b_2) if $a, b \in \Delta_n$ are composable in **A**, then $ba \in \Delta_{n+1}$,

(b_3) if $e \in \Delta_n$ is a projection of **A**, then $\underline{n}e \in \Delta_{n+1}$.

The construction of $\mathbf{A}'' = \overline{\Delta}$ needs one inductive rule more

(b_4) if $x, y \in \Delta_n$ and $y \triangleleft x$ in **A**, then $x/y \in \Delta_{n+1}$.

As a consequence

(2) $\mathrm{card}(\mathrm{Mor}\mathbf{A}') \leq \mathrm{card}(\mathrm{Mor}\mathbf{A}'') \leq \max(\mathrm{card}(\mathrm{Ob}\Delta), \mathrm{card}(\mathrm{Mor}\Delta), \aleph_0)$.

2.7. Completeness of RO. We recall (from I.9) that the 2-category **RO** is *strictly 2-complete*. 2-products and 2-equalisers are constructed as in **CAT**, and provided with the obvious involution and order. The comma square $\mathbf{Z} = (F{\downarrow}G)$ of two RO-functors has the following construction

(1) $$\begin{array}{ccc} \mathbf{Z} & \xrightarrow{D'} & \mathbf{A} \\ {\scriptstyle D''}\downarrow & \swarrow \delta & \downarrow {\scriptstyle F} \\ \mathbf{B} & \xrightarrow[G]{} & \mathbf{C} \end{array}$$

the objects of **Z** are triples

(2) $(A, B; u: FA \to GB)$ $\qquad$ $(u \in \mathrm{Prp}\mathbf{C})$,

where A and B are in **A** and **B**; the morphisms are pairs

(3) $(a, b): (A, B; u) \to (A', B'; u')$,

$a \in \mathbf{A}(A, A')$, $\quad b \in \mathbf{B}(B, B')$, $\quad u'.Fa \leq Gb.u$;

the composition, involution and order are obvious, as well as the RO-functors D', D" and the (lax!) RO-transformation

(4) $\delta: FD' \to GD'': \mathbf{Z} \to \mathbf{C}$, $\qquad$ $\delta(A, B; u) = (u: FA \to GB)$.

Clearly the 2-functor $\mathrm{Prp}: \mathbf{RO} \to \mathbf{CAT}$ preserves 2-limits.

2.8. Projections in comma squares. Consider again a comma category $\mathbf{Z} = (F{\downarrow}G)$ in **RO**. In order to study the RW-case, we need to characterise the projections and the restrictions of the object $(A, B; u: FA \to GB)$ in $\mathbf{Z}$. The *projections* are clearly the pairs

(1) $(e, f) \in \mathrm{Prj}_{\mathbf{A}}(A) \times \mathrm{Prj}_{\mathbf{B}}(B)$

which satisfy

(a) $u.Fe \leq Gf.u$,

or also the following more explicit conditions, both equivalent to (a)

(b) $u_P(Fe) \leq Gf$,

(c) $Fe \leq u^P(Gf)$.

Indeed, recalling that $u \in \mathrm{Prp}\mathbf{C}$, if (a) holds

(2) $u_P(Fe) = u.Fe.u^{\#} \leq Gf.\, uu^{\#} \leq Gf$,

while from (b) it follows that

(3) $Fe \leq u^{\#}u.Fe.u^{\#}u = u^P u_P(Fe) \leq u^P(Gf)$,

and from (c)

(4) $u.Fe \leq u.u^P(Gf) = uu^{\#}.Gf.u \leq Gf.u$.

Thus the *restrictions* of $(A, B; u)$ in $\mathbf{Z}$ are exactly those pairs $(x, y) \in \mathrm{Rst}_{\mathbf{A}}(A) \times \mathrm{Rst}_{\mathbf{B}}(B)$ which satisfy the equivalent conditions

(a') $u.Fx \leq Gy.u$,

(b') $u_R(Fx) \leq Gy$,

(c') $Fx \leq u^R(Gy)$,

since $u_R(F\omega) = u_R(\omega) = \underline{\mathrm{ind}}(u) = \omega = G\omega$ and $F\omega = \omega \leq u^R(\omega) = u^R(G\omega)$.

2.9. Completeness of RW. Clearly the 2-embedding $\mathbf{RW} \to \mathbf{RO}$ creates 2-limits and also **RW** *is strictly 2-complete*. We shall need the two following results concerning 2-limits and closure.

First, the equaliser of two parallel RW-functors is easily seen to be a *closed* embedding.

Second, if $\mathbf{Z} = (F{\downarrow}G)$ is the comma square of the converging RW-functors F and G (2.7), the faithful RW-functor

(1) $J: \mathbf{Z} \to \mathbf{A}{\times}\mathbf{B}, \qquad (A, B; u) \mapsto (A, B), \quad (a, b) \mapsto (a, b)$,

is *closed* because of the previous characterisation of projections and restrictions in $\mathbf{Z}$ (2.8): if $(x_0, y_0) \leq (x, y)$ in $\mathrm{Rst}_{\mathbf{Z}}(A, B; u)$ and $(x_0, y_0) \triangleleft (x, y)$ in $\mathrm{Rst}(A, B)$, we have

(2) $u_R(Fx_0) \leq Gy_0, \qquad u_R(Fx) \leq Gy, \qquad x_0 \triangleleft x$ in $\mathbf{A}$, $\quad y_0 \triangleleft y$ in $\mathbf{B}$;

therefore, by letting $e = x/x_0 \in \mathrm{Prj}_{\mathbf{A}}(A)$ and $f = y/y_0 \in \mathrm{Prj}_{\mathbf{B}}(B)$, it results

(3) $u_P(Fe) = (u_R(Fx))/(u_R(Fx_0)) \leq Gy/Gy_0 = Gf$,

hence $(e, f) \in \mathrm{Prj}_{\mathbf{Z}}(A, B; u)$ and $(x_0, y_0) \triangleleft (x, y)$ in $\mathbf{Z}$.

3. Factorisations of RW-functors

The factorisation of RE-functors via RE-quotients and faithful RE-functors (I.5.10) has an obvious extension to RW-functors (3.2) and a less obvious extension via "weak quotients" and closed faithful RW-functors (3.6), which will result to be more useful. We also extend here the factorisation of graph-morphisms considered in [G11], Section 1. **A** is always an RW-category.

3.1. Strict quotients and faithful functors. A (strict) *RW-quotient* $F: \mathbf{A} \to \mathbf{B}$ will be an RW-functor which is bijective on the objects and full; by the order-reflection lemma (2.4), the RW-structure of **B** (i.e. composition, involution and order) is determined by the one of **A** and by the *mapping* F.

A *faithful* RW-functor $F: \mathbf{A} \to \mathbf{B}$, again by 2.4, reflects the *order* between parallel maps; it also reflects *proper* and *null* morphisms; its restriction to endomorphisms reflects *restrictions* and *corestrictions*. The RW-structure of **A** is determined by the one of **B** and by the mapping F together with the "domain" and "codomain" mappings of **A**.

3.2. The ordinary factorisation. The factorisation of RE-functors (I.5.10) extends trivially to the *ordinary RW-factorisation* of an RW-functor F

(1) $\qquad \mathbf{A} \xrightarrow{F_1} \mathbf{C} \xrightarrow{F_2} \mathbf{B}, \qquad\qquad F = F_2F_1,$

where F_1 is an RW-quotient and F_2 a faithful RW-functor. Such a factorisation is essentially unique. If **A** is projection complete, so is **C**.

3.3. Local properties of RW-functors. Also here (see I.5.11) an RW-functor $F: \mathbf{A} \to \mathbf{B}$ will be said to be *Prj-faithful* (resp. *Prj-full*) whenever the mappings

(1) $\quad \mathrm{Prj}_{\mathbf{A}}(A) \to \mathrm{Prj}_{\mathbf{B}}(FA), \qquad\qquad e \mapsto F(e),$

are injective (resp. surjective) for all objects A. Analogously we define the *Rst-faithful* and *Rst-full* RW-functors, by the same conditions on the mappings

(2) $\quad \mathrm{Rst}_{\mathbf{A}}(A) \to \mathrm{Rst}_{\mathbf{B}}(FA), \qquad\qquad x \mapsto F(x).$

It is easily seen that F *is Prj-faithful if and only if it is Rst-faithful* (if and only if, in the ordinary factorisation $F = F_2F_1$, the first functor F_1 is so). On the other hand, F is Prj-full if and only if it is both Rst-full and $\lhd$-closed (if and only if F_2 is so).

A Rst-faithful functor reflects the order $\prec$ of projections, hence also their order $\leq$ (RW.3b). It reflects also monos, epis, proper morphisms, null morphisms; when acting on endomorphisms, it reflects projections and restrictions.

3.4. Dense subgraphs. We say that the subgraph Δ of **A** is $\lhd$-*dense* (or *dense*) in **A** whenever the closed RW-subcategory spanned by Δ, i.e. $\overline{\Delta} = \mathrm{cl}_{\mathbf{A}}\Delta$ (2.6), coincides with **A**.

For every RW-functor $F: \mathbf{A} \to \mathbf{B}$ and every subgraph Δ of **A**, we have a sort of continuity property

(1) $F(\bar{\Delta}) \subset (F\Delta)^{-}$,

because, following the inductive construction of $\bar{\Delta}$ (2.6) and $(F\Delta)^{-}$, it is easy to prove that $F(\Delta_n) \subset (F\Delta)_n$, for every $n \geq 0$. Analogously one proves that, if **A** is an RW-subcategory of **B**

(2) $cl_{\mathbf{A}}(\Delta) = cl_{\mathbf{B}}(\Delta) \cap \mathbf{A}$.

3.5. Weak quotients. The RW-functor $F: \mathbf{A} \to \mathbf{B}$ will be said to be a *weak quotient* if

(a) F is injective on the objects,

(b) the graph $F(\mathbf{A})$ is dense in **B**.

In such a case $F(\mathbf{A})$ is an RW-subcategory of **B** and F is actually bijective on the objects. By 3.4.1, weak quotients are stable for composition: if $G: \mathbf{B} \to \mathbf{C}$ is also so, then

(1) $(GF\mathbf{A})^{-} \supset G((F\mathbf{A})^{-}) = G(\mathbf{B})$, $\qquad (GF\mathbf{A})^{-} \supset (G(\mathbf{B}))^{-} = \mathbf{C}$.

Conversely if GF is a weak quotient and G is injective on the objects then G itself is a weak quotient as $(G(\mathbf{B}))^{-} \supset (GF\mathbf{A})^{-} = \mathbf{C}$.

3.6. Theorem (The closed factorisation). Every RW-functor $F: \mathbf{A} \to \mathbf{B}$ has an essentially unique *closed RW-factorisation*, or *closed factorisation*

(1) $\mathbf{A} \xrightarrow{F_1} \mathbf{C} \xrightarrow{F_2} \mathbf{B}$, $\qquad F = F_2F_1$,

where F_1 is a *weak quotient* and F_2 a *closed faithful* RW-functor.

Proof. To establish the existence, consider first the following decomposition of F in RW-functors

(2) $\mathbf{A} \xrightarrow{G} \mathbf{B}' \xrightarrow{H} \mathbf{B}$, $\qquad F = HG$,

$Ob\mathbf{B}' = Ob\mathbf{A}$, $\qquad \mathbf{B}'(A_1, A_2) = \mathbf{B}(FA_1, FA_2)$,

$G(A) = A$, $\qquad G(a) = F(a)$,

$H(A) = F(A)$, $\qquad H(b: A_1 \to A_2) = (b: FA_1 \to FA_2)$.

Here G is injective on the objects, hence $G(\mathbf{A})$ is an RW-subcategory of $\mathbf{B}'$. Now let

(3) $\mathbf{C} = cl_{\mathbf{B}'}(G\mathbf{A})$,

be the closed RW-subcategory of $\mathbf{B}'$ spanned by $G(\mathbf{A})$, and define our functors F_1 and F_2 (3.6.1) as restrictions of G and H respectively.

Trivially, F_1 is a weak quotient and F_2 is faithful. Moreover F_2 is closed: let $y' \leq x'$ in $Rst_{\mathbf{C}}(A)$ with $y' = F_2(y') \triangleleft x' = F_2(x')$ in $Rst_{\mathbf{B}}(FA)$; then $e' = x'/y' \in Prj_{\mathbf{B}}(FA)$ and therefore $e' \in Prj_{\mathbf{B}'}(A)$; since $\underline{n}e = x'$ and $\underline{d}e = y'$ belong to **C** which is closed in $\mathbf{B}'$, it follows that $e' \in Prj_{\mathbf{C}}(A)$.

Now, for uniqueness, assume that $F = F_2F_1$ is any closed factorisation of F and define $K: \mathbf{C} \to \mathbf{B}'$ as follows

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{F_1} & \mathbf{C} & \xrightarrow{F_2} & \mathbf{B} \\ \| & & \downarrow K & & \| \\ \mathbf{A} & \xrightarrow[G]{} & \mathbf{B}' & \xrightarrow[H]{} & \mathbf{B} \end{array} \tag{4}$$

$K(F_1A) = G(A) = A, \qquad\qquad K(c: F_1AF_1A') = F_2(c): A \to A'.$

Thus K is an RW-functor and (4) commutes. Moreover K is bijective on the objects (F_1 and G are so) and faithful (F_2 is so): this proves that $\mathbf{C}$ is isomorphic to $K(\mathbf{C})$, and it suffices to prove that $K(\mathbf{C}) = (G\mathbf{A})^{-}$. In fact, $K(\mathbf{C})$ is closed in $\mathbf{B}'$ (since F_2 is closed and H is faithful) and $G(\mathbf{A})$ is dense in $K(\mathbf{C})$, because

(5) $(G\mathbf{A})^{-} \supset (KF_1(\mathbf{A}))^{-} \supset K(F_1(\mathbf{A})^{-}) = K(\mathbf{C}).$ □

3.7. Remarks. (a) If $F = F_2 F_1$ is a closed RW-factorisation, then F is closed iff F_1 is so, iff F_1 is a quotient (by 3.4.2).

(b) Weak quotients are epi in the category **RW**. Actually, if $F: \mathbf{A} \to \mathbf{B}$ is a weak quotient and $G_1F = G_2 F$ in **RW** (with $G_i: \mathbf{B} \to \mathbf{C}$), write $H: \mathbf{B}_0 \to \mathbf{B}$ the equaliser of G_1 and G_2 (a closed embedding, by 2.9) and factor $F = HG$. Thus $F\mathbf{A} \subset \mathbf{B}_0$ and $\mathbf{B}_0 = cl_{\mathbf{B}}(\mathbf{B}_0) \supset cl_{\mathbf{B}}(F\mathbf{A}) = \mathbf{B}$, i.e. $G_1 = G_2$.

3.8. Factorisation of graph-morphisms. The closed factorisation can be easily generalised to a graph morphism $F: \Delta \to \mathbf{B}$, defined on a graph, with values in an RW-category: F factors uniquely as

(1) $\Delta \xrightarrow{F_1} RW(F) \xrightarrow{F_2} \mathbf{B}, \qquad\qquad F = F_2F_1,$

where F_1 is a *q-morphism* (a graph morphism which satisfies the conditions 3.5a, 3.5b) and F_2 is a *closed faithful* RW-functor (same proof as above, for 3.6). Finally, say that $F: \Delta \to \mathbf{B}$ is *Rst-spanning* if this functor F_2 is Rst-full (or, equivalently, Prj-full; cf. 3.3).

4. A review of W-categories and W-relations

Because of the lack of general pullbacks in exact categories, and a fortiori in their generalisations, relations over these categories cannot be constructed as equivalence classes of span diagrams ("V-relations"), but as equivalence classes of *four-arrow* diagrams (W-relations, see 4.3) as first established by Calenko [C1-2] for exact categories, and extended in various ways by Brinkmann-Puppe [BP], Burgin [Bu], Calenko-Gisin-Raikov [CGR] and others. We recall here (from [G1]), briefly and without proofs, a construction of W-relations based on minimal assumptions, together with its basic properties.

4.1. Factorisation systems. A category with *factorisation system* (see, for instance, [AHS, CJKP]) is a category **E** equipped with subcategories **P** and **M** which contain all isomorphisms, so that each morphism u in **E** has a *canonical factorisation*

(1)   u = mp   (p ∈ **P**; m ∈ **M**),

essentially unique in the usual sense. Then **P**∩**M** is the subcategory of the isomorphisms of **E** and

(2)   vu ∈ **P** ⇒ v ∈ **P**,   vu ∈ **M** ⇒ u ∈ **M**.

Here we are only interested in *proper* factorisation systems, where **P** and **M** are contained in the subcategories of epis and monos, respectively. In the sequel, the arrows $\twoheadrightarrow$, $\rightarrowtail$ will always stand for morphisms of **P** and **M**, with regard to some (specified) factorisation system. The terms *subobject* and *quotient* will always refer to **M**-subobjects and **P**-quotients; similarly for *well powered* and *well copowered*.

Of course, a category with unique epi-mono factorisations will always be provided with its unique proper factorisation system.

4.2. W-categories. A *W-category* **E** = (**E**, **P**, **M**) is a category **E** equipped with subcategories **P** and **M**, so that:

(W.1)   (**E**, **P**, **M**) is a category with proper factorisation system, well powered and copowered,

(W.2)   every pair of arrows of **M** with the same codomain has a pullback in **M** (which is still a pullback in **E**, by (W.1)),

(W.2*) every pair of arrows of **P** with the same domain has a pushout in **P** (which is still so in **E**),

(W.3)   every diagram $A \twoheadrightarrow \bullet \leftarrowtail B$ has a "mixed pullback" $A \leftarrowtail \bullet \twoheadrightarrow B$ in **E**,

(W.4)   (*modular cubic axiom*) every commutative diagram, as at the left-hand in (1)

(1)
$$\begin{array}{ccccc} A & \rightarrowtail & \bullet & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ C & \rightarrowtail & \bullet & \twoheadrightarrow & D \end{array} \qquad\qquad \begin{array}{ccccc} A & \twoheadrightarrow & \bullet & \rightarrowtail & B \\ \downarrow & & \downarrow & & \downarrow \\ C & \twoheadrightarrow & \bullet & \rightarrowtail & D \end{array}$$

formed of a mixed pullback and a pushout of **P**-epis, yields – by canonical factorisation of its rows – the commutative diagram at the right hand, formed of a *pushout* of **P**-epis and a mixed *pullback*.

These categories were introduced (as *quaternary* categories) in [G1], extending the axioms of Brinkmann-Puppe [BP]; actually the original formulation is written for categories with unique epi-mono factorisation, but the generalisation to a proper factorisation system is obvious and useful (e.g., to include **Rng**). Note that a zero object need not exist: $\mathbf{Set}^{\mathrm{op}}$ is a W-category.

4.3. W-relations. We recall now, briefly and without proofs, the construction of the *RO-category of relations* of **E** [G1], which we write here as $\mathrm{Rel}_W(\mathbf{E})$.

The objects are the ones of **E**; a *W-relation* a = [m, p, q, n]: A → B is a class of equivalence of *W-diagrams* of **E**

(1)
$$A \overset{m}{\leftarrowtail} \bullet \overset{p}{\twoheadrightarrow} \bullet \overset{q}{\twoheadleftarrow} \bullet \overset{n}{\rightarrowtail} B$$

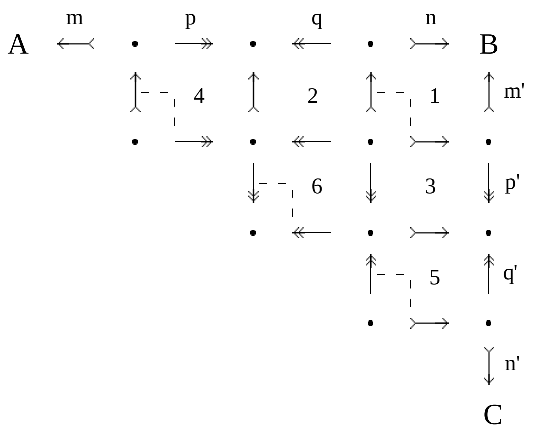
two such diagrams being identified when there is a commutative diagram of **E**

$$(2) \qquad \begin{array}{ccccccccc} A & \xleftarrow{m} & \bullet & \xrightarrow{p} & \bullet & \xleftarrow{q} & \bullet & \xrightarrow{n} & B \\ \| & & \downarrow u & & \downarrow w & & \downarrow v & & \| \\ A & \xleftarrow[m']{} & \bullet & \xrightarrow[p']{} & \bullet & \xleftarrow[q']{} & \bullet & \xrightarrow[n']{} & B \end{array}$$

where u, v, w are isomorphisms (uniquely determined).

The *composition* of W-relations a = [m, p, q, n]: A → B and b = [m', p', q', n']: B → C is constructed (as for p-exact categories) by means of the following diagram of **E**

(3)



where the square <1> is a pullback of **M**-monos (W.2), <2> and <3> are commutative (W.1), <4> and <5> are mixed pullbacks (W.3), <6> is a pushout of **P**-epis (W.2*).

Of course, the *involution* is obtained by reversing diagram (1)

(4) $[m, p, q, n]^{\#} \ = \ [n, q, p, m]: B \to A,$

while the *order* between parallel W-relations

(5) $[m, p, q, n] \ \leq \ [m', p', q', n'],$

is defined by the existence of *morphisms* u, v, w of **E** making (2) commutative.

$\mathrm{Rel}_W(\mathbf{E})$ is thus a RO-category. The crucial part of the proof is the associativity of the composition; it uses heavily the "modular cubic axiom" (W.4).

4.4. W-factorisations. The proper morphisms of $\mathbf{A} = \mathrm{Rel}_W(\mathbf{E})$ are the W-relations of type u = [1, p, 1, n]

$$(1) \qquad A \; = \; \bullet \; \xrightarrow{p} \; \bullet \; = \; \bullet \; \xrightarrow{n} \; B$$

We shall identify **E** and Prp(**A**), identifying the morphism u = mp (canonical factorisation in **E**) with the W-relation (1); this is coherent with the composition.

Thus the W-relation a = [m, p, q, n]: A → B has a *W-factorisation*

(2) $a \ = \ nq^{\#}pm^{\#}$ $\qquad (m, n \in \mathbf{M}, \quad p, q \in \mathbf{P}),$

essentially unique, up to three isomorphisms uniquely determined (4.3.2).

We also recall ([G1], 3.7-10) that

(3) a is a *monorelation* (i.e., mono in **A**) iff m and p are iso in **E**,

(4) a is *proper* iff m and q are iso in **E**,

(5) a is an *isomorphism* of **A** iff it is mono and epi in **A**, iff a and $a^\#$ are both proper, iff a is iso in **E**, iff m, p, q, n are iso in **E**,

(6) $a \in \mathbf{M}$ iff m, p, q are iso, iff a is a proper mono of **A**,

(7) $a \in \mathbf{P}$ iff m, q, n are iso, iff a is a proper epi of **A**.

The RO-category $\mathbf{A} = \mathrm{Rel}_W(\mathbf{E})$ is always *projection complete* (0.5); the canonical factorisation of the relation (2) is

(8) $a = (nq^\#).(pm^\#)$.

By I.3.3, for each object A, the mapping

(9) $\mathrm{Sub}_{\mathbf{A}}(A) \to \mathrm{Prj}_{\mathbf{A}}(A)$, $\qquad s \mapsto ss^\#$,

is an isomorphism of ordered sets (with regard to $\prec$). By (W.1) and the characterisation of monorelations in (3) this also proves that **A** is Prj-small. Note that, by (3), an **A**-subobject s: L → A is actually an ***E**-subquotient*

(10) $L \overset{p}{\twoheadleftarrow} \bullet \overset{m}{\rightarrowtail} A \qquad s = mp^\#$,

i.e. a quotient (p) of a subobject (m) of A with respect to **E**.

4.5. Projections. Thus, an endorelation e: A → A is a *projection* (i.e., a symmetric idempotent) if and only if it has a W-factorisation of the following type

(1) $e = mp^\#pm^\# \qquad (m \in \mathbf{M},\ p \in \mathbf{P})$.

By I.3.2, e is a *restriction* ($e \leq 1$) iff p is iso, iff e has a W-factorisation

(2) $e = mm^\# \qquad (m \in \mathbf{M})$,

while e is a *corestriction* ($e \geq 1$) iff m is iso, iff e has a W-factorisation

(3) $e = pp^\# \qquad (p \in \mathbf{P})$.

In other words, the isomorphism 4.4.9 produces the following isomorphisms of semilattices (*with regard to* $\prec$)

(4) $\underline{i}_A$: $\mathrm{Sub}_{\mathbf{E}}(A) \to \mathrm{Rst}_{\mathbf{A}}(A)$, $\qquad \underline{i}(m) = mm^\#$,

(5) $\underline{c}_A$: $\mathrm{Quo}_{\mathbf{E}}(A) \to \mathrm{Crs}_{\mathbf{A}}(A)$, $\qquad \underline{c}(p) = p^\#p$.

4.6. Definition and values. Last, every W-relation $a = nq^\#pm^\#: A \to B$ determines the following *subobjects* of A and B ([G1], 3.3)

(1) def(a): Def(a) → A, $\qquad$ def(a) ~ m,

(2) val(a): Val(a) → B, $\qquad$ val(a) ~ n,

and we recall that ([G1], 3.3-4)

(3) $\mathrm{def}(a) = \mathrm{val}(a^\#) = \mathrm{def}(a^\# a) = \mathrm{val}(a^\# a)$,

(4) $a \leq a' \;.\!\Rightarrow\!.\; \mathrm{def}(a) \prec \mathrm{def}(a')$ and $\mathrm{val}(a) \prec \mathrm{val}(a')$,

(5) $a = cb \;.\!\Rightarrow\!.\; \mathrm{def}(a) \prec \mathrm{def}(b)$ and $\mathrm{val}(a) \prec \mathrm{val}(c)$,

(6) $a^\# a \geq 1 \;\Leftrightarrow\; \mathrm{def}(a) = 1$, $\qquad aa^\# \geq 1 \;\Leftrightarrow\; \mathrm{val}(a) = 1$.

4.7. SW-categories. To characterise the categories of W-relations we introduce the following definition (rephrasing the "quaternary symmetrisations" of [G1]): an *SW-category* is a triple **A** = (**A**, $^\#$, ≤) satisfying:

(SW.0) **A** is a RO-category,

(SW.1) every morphism a of **A** has a *W-factorisation*

(1) $a = nq^\# pm^\#$,

where m, n are proper monos of **A** and p, q are proper epis of **A**,

(SW.2) this factorisation is *essentially unique*: if $nq^\# pm^\# = n'q'^\# p'm'^\#$, there exist isomorphisms u, v, w of **A** making diagram 4.3.2 commutative.

These axioms are clearly equivalent to the following ones:

(a) **A** is a projection complete RO-category,

(b) every monomorphism s of **A** has a *W-factorisation* $s = nq^\#$ where n is a proper mono and q a proper epi of **A**,

(c) such factorisations are essentially unique.

4.8. W-Symmetrisation Theorem, I. Let **A** = (**A**, $^\#$, ≤) be a RO-category and **E** = Prp(**A**); write **P** and **M** the subcategories of proper epis and proper monos of **A**, characterised respectively by

(1) $p^\# p \geq 1$, $\qquad pp^\# = 1$,

(2) $m^\# m = 1$, $\qquad mm^\# \leq 1$.

The following conditions are equivalent

(a) **E** = (**E**, **P**, **M**) is a W-category and **A** is RO-isomorphic to Rel_W(**E**),

(b) **A** is an SW-category,

(c) **A** is projection complete and satisfies (RW.1).

When they hold, for every morphism $a \in$ **A**(A, B)

(3) $\underline{\mathrm{def}}(a) = \underline{i}_A(\mathrm{def}(a))$, $\qquad \underline{\mathrm{val}}(a) = \underline{i}_B(\mathrm{val}(a))$.

Proof. The conditions (a), (b) are equivalent by [G1; 2.9, 2.11].

(b) ⇒ (c). Let $e \in \mathrm{Prj}(A)$ have a W-factorisation (4.5.1)

(4) $e = mp^\# pm^\#$ $\qquad (m \in$ **M**, $p \in$ **P**),

and take $x = mm^{\#} = \underline{i}(def(e))$, so that $x \leq m(p^{\#}p)m^{\#} = e$ and $x.e = e$. Conversely, if $y \in Rst(A)$ and $y \leq e \prec y$ then (4.5.4) $y = nn^{\#}$ with $n \sim def(y)$ and, by 4.6.4-5

(5) $def(y) \prec def(e) = def(ey) \prec def(y)$,

therefore $n \sim def(e)$ and $y = x$. Thus **A** satisfies (RW.1); we have also proved that $\underline{n}e = \underline{i}(def(e))$, from which property (3) follows at once

(6) $\underline{def}(a) = \underline{n}(a^{\#}a) = \underline{i}(def(a^{\#}a)) = \underline{i}(def(a))$.

(c) ⇒ (b). We need only to prove that **A** satisfies 4.7b and 4.7c.

Let $a: A_0 \to A$ be a monorelation and take $e = aa^{\#} \in Prj(A)$, $x = \underline{n}(e) \in Rst(A)$; by 4.5 there exists $m \in \mathbf{M}$ such that $x = mm^{\#}$. Take now $p = a^{\#}m$, so that

(7) $mp^{\#} = mm^{\#}a = (mm^{\#})ea = ea = a$,

which also proves that $p^{\#}$ is a monorelation and p is epi. To prove that p is proper (hence $p \in \mathbf{P}$) it is sufficient to consider the restriction $y = \underline{n}(p^{\#}p)$ and prove that $y = 1$ (1.2.5)

(8) $mym^{\#} \leq mm^{\#} = x \leq e$,

$(mym^{\#})e = my(m^{\#}a)a^{\#} = myp^{\#}(pm^{\#}) = m(y.p^{\#}p)m^{\#} = mp^{\#}pm^{\#} = aa^{\#} = e$,

thus the restriction $mym^{\#}$ is the numerator of e: $mym^{\#} = x = m1m^{\#}$, which proves that $y = 1$.

Last, for the uniqueness of the W-factorisation of a, let us take two of them

(9)
$$\begin{array}{ccccc} L & \overset{p}{\twoheadleftarrow} & \bullet & \overset{m}{\rightarrowtail} & A \\ \| & & \downarrow{\scriptstyle i} & & \| \\ L & \underset{q}{\twoheadleftarrow} & \bullet & \underset{n}{\rightarrowtail} & A \end{array} \qquad\qquad a = mp^{\#} = nq^{\#}.$$

As $mm^{\#} \leq mp^{\#}pm^{\#} = aa^{\#}$ and $mm^{\#}.aa^{\#} = aa^{\#}$, we have

(10) $mm^{\#} = \underline{n}(aa^{\#}) = nn^{\#}$.

Thus the relation $i = n^{\#}m$ is an isomorphism (of **A** and **E**)

(11) $i^{\#}i = m^{\#}nn^{\#}m = m^{\#}(mm^{\#})m = 1$,

and similarly $ii^{\#} = 1$. Finally

(12) $qi = qn^{\#}n = (pm^{\#})m = p$, $\qquad ni = nn^{\#}m = mm^{\#}m = m$.

4.9. The 2-categories W and SW. We also recall ([G1], 4.6) that a functor $F: \mathbf{E} \to \mathbf{E}'$ between W-categories has a (necessarily unique) extension $RelF: Rel\mathbf{E} \to Rel\mathbf{E}'$ in **RO** if and only if F is a *W-functor*; i.e.

(a) F preserves the factorisation system,

(b) F preserves pullbacks of **M**, mixed pullbacks and pushouts of **P**.

Moreover $RelF$ is faithful if and only if F is so ([G1], 4.10). It is also easy to prove, by the same argument as in I.2.7, that every *W-transformation* $\alpha: F \to G: \mathbf{E} \to \mathbf{E}'$ (natural transformation of W-functors) yields a RO-transformation

(1) $\mathrm{Rel}(\alpha)\colon \mathrm{Rel}F \to \mathrm{Rel}G\colon \mathrm{Rel}\mathbf{E} \to \mathrm{Rel}\mathbf{E}'$, $\qquad (\mathrm{Rel}(\alpha))A = \alpha A\colon FA \to GA$,

which has the same components as α, but is just lax-natural with respect to the morphisms of Rel**E**.

Thus, we have established a 2-adjoint 2-equivalence

(2) $\mathbf{W} \xrightarrow{\mathrm{Rel}} \mathbf{SW} \xrightarrow{\mathrm{Prp}} \mathbf{W}$,

$\eta = 1\colon 1_{\mathbf{W}} \to \mathrm{Prp.Rel}$, $\qquad \varepsilon\colon \mathrm{Rel.Prp} \to 1_{\mathbf{SW}}$,

between the 2-category **W** of W-categories, W-functors, W-transformations and the 2-category **SW** of SW-categories, *SW-functors* (i.e., RO-functors between SW-categories) and *SW-transformations* (i.e., RO-transformations between SW-functors).

4.10. W-symmetrisation theorem, II. Let **A** be an SW-category and **E** = Prp**A** the associated W-category, with factorisation system (**P**, **M**). The following conditions are equivalent

(a) **A** is connected, non empty and satisfies (RW.2),

(b) **E** has a *zero-object* 0 (i.e. initial and terminal) *coherent with the factorisation system*, in the sense that all morphisms $A \to 0 \to A$ are respectively in **P** and **M**.

In such a case the zero objects of **E** coincide with the null objects of **A**; the projections ω_A and Ω_A are respectively given by

(1) $A \leftarrowtail 0 \rightarrowtail A$, $\qquad A \twoheadrightarrow 0 \twoheadleftarrow A$.

If $e = mp^{\#}pm^{\#} \in \mathrm{Prj}(A)$ and $h = \ker(p)$ (existing by (W.2)), then

(2) $\underline{d}e = (mh)(mh)^{\#}$.

Further, *one* implication of (RW.3a, b) holds:

(RW.3a') if $e \prec f$ in Prj(A), then $\underline{n}e \prec \underline{n}f$ and $\underline{d}e \succ \underline{d}f$,

(RW.3b') if $e \leq f$ in Prj(A), then $\underline{n}e \prec \underline{n}f$ and $\underline{d}e \prec \underline{d}f$.

Proof. If **E** has a zero object with the required conditions, for each object A the relations ω_A and Ω_A defined in (1) are indeed the least and the greatest morphism in **A**(A, A)

$$(3)\qquad \begin{array}{ccccccccc}
A & \leftarrowtail & 0 & = & 0 & = & 0 & \rightarrowtail & B \\
\| & m & \downarrow & p & \downarrow & q & \downarrow & n & \| \\
A & \leftarrowtail & \bullet & \twoheadrightarrow & \bullet & \twoheadleftarrow & \bullet & \rightarrowtail & B \\
\| & & m\downarrow & & \downarrow & & \downarrow n & & \| \\
A & = & A & \twoheadrightarrow & 0 & \twoheadleftarrow & B & = & B
\end{array}$$

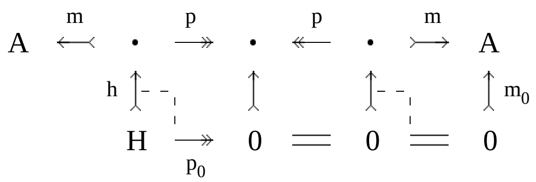
Conversely, if **A** is connected, non empty and satisfies (RW.2), let A be any object: the W-factorisation of the restriction $\omega_A = mm^{\#}$, with $m\colon Z \rightarrowtail A$, yields an object Z which is null in **A**, since its identity is a null morphism

(4) $1_Z = m^{\#}m = m^{\#}.mm^{\#}.m = m^{\#}.\omega_A.m$;

then, by I.4.11-12 (which only depend on (RE.2), i.e. (RW.2)), Z is a zero-object for **E**.

From now on, we assume that (a) and (b) hold. In order to verify (2), let $e = mp^{\#}pm^{\#} \in \mathrm{Prj}(A)$ and $h = \ker(p): H \rightarrowtail A$; then the projection $e\omega e$ can be calculated by the following diagram, as in 4.3.3

(5)



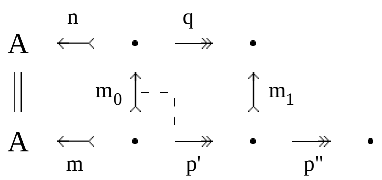
(6) $(mp^{\#}pm^{\#})m_0 = (mh)p_0^{\#}$,

$e\omega_A e = (mp^{\#}pm^{\#})(m_0m_0^{\#})(mp^{\#}pm^{\#}) = (mh).p_0^{\#}p_0.(mh)^{\#}$,

$\underline{d}e = \underline{n}(e\omega e) = (mh)(mh)^{\#}$.

Consider now a second projection $f = nq^{\#}qn^{\#} \in \mathrm{Prj}(A)$. For (RW.3b'), let $e \leq f$; then $\mathrm{def}(e) \prec \mathrm{def}(f)$ (4.6.4) and $\underline{n}e \leq \underline{n}f$; moreover $e\omega e \leq f\omega f$ and $\underline{d}e \leq \underline{d}f$. For (RW.3a'), let $e \prec f$. By [G1], 3.12, this condition is equivalent to the existence of a commutative diagram in **E**

(7)



$p = p''p'$,

with cartesian right square. By the commutativity of the left square, $\underline{n}e \leq \underline{n}f$. By the universal property of the pullback, it follows that

(8) $m_0.\ker(p') \sim \ker(q)$,

$n.\ker(q) \sim nm_0.\ker(p') = m.\ker(p') \prec m.\ker(p)$,

$\underline{d}(f) = \underline{i}(n.\ker(q)) \leq \underline{i}(m.\ker(p)) = \underline{d}e$. □

5. RW-categories and w-exact categories

We show here that RW-categories coincide with the Prj-full involutive subcategories of the categories of relations over Burgin's γ-*categories* [Bu]; the latter we also call *w-exact* categories.

5.1. γ-categories. A γ-*category* [Bu] **E** is assumed to satisfy the following axioms

(A.0) **E** has a zero object 0; every morphisms u has a *canonical factorisation* $u = mp$ where p is a *conormal epi* (i.e. a cokernel of some map) and m is *mono*.

Such a factorisation is necessarily unique up to isomorphism; **E** will always be provided with this canonical factorisation system (**P**, **M**): therefore (4.1) the terms "quotient" and "subobject", as well as the arrows "↠" and "↣", will always be used for conormal epis and monos, respectively. (This axiom (A.0) is equivalent to the conjunction of the original axioms (A.2, 3, 5) of Burgin. We recall that a *zero object* is, by definition, both initial and terminal; then for all objects A and B, the zero morphism $0_{AB} = (A \to 0 \to B)$ is determined.)

(A.1) **E** is well powered;

(A.4a) **E** has *counterimages* of monos (pullbacks of diagrams $A \to \bullet \leftarrowtail B$);

(A.4b) the pullback of two arrows $A \twoheadrightarrow \bullet \leftarrowtail B$ is of the form $A \leftarrowtail \bullet \twoheadrightarrow B$ (preserving conormal epis);

(A.6) the image of a *normal* mono by a conormal epi is a normal mono;

(A.7) if the diagram

$$(1) \qquad \begin{array}{ccccc} \bullet & \overset{h}{\rightarrowtail} & \bullet & \overset{p}{\twoheadrightarrow} & \bullet \\ \| & & m \uparrow & & \| \\ \bullet & \underset{k}{\rightarrowtail} & \bullet & \underset{q}{\twoheadrightarrow} & \bullet \end{array}$$

is commutative with *exact rows* (i.e. $h \sim \ker(p)$ and $k \sim \ker(q)$ (4.2)) then m is iso.

Here a γ-category will also be called a *w-exact* category, for the sake of uniformity of the present terminology.

5.2. Kernels, cokernels, exact sequences. Let **E** be w-exact (or, more generally, any category satisfying (A.0, 4)). It is easy to see that the zero-object is coherent with the factorisation system, in the sense that all zero morphisms $A \to 0 \to A$ are respectively conormal epis (the cokernel of 1_A) and (normal) monos. **E** has kernels, by (A.4a).

For each object A, by well-known arguments, there is an anti-isomorphism between the ordered sets of quotients and of normal subobjects

(1) ker: Quo(A) $\to$ NrmSub(A), cok: NrmSub(A) $\to$ Quo(A),

proving that (if (A.1) holds) **E** is also well copowered.

Actually, this mapping "ker" clearly reverses the order. Now, let $h = \ker(u)$ ($u: A \to B$) be a normal subobject of A and $u = mp$ a canonical factorisation: trivially, $h = \ker(p)$; we want to prove that p is a cokernel of h. Indeed $p \sim \mathrm{cok}(v)$, for some morphism v

$$(2) \qquad \begin{array}{ccccc} \bullet & \overset{h}{\rightarrowtail} & A & \overset{p}{\twoheadrightarrow} & \bullet \\ \uparrow & & \| & & \downarrow \\ \bullet & \underset{v}{\longrightarrow} & A & \underset{w}{\longrightarrow} & \bullet \end{array}$$

hence $pv = 0$ and v factors through $h = \ker(p)$; therefore, for any w, $wh = 0$ implies $wv = 0$ and w factors through p. This proves that the mapping "cok" in (1) is well defined, and $\mathrm{cok}(\ker(p)) = p$, for all quotients p. Since "ker" is easily seen to be surjective in (1), the conclusion follows.

As usual, the sequence

$$(3) \qquad A \overset{u}{\longrightarrow} B \overset{v}{\longrightarrow} C,$$

is said to be exact in B if im(u) = ker(v) (in which case the morphism u is *normal*, i.e. its image is so); the sequence is exact if it is so in each term. In particular the *short sequence*

(4) $$\bullet \overset{h}{\rightarrowtail} \bullet \overset{p}{\twoheadrightarrow} \bullet \qquad\qquad (h \in \mathbf{M};\ p \in \mathbf{P}),$$

is exact if and only if $h \sim \ker(p)$, if and only if h is normal and $p \sim \mathrm{cok}(h)$.

5.3. Proposition (Pushouts of conormal epis [Bu]). In the w-exact category **E**, given the commutative diagram with exact rows

(1) $$\begin{array}{ccccc} \bullet & \overset{h'}{\rightarrowtail} & \bullet & \overset{p'}{\twoheadrightarrow} & \bullet \\ {\scriptstyle q_0}\uparrow & & \uparrow{\scriptstyle q} & & \uparrow{\scriptstyle q'} \\ \bullet & \underset{h}{\rightarrowtail} & \bullet & \underset{p}{\twoheadrightarrow} & \bullet \end{array}$$

(PE) the right square is a pushout if and only if there is a conormal epi q_0 which fills-in commutatively.

More precisely, for every category **E** satisfying (A.0, 4), the axiom (A.6) is equivalent to the existence of pushouts of conormal epis together with (PE).

Proof. First assume that **E** satisfies (A.0, 4, 6).

If the right square of (1) is a pushout, consider the canonical factorisation

(2) $qh = h_1q_1$.

By the universal property of pushouts it is easy to see that $\mathrm{cok}(h_1) \sim p'$; since h_1 is normal by (A.6), it follows that $h_1 \sim \ker(p') \sim h'$. Thus q_1 (more precisely, an equivalent epi) satisfies our condition.

Conversely, given a conormal epi q_0 making (1) commutative, let us prove that the right square is cocartesian. Consider a commutative square $p''q = q''p$, where we may assume that p'' and q'' are conormal epis, because of (A.0). Now $p''h' = 0$ as

(3) $p''h'.q_0 = p''qh = q''ph = 0$,

therefore p'' factors through $\mathrm{cok}(h') \sim p'$ and the conclusion follows. The existence of pushouts of **P** is a trivial consequence.

Last, if **E** satisfies (A.0) and these pushouts do exist, (PE) implies clearly (A.6). □

5.4. Proposition (Mixed pullbacks, [Bu]). In the w-exact category **E**, given the commutative diagram with exact rows

(1) $$\begin{array}{ccccc} \bullet & \overset{h}{\rightarrowtail} & \bullet & \overset{p}{\twoheadrightarrow} & \bullet \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle n} \\ \bullet & \underset{k}{\rightarrowtail} & \bullet & \underset{q}{\twoheadrightarrow} & \bullet \end{array}$$

(MP) the right square is a pullback if and only if there is an isomorphism i which fills-in commutatively.

More precisely, for every category $\mathbf{E}$ satisfying (A.0, 4), the axiom (A.7) is equivalent to (MP).

Proof. First assume that $\mathbf{E}$ satisfies (A.0, 4, 7).

If the right square of (1) is a pullback, it is easy to see that mk is a kernel of p, hence $mk \sim h$. Conversely, given the isomorphism i, form the commutative diagram

$$(2) \qquad \begin{array}{ccccc} \bullet & \overset{h}{\rightarrowtail} & \bullet & \overset{p''}{\twoheadrightarrow} & \bullet \\ {\scriptstyle n''}\uparrow & & \uparrow{\scriptstyle n'} & & \uparrow{\scriptstyle n} \\ \bullet & \overset{h'}{\rightarrowtail} & \bullet & \overset{p'}{\twoheadrightarrow} & \bullet \\ {\scriptstyle m''}\uparrow & & \uparrow{\scriptstyle m'} & & \parallel \\ \bullet & \underset{k}{\rightarrowtail} & \bullet & \underset{q}{\twoheadrightarrow} & \bullet \end{array}$$

where the upper squares are pullbacks (A.4), $n'm' = m$ (universal property of the right pullback) and $n''m'' = i$ (universal property of the upper rectangle). Since i is iso, n'' is a conormal epi; thus n'' is iso and m'' too. By (A.7), applied to the lower rectangle, m' is iso: thus the right square of (1) is a pullback.

Last, if $\mathbf{E}$ satisfies (A.0), (MP) implies trivially (A.7). □

5.5. W-Symmetrisation Theorem, III. Let $\mathbf{A}$ be a RO-category and $\mathbf{E} = \mathrm{Prp}\mathbf{A}$. The following conditions are equivalent:

(a) $\mathbf{A}$ is a projection complete, connected, non empty RW-category;

(b) $\mathbf{A}$ is a projection complete, connected, non empty R0-category satisfying (RW.1, 2) together with:

(RW.3a") for each $e, f \in \mathrm{Prj}_{\mathbf{A}}(A)$, if $\underline{n}e \prec \underline{n}f$ and $\underline{d}e \succ \underline{d}f$ then $e \prec f$;

(c) $\mathbf{A}$ is a connected, non empty SW-category (4.7) satisfying (RW2, 3a");

(d) $\mathbf{E}$ is w-exact (i.e., a γ-category) and $\mathbf{A}$ is RO-isomorphic to the Burgin's category of relations over $\mathbf{E}$;

(e) $\mathbf{E}$ satisfies (A.0), is a W-category with regard to this factorisation structure and satisfies (A.7); moreover $\mathbf{A}$ is RO-isomorphic to $\mathrm{Rel}_W(\mathbf{E})$.

When these conditions hold, there are two commutative squares of (vertical) isomorphisms and (horizontal) anti-isomorphisms of ordered sets, with respect to the relations $\prec$

$$(1) \qquad \begin{array}{ccc} \mathrm{Quo}_{\mathbf{E}}(A) & \underset{\mathrm{cok}}{\overset{\mathrm{ker}}{\rightleftarrows}} & \mathrm{NrmSub}_{\mathbf{E}}(A) \\ {\scriptstyle \underline{c}}\downarrow & & \downarrow{\scriptstyle \underline{i}} \\ \mathrm{Crs}_{\mathbf{A}}(A) & \underset{\underline{d}'}{\overset{\underline{d}}{\rightleftarrows}} & \mathrm{NrmRst}_{\mathbf{A}}(A) \end{array}$$

where $\underline{d}$ is the denominator-mapping and

(2) $\underline{d}'(x) \; = \; 1/x, \quad \text{for} \quad x \lhd 1_A.$

Proof. (a) ⇒ (b). Obvious.

(b) ⇒ (c). Follows from the W-symmetrisation theorem I (4.8).

(c) ⇒ (e). From the same theorem we know that **E** is a W-category with respect to the factorisation system (**P**, **M**) of proper epis and proper monos of **A**; moreover **A** and $\mathrm{Rel}_W(\mathbf{E})$ are RO-isomorphic.

Thus we only need to prove that **E** satisfies (A.0) and (A.7). Note that, by the W-symmetrisation theorem II (4.10), **A** satisfies (RW2, 3a).

For (A.0), we know (4.10) that **E** has a zero object, which determines the projections ω and Ω as in the formulas 4.10.1. Consider now the above factorisation system (**P**, **M**) of **E**, and let us first prove that all $p \in \mathbf{P}$ are conormal epis.

Take some $p \in \mathbf{P}$ and $h = \ker(p)$ (W.2): we want to prove $p \sim \mathrm{cok}(h)$. Trivially $hp = 0$; assume that also $hq = 0$ (it is sufficient to consider $q \in \mathbf{P}$) and build the left diagram below

(3)
$$\begin{array}{ccccc} \bullet & \overset{k}{\rightarrowtail} & \bullet & \overset{q}{\twoheadrightarrow} & \bullet \\ \downarrow & & \downarrow p & & \downarrow p' \\ 0 & \rightarrowtail & \bullet & \underset{q'}{\twoheadrightarrow} & \bullet \end{array} \qquad\qquad \begin{array}{ccccc} \bullet & \overset{q_0}{\twoheadrightarrow} & A_0 & \overset{k_0}{\rightarrowtail} & \bullet \\ \downarrow & & \downarrow & & \downarrow p' \\ 0 & \twoheadrightarrow & A_1 & \rightarrowtail & \bullet \end{array}$$

where the left square is a pullback and the right one a pushout (W.2*). By (W.4) the factorisation of the rows yields the pushout/pullback diagram at the right, where A_0 is null for **A**

(4) $1_{A_0} \; = \; (k_0^{\#} k_0)(q_0 q^{\#}_0) \; = \; k_0^{\#} 0 q_0^{\#} \; \in \; \mathrm{Nul}(\mathbf{A}),$

hence a zero object for **E**. Analogously A_1 is a zero object and $\mathrm{Ker}(p') = 0$. It follows that (4.10.1)

(5) $\underline{d}(p'^{\#}p') \; = \; k_0 k_0^{\#} \; = \; \omega \; = \; \underline{d}(1),$

hence, by (RW.3a), $p'^{\#}p' = 1$; therefore p' is an isomorphism and q factors through p.

It is now easy to deduce that **P** coincides with the subcategory of conormal epis: if p is so and $p = mq$ with $q \in \mathbf{P}$ and $m \in \mathbf{M}$, then m is iso, which gives $p \in \mathbf{P}$. Similarly **M** coincides with the subcategory of monos.

Last, we check the axiom (A.7): let the commutative diagram 5.1.1 be given, with exact rows, and consider the projections of A (the codomain of h)

(6) $e \; = \; mq^{\#}qm^{\#} \; = \; m_p(q^{\#}q) \; = \; m_R(1)/m_R(kk^{\#}) \; = \; (mm^{\#})/(hh^{\#}),$

$\quad z \; = \; p^{\#}p \; = 1/(hh^{\#}).$

Since $e \lhd z$ (RW.3a), it follows that

(7) $mq^{\#}qm^{\#} \; = \; e \; = \; ez \; = \; mq^{\#}q.m^{\#}p^{\#}.p \; = \; mq^{\#}q.q^{\#}.p \; = \; mq^{\#}p1^{\#},$

which, by the uniqueness of W-factorisations, proves that m is iso.

(e) ⇒ (d). Since the symmetrisation procedure is the same in both cases, we have only to verify (A.6). Let $h = \ker(p)\colon H \to A$ be a normal mono and $q\colon A \to Q$ some conormal epi. Form the commutative diagram at the left

$$(8)\qquad \begin{array}{ccccc} H & \overset{h}{\rightarrowtail} & A & \overset{q}{\twoheadrightarrow} & \bullet \\ \downarrow & & \downarrow p & & \downarrow p' \\ 0 & \rightarrowtail & \bullet & \underset{q'}{\twoheadrightarrow} & \bullet \end{array} \qquad\qquad \begin{array}{ccccc} \bullet & \twoheadrightarrow & \bullet & \overset{k}{\rightarrowtail} & \bullet \\ \downarrow & & \downarrow & & \downarrow p' \\ 0 & \twoheadrightarrow & 0 & \rightarrowtail & \bullet \end{array}$$

where the left square is cartesian and the right one cocartesian. Apply now (W.4) and form the right-hand diagram above: its right square is cartesian, hence k (the image of h by q) is a normal mono.

(d) ⇒ (a). By the W-symmetrisation theorem II (4.10) we have just to prove that **A** satisfies (RW.3a", b"). Consider the projections of A

$$(9)\qquad e = mp^{\#}pm^{\#}, \qquad\qquad f = nq^{\#}qn^{\#}.$$

Let $\underline{n}e \prec \underline{n}f$ and $\underline{d}e \succ \underline{d}f$ and let us prove that $e \prec f$ building the following diagram (as in 4.10.7)

$$(10)\qquad \begin{array}{ccccccc} A & \overset{n}{\leftarrowtail} & \bullet & \overset{q}{\twoheadrightarrow} & \bullet & & \\ \| & & m_0 \uparrow & & \uparrow m_1 & & \\ A & \underset{m}{\leftarrowtail} & \bullet & \underset{p'}{\twoheadrightarrow} & \bullet & \underset{p''}{\twoheadrightarrow} & \bullet \end{array} \qquad\qquad p = p''p',$$

The first condition, $\underline{n}e \prec \underline{n}f$, allows us to build its left square. Take $h = \ker(p)$ and $k = \ker(q)$

$$(11)\qquad nk \sim \mathrm{ind}(f) \prec \mathrm{ind}(e) \sim mh = nm_0h.$$

Thus $k \prec m_0h$ and there exists one mono k_0 such that $k = m_0hk_0$

$$(12)\qquad \begin{array}{ccccc} \bullet & \overset{k}{\rightarrowtail} & \bullet & \overset{q}{\twoheadrightarrow} & \bullet \\ \| & & m_0 \uparrow & & \uparrow m' \\ \bullet & \underset{hk_0}{\rightarrowtail} & \bullet & \underset{p'}{\twoheadrightarrow} & \bullet \end{array}$$

now factor $qm_0 = m'p'$: it is easy to see that $hk_0 \sim \ker(p')$ and, by 5.4, the right square is a pullback. Moreover $p(hk_0) = 0k_0 = 0$, hence p factors through $p' \sim \mathrm{cok}(hk_0)$ and diagram (10) is built.

Now, for (RW.3b"), let $\underline{n}e \leq \underline{n}f$, $\underline{d}e \leq \underline{d}f$ and form the commutative diagrams

$$(13)\qquad \begin{array}{ccc} N & \overset{n}{\rightarrowtail} & A \\ m_0 \uparrow & & \| \\ M & \underset{m}{\rightarrowtail} & A \end{array} \qquad\qquad \begin{array}{ccccc} \bullet & \overset{k}{\rightarrowtail} & N & \overset{q}{\twoheadrightarrow} & \bullet \\ n_0 \uparrow & & \uparrow m_0 & & \uparrow \\ \bullet & \underset{h}{\rightarrowtail} & M & \underset{p}{\twoheadrightarrow} & \bullet \end{array}$$

with $h = \ker(p)$ and $k = \ker(q)$. By an obvious lemma there exists exactly one morphism w which fills-in commutatively (factor $qm_0 = m'p'$; then $p' = qp$, because $p = \mathrm{cok}(h)$; take $w = m'q$). This proves that $e \leq f$, by definition of the order $\leq$ in $\mathrm{Rel}_W(\mathbf{E})$.

Finally, consider diagram (1). As we already know that their rows are formed by inverse anti-isomorphisms of ordered sets (5.2.1; 1.8.1-2), it is sufficient to check the commutativity of the "ker-square"; this follows from 4.10.1: if $p \in \mathrm{Quo}(A)$ and $k = \ker(p)$, then $\underline{d}(p^\# p) = kk^\#$. □

5.6. Annihilator and indetermination. Let $\mathbf{E}$ be a w-exact category and $\mathbf{A} = \mathrm{Rel}_W(\mathbf{E})$. Each relation $a = nq^\# pm^\#: A' \to A''$ determines, together with the subobjects $\mathrm{def}(a) \sim m$ and $\mathrm{val}(a) \sim n$, two other subobjects of A' and A'', called *annihilator* and *indetermination*

(1) $\mathrm{ann}(a): \mathrm{Ann}(a) \rightarrowtail A'$, $\qquad \mathrm{ann}(a) \sim m.\ker(p)$,

(2) $\mathrm{ind}(a): \mathrm{Ind}(a) \rightarrowtail A''$, $\qquad \mathrm{ind}(a) \sim n.\ker(q)$,

satisfying

(3) $\mathrm{ann}(a) \prec \mathrm{def}(a)$, $\qquad \mathrm{ind}(a) \prec \mathrm{val}(a)$,

(4) $\mathrm{ann}(a) = \mathrm{ind}(a^\#) = \mathrm{ann}(a^\# a) = \mathrm{ind}(a^\# a)$.

By the W-symmetrisation theorem II (4.10) it follows easily that

(5) $\underline{\mathrm{ann}}(a) = \underline{\mathrm{ann}}(a^\# a) = \underline{d}(a^\# a) = \underline{i}(\mathrm{ann}(a))$,

(6) $\underline{\mathrm{ind}}(a) = \underline{\mathrm{ind}}(a^\# a) = \underline{d}(aa^\#) = \underline{i}(\mathrm{ind}(a))$.

5.7. W-exact functors. A functor $F: \mathbf{E} \to \mathbf{E}'$ between w-exact categories will be said to be *w-exact* if it satisfies these equivalent conditions:

(a) F is a W-functor (4.9) and preserves zero objects;

(b) F preserves zero objects, monos, conormal epis, finite intersections of monos, kernels;

(c) F preserves monos, their finite intersections and short exact sequences;

(d) F has a (necessarily unique) RW-extension $\mathrm{Rel}_W F: \mathrm{Rel}_W\mathbf{E} \to \mathrm{Rel}_W\mathbf{E}'$.

Indeed (a) ⇒ (b) ⇒ (c) is trivial. (c) ⇒ (a) follows from the characterisations in terms of short exact sequences: - of the zero object ($0 \to 0 \to 0$ is short exact), - of kernels and cokernels, - of "mixed pullbacks" (5.4), - of pushouts of conormal epis (5.3). Last (a) ⇔ (d) by 4.9-10.

5.8. The equivalence. It follows that the 2-adjoint 2-equivalence between the 2-categories $\mathbf{W}$ and $\mathbf{SW}$ described in 4.9.2 restricts to an equivalence

$$(1) \qquad \mathbf{WE} \xrightarrow{\mathrm{Rel}} \mathbf{RWE} \xrightarrow{\mathrm{Prp}} \mathbf{WE}$$

where $\mathbf{WE}$ is the 2-category of *w-exact categories, w-exact functors and w-transformations* (i.e., natural transformations of w-exact functors), while $\mathbf{RWE}$ is the full sub-2-category of $\mathbf{RW}$ containing the projection complete, connected, non empty RW-categories.

The equivalence extends trivially to $\mathbf{WE}'$ (*componentwise* w-exact categories) and $\mathrm{F}\mathbf{RW}$ (projection complete RW-categories).

5.9. Weak adjunctions. From this equivalence and the biuniversal embedding **A** → Fct**A** (3.3) one gets a full RW-embedding

(1) η: **A** → Rel_W(**E**) (**E** = Prp(Fct**A**)),

which is a biuniversal arrow from **A** to the 2-functor Rel: **WE'** → **RW**.

Thus, also because of 5.5, RW-categories coincide up to isomorphism with the full subcategories of categories of relations over componentwise w-exact categories.

Analogously to I.6.7, one forms biuniversal arrows

(2) η: **A** → Rel_W(**E**), (**E** = Z(Prp(Fct**A**))),

where Z(**E**') is a suitable W-exact category associated to the componentwise W-exact category **E**' (and **E**' is ◃-*closed* in Z(**E**')).

Thus RW-categories can also be considered as Prj-full involutive subcategories of categories of relations over w-exact categories.

6. W-MODULAR W-LATTICES, RELATIONS, CONNECTIONS

We introduce here the RW-category w**Mlr** of *wm-lattices and wm-relations* and its w-exact subcategory w**Mlc** = Prp(w**Mlr**) of *wm-lattices and wm-connections*. These categories will be shown (Section 7) to model, respectively, the transfer of restrictions for RW-categories and the transfer of subobjects for w-exact categories. X, Y, Z are always *wm-lattices* (see the appendix, Section 11).

6.1. Wm-relations. A *wm-relation* a: X → Y between (small) wm-lattices will be a pair $a = (a_\bullet, a^\bullet)$ such that (for $x, x' \in X$ and $y, y' \in Y$)

(a) $a_\bullet: X \to Y$ and $a^\bullet: Y \to X$ are mappings preserving ≤ and ◃,

(b) if $y' ◃ y$ then $a^\bullet((a_\bullet x \wedge y) \vee y') = (x \wedge a^\bullet y) \vee a^\bullet y'$,

(c) if $x' ◃ x$ then $a_\bullet((a^\bullet y \wedge x) \vee x') = (y \wedge a_\bullet x) \vee a_\bullet x'$.

Other characterisations will be given in 6.6. Define the composition of a: X → Y and b: Y → Z as

(1) $ba = (b_\bullet, b^\bullet)(a_\bullet, a^\bullet) = (b_\bullet a_\bullet, a^\bullet b^\bullet)$,

which is possible because, for $z ◃ z'$ in Z

(2) $a^\bullet b^\bullet((b_\bullet a_\bullet x \wedge z) \vee z') = a^\bullet((a_\bullet x \wedge b^\bullet z) \vee b^\bullet z') = (x \wedge a^\bullet b^\bullet z) \vee a^\bullet b^\bullet z'$.

This category w**Mlr**, *of wm-latices and wm-relations*, has obvious involution and order

(3) $(a_\bullet, a^\bullet)^\# = (a^\bullet, a_\bullet): Y \to X$,

(4) $(a_\bullet, a^\bullet) \le (b_\bullet, b^\bullet)$ if a and b are parallel morphism and $a_\bullet \le b_\bullet$, $a^\bullet \le b^\bullet$.

(Where $a_\bullet \le b_\bullet$ obviously means $a_\bullet(x) \le b_\bullet(x)$, for all $x \in X$.)

6.2. Regularity. The ordered involutive category w**Mlr** *is a RO-category*: the regularity of the involution follows from

(1) $a_\bullet a^\bullet a_\bullet(x) = a_\bullet((a^\bullet a_\bullet(x) \wedge 1) \vee 0) = (a_\bullet(x) \wedge a_\bullet(1)) \vee a_\bullet(0) = a_\bullet(x)$.

Therefore

(2) a is *mono* $\Leftrightarrow$ $a^\# a = 1$ $\Leftrightarrow$ $a^\bullet a_\bullet = 1_X$ $\Leftrightarrow$ $a_\bullet$ is injective $\Leftrightarrow$ $a^\bullet$ is surjective,

(3) a is *iso* $\Leftrightarrow$ ($a^\# a = 1$ and $aa^\# = 1$) $\Leftrightarrow$ $a_\bullet$ is bijective $\Leftrightarrow$ $a_\bullet$ is an iso of w**Mlh** (11.6).

We usually identify an isomorphism $a: X \to Y$ of w**Mlr** with its "covariant part" $a_\bullet: X \to Y$, an isomorphism of the category w**Mlh** of wm-lattices and homomorphisms (11.6): thus the two categories "have the same isomorphisms". The subobjects of w**Mlr** will be characterised in 6.4.

6.3. Null relations. Every set w**Mlr**(X, Y) has a minimum ω_{XY} and a maximum Ω_{XY}

(1) $\omega_{XY}: X \to Y$, $\qquad x \mapsto 0_Y, \quad y \mapsto 0_X$,

(2) $\Omega_{XY}: X \to Y$, $\qquad x \mapsto 1_Y, \quad y \mapsto 1_X$.

A null wm-relation $a: X \to Y$ is characterised by the condition: $a = a\omega_{YX}a$, or equivalently $a = a\Omega_{YX}a$; thus a is a pair of *constant* mappings

(3) $a_\bullet(x) = a_\bullet(\omega_{YX\bullet}(a_\bullet(x))) = a_\bullet(0_X)$, $\qquad$ for every $x \in X$,

(4) $a^\bullet(y) = a^\bullet(\omega_{YX}{}^\bullet(a^\bullet(y))) = a^\bullet(0_Y)$, $\qquad$ for every $y \in Y$.

Conversely every pair $(x_0, y_0) \in X{\times}Y$ determines two constant mappings

(5) $a_\bullet(x) = y_0$, $\qquad a^\bullet(y) = x_0$,

which are easily seen to form a null wm-relation $a: X \to Y$. Accordingly, *null wm-relations coincide with pairs of constant mappings*.

6.4. Theorem (Projections and monorelations). The RO-category w**Mlr** is projection complete. For a wm-lattice X, there is a biunivocal correspondence among: (a) equivalence classes of monorelations $m: \bullet \to X$, (b) projections $e: X \to X$, and (c) *normal intervals* of X

(1) $[x_0, x_1] = \{y \in X \mid x \leq y \leq x_1\}$, $\qquad x_0 \lhd x_1$.

More precisely, given $x_0 \lhd x_1$, one constructs a *canonical monorelation*

(2) $m: Y = [x_0, x_1] \to X$, $\qquad m_\bullet(y) = y, \quad m^\bullet(x) = (x \wedge x_1) \vee x_0$;

given a monorelation $m: \bullet \to X$, one takes the projection $e = \underline{c}(m) = mm^\#: X \to X$; finally, a projection $e \in \mathrm{Prj}(X)$ yields a *normal pair* of elements of X (and a normal interval)

(3) $x_0 = e_\bullet(0) \lhd x_1 = e_\bullet(1)$.

The loop is closed: starting from $x_0 \lhd x_1$, the projection $e = mm^\#$ gives back the original pair

(4) $e_\bullet(x) = e^\bullet(x) = (x \wedge x_1) \vee x_0$, $\qquad e_\bullet(0) = x_0, \quad e_\bullet(1) = x_1$.

We shall often write e(x) for $e_\bullet(x) = e^\bullet(x)$. We have

(5) e is a restriction $\Leftrightarrow$ $x_0 = 0$ $\Leftrightarrow$ $e(x) = x \wedge x_1$,

(6) e is a corestriction $\Leftrightarrow$ $x_1 = 1$ $\Leftrightarrow$ $e(x) = x \vee x_0$ $\qquad (x_0 = e(0),\ x_1 = e(1))$,

(7) $e \prec e'$ $\Leftrightarrow$ $(x_1 \leq x'_1$ and $x_0 \geq x'_0)$ $\Leftrightarrow$ $[x_0, x_1] \subset [x'_0, x'_1]$,

(8) $e \leq e'$ $\Leftrightarrow$ $(x_1 \leq x'_1$ and $x_0 \leq x'_0)$ $\qquad (x'_0 = e'(0),\ x'_1 = e'(1))$.

Proof. The biunivocal correspondence needs only to be checked at its first step, from $x_0 \lhd x_1$.

The normal interval $[x_0, x_1]$, equipped with the induced relations $\leq$ and $\lhd$, is clearly a wm-lattice. The wm-relation (2) is well defined, since $m^\bullet$ preserves $\leq$ and $\lhd$ ((wl.2,4), see 11.1); moreover, for $x' \lhd x$ in X and $y' \lhd y$ in $Y = [x_0, x_1]$

(9) $m^\bullet((m_\bullet y \wedge x) \vee x') = (((y \wedge x) \vee x') \wedge x_1) \vee x_0 = (y \wedge x) \vee (x' \wedge x_1) \vee x_0$ $\qquad$ (by (wm.1))

$\qquad = ((y \wedge (x \wedge x_1)) \vee x_0) \vee ((x' \wedge x_1) \vee x_0)$

$\qquad = (y \wedge ((x \wedge x_1) \vee x_0)) \vee ((x' \wedge x_1) \vee x_0) = (y \wedge m^\bullet(x)) \vee m^\bullet(x')$ $\qquad$ (by (wm.2)),

(10) $m_\bullet((m^\bullet x \wedge y) \vee y') = (((x \wedge x_1) \vee x_0) \wedge y) \vee y') = ((x \wedge x_1 \wedge y) \vee x_0) \vee y'$ $\qquad$ (by (wm.2))

$\qquad = (x \wedge y) \vee y' = (x \wedge m_\bullet y) \vee m_\bullet y'$.

Last, m is mono since $m^\bullet m_\bullet(y) = (y \wedge x_1) \vee x_0 = y$ (for $y \in Y$) and $m^\# m = (m^\bullet m_\bullet, m^\bullet m_\bullet) = 1_Y$. The properties (5)-(8) are now straightforward. We have also proved that all projections split. □

6.5. Theorem (The RW axioms). The RO-category w**Mlr** is a projection complete, connected RW-category. (Its category of proper morphisms will be considered in 6.9-10.)

For any relation $a: X \rightarrow Y$

(1) $\underline{def}(a) = \underline{n}(a^\# a) = -\wedge(a^\bullet a_\bullet 1) = -\wedge a^\bullet 1: X \rightarrow X$,

$\quad \underline{ann}(a) = \underline{d}(a^\# a) = -\vee(a^\bullet a_\bullet 0) = -\vee a^\bullet 0: X \rightarrow X$.

Proof. To prove (RW.1), let $e \in Prj(X)$

(2) $e(x) = (x \wedge x_1) \vee x_0$, $\qquad (x_0 = e(0) \lhd x_1 = e(1))$.

Then the restriction $\underline{n}e$

(3) $\underline{n}e(x) = x \wedge x_1 = x \wedge e(1)$,

is the only restriction of X satisfying: $e \prec \underline{n}e \leq e$ (by 6.4.7-8). The axiom (RW.2) follows from 6.3, with

(4) $\omega_X: X \rightarrow X$, $\quad \omega_X(x) = 0_X$, $\qquad \Omega_X: X \rightarrow X$, $\quad \Omega_X(x) = 1_X$.

Then

(5) $e\omega e(x) = e(0_X) = x_0$,

$\quad \underline{d}e(x) = \underline{n}(e\omega e)(x) = x \wedge (e\omega e(1)) = x \wedge x_0$,

and the last axiom (RW.3) follows straightforwardly from 6.4.7-8.

Finally, we already know that w**Mlr** is projection complete (6.4); it is connected because of 6.3.1-2. The relations (1) are obvious. □

6.6. Proposition (Characterisation of wm-relations). Let $a_\bullet: X \rightarrow Y$ and $a^\bullet: Y \rightarrow X$ be a pair of mappings preserving $\leq$ and $\lhd$; the following conditions are equivalent:

(a) $(a_\bullet, a^\bullet)$ is a wm-relation (i.e., $a_\bullet$ and $a^\bullet$ satisfy 6.1b, 6.1c),

(b) - for all $x \in X$: $a^\bullet a_\bullet(x) = (x \wedge a^\bullet 1) \vee a^\bullet 0$,

- for all $y \in Y$: $a_\bullet a^\bullet(y) = (y \wedge a_\bullet 1) \vee a_\bullet 0$,

- $a_\bullet a^\bullet a_\bullet = a_\bullet$, $\quad a^\bullet a_\bullet a^\bullet = a^\bullet$,

(c) the mappings $a_\bullet$ and $a^\bullet$ factor respectively as

$$(1) \quad X \underset{p^\bullet}{\overset{p_\bullet}{\rightleftarrows}} [a^\bullet 0, a^\bullet 1] \underset{b^\bullet}{\overset{b_\bullet}{\rightleftarrows}} [a_\bullet 0, a_\bullet 1] \underset{m^\bullet}{\overset{m_\bullet}{\rightleftarrows}} Y$$

where $m = (m_\bullet, m^\bullet)$ and $p^\# = (p^\bullet, p_\bullet)$ are canonical monorelations (6.4.2) and

(2) $b_\bullet(x) = a_\bullet(x)$, $\qquad$ for $a^\bullet 0 \leq x \leq a^\bullet 1$,

(3) $b^\bullet(y) = a^\bullet(y)$, $\qquad$ for $a_\bullet 0 \leq y \leq a_\bullet 1$,

are inverse isomorphisms of wm-lattices (hence $b = (b_\bullet, b^\bullet)$ is an isomorphism of w**Mlr** (6.2)).

Proof. (a) $\Rightarrow$ (b). Trivial, as $0 \lhd 1$ in X and Y.

(b) $\Rightarrow$ (c). First note that $b_\bullet$ and $b^\bullet$ are inverse mappings by (b) and preserve $\leq$ and $\lhd$, because $a_\bullet$ and $a^\bullet$ do; moreover

(4) $a_\bullet(x) = a_\bullet(a^\bullet a_\bullet x) = a_\bullet((x \wedge a^\bullet 1) \vee a^\bullet 0) = m_\bullet b_\bullet p_\bullet(x)$,

and similarly $a^\bullet(y) = p^\bullet b^\bullet m^\bullet(y)$.

(c) $\Rightarrow$ (a). Since $p = (p_\bullet, p^\bullet)$, $b = (b_\bullet, b^\bullet)$ and $m = (m_\bullet, m^\bullet)$ are wm-relations, by 6.2 and 6.4, so is their composite $mbp = (a_\bullet, a^\bullet)$. □

6.7. Full subcategories. We are interested in the following *full* subcategories of **Mrw** (hence RW-subcategories; other subcategories will be considered in 7.4)

(a) w**Dlr**, containing the *w-distributive* w-lattices (11.2); this category will be shown in Section 8 to model the transfer of restrictions for w-distributive RW-categories;

(b) **Mlr**, containing the *modular* lattices (i.e., normal wm-lattices); it is an RE-category, modelling the transfer of restrictions for RE-categories (i.e., normal RW-categories): see I.7;

(c) **Dlr** = w**Dlr** ∩ **Mlr**, containing the *distributive* lattices (i.e., normal wd-lattices).

6.8. Double categories of w-lattices. Also here, in order to model the action of RW-functors on restrictions (7.1), we need to combine homomorphisms and relations, forming the double category w𝕄lhr having for objects the wm-lattices, horizontal morphisms in w**Mlh**, vertical ones in w**Mlr** and cells given by "bicommutative squares" of type

$$(1) \qquad \begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Y & \xrightarrow[k]{} & Y' \end{array} \qquad\qquad k a_\bullet = b_\bullet h; \quad h a^\bullet = b^\bullet k;$$

(composition of *mappings*). The underlying vertical category (homomorphisms as objects and bicommutative squares as morphisms) has an obvious RW-structure.

The full double subcategories determined by wd-lattices, or modular lattices, or distributive lattices will be written here as $\mathrm{w}\mathbb{D}\mathrm{lhr}$, $\mathbb{M}\mathrm{lhr}$, $\mathbb{D}\mathrm{lhr}$ (the last two were introduced in I.7.2).

6.9. **Wm-connections.** We proved in 6.5 that w**Mlr** is a projection complete, connected RW-category. By the W-symmetrisation theorem (5.5) its category of proper morphisms

(1) $\quad \mathrm{w}\mathbf{Mlc} \; = \; \mathrm{Prp}(\mathrm{w}\mathbf{Mlr}),$

is w-exact, and will be called the category of wm-lattices and wm-connections. Thus a wm-connection

(2) $\quad u \; = \; (u_\bullet, u^\bullet)\colon X \to Y,$

is characterised, among wm-relations, by the conditions

(3) $\quad \underline{\mathrm{def}}(u) \; = \; \underline{n}(u^\# u) \; = \; 1, \qquad\qquad \underline{\mathrm{ind}}(u) \; = \; \underline{d}(uu^\#) \; = \; 0,$

which, by 6.5.1 are equivalent to

(4) $\quad u^\bullet(1) \; = \; 1, \qquad\qquad u_\bullet(0) \; = \; 0.$

In other words, by the characterisation 6.6b of wm-relations, a wm-connection (2) is a pair $(u_\bullet, u^\bullet)$ such that

(a) $\quad u_\bullet\colon X \to Y$ and $u^\bullet\colon Y \to X$ are mappings, preserve $\leq$ and $\triangleleft$, and satisfy $u_\bullet 0 = 0$, $u^\bullet 1 = 1$,

(b) $\quad u^\bullet u_\bullet(x) \; = \; x \vee u^\bullet(0) \geq x,$

(c) $\quad u_\bullet u^\bullet(y) \; = \; y \wedge u_\bullet(1) \leq y.$

In particular $u_\bullet \dashv u^\bullet$ (u is a "covariant Galois connection" between the ordered sets X and Y, which implies $u_\bullet = u_\bullet u^\bullet u_\bullet$, $u^\bullet = u^\bullet u_\bullet u^\bullet$), and each of these mappings determines the other; $u_\bullet$ preserves the existing unions while $u^\bullet$ preserves intersections. w**Mlc** is concrete and coconcrete. Note that the last two properties in (a) are obviously a consequence of the adjunction $u_\bullet \dashv u^\bullet$ (but we prefer to state them explicitly to avoid doubts on the meaning of (b), where the a priori existence of $x \vee u^\bullet(0)$ depends on $u^\bullet(0) \triangleleft u^\bullet(1) = 1$). We identify w**Mlr** with the isomorphic RW-category Rel(w**Mlc**).

Equivalently, one can replace (b), (c) with

(b') $\quad u^\bullet(u_\bullet x \vee y) \; = \; x \vee u^\bullet y, \qquad\qquad$ for each $x \in X$ and each $y \triangleleft 1$ in Y,

(c') $\quad u_\bullet(u^\bullet y \wedge x) \; = \; y \wedge u_\bullet x, \qquad\qquad$ for each $x \in X$ and each $y \in Y$.

6.10. Exactness. The zero object 0 of the w-exact category w**Mlc** of wm-lattices and wm-connections is the one-point lattice; zero morphisms are given by

(1) $\quad 0_{XY}\colon X \to Y, \qquad\qquad x \mapsto 0_Y, \quad y \mapsto 1_X.$

The subobjects (resp. quotients) of X are determined by proper canonical monorelations in X (resp. epirelations from X); accordingly to 6.4, the latter are as in (2) (resp. in (3))

(2) $\quad m\colon Y = [0, x_1] \to X, \qquad\qquad m_\bullet(y) \; = \; y, \quad m^\bullet(x) \; = \; x \wedge x_1,$

(3) $\quad p\colon X \to [x_0, 1] = Y, \qquad\qquad p_\bullet(x) \; = \; x \vee x_0, \quad p^\bullet(y) \; = \; y.$

Therefore the set of subobjects of X is in biunivocal correspondence with X itself via $m \mapsto x_1 = m_\bullet(1_Y)$, while the set of quotients of X is in biunivocal correspondence with the set Nrm(X) of normal elements of X, via $p \mapsto x_0 = p^\bullet(0_Y)$).

The wm-connection $u: X \to Y$ has canonical factorisation (6.6.1)

$$(4) \quad X \xrightarrow{p} [u^\bullet 0, 1_X] \xrightarrow{b} [0_Y, u_\bullet 1] \xrightarrow{m} Y$$

and

$$(5) \quad \ker(u) = ([0_X, u^\bullet 0] \rightarrowtail X).$$

Each short exact sequence of w**Mlc** is of the following kind, up to isomorphism

$$(6) \quad [0_X, x_0] \rightarrowtail X \twoheadrightarrow [x_0, 1_X] \qquad\qquad (x_0 \lhd 1_X).$$

Finally, we write

(7) w**Dlc** = Prp(w**Dlr**),

the full w-exact subcategory of w**Mlc** determined by wd-lattices. Analogously

(8) **Mlc** = Prp(**Mlr**), **Dlc** = Prp(**Dlr**),

are full w-exact subcategories of w**Mlc**, and exact categories in their own right.

The double, vertically w-exact, categories w$\mathbb{M}$lhc, w$\mathbb{D}$lhc, $\mathbb{M}$lhc, $\mathbb{D}$lhc have homomorphisms as horizontal arrows, connections as vertical ones, bicommutative squares as cells. The last two were introduced in [G8] and are vertically exact.

7. Transfer of restrictions and lattice properties

Extending I.7 and [G8], we define here the transfer functor Rst: **A** → w**Mlr** of the RW-category **A** and deduce the transfer functor Sub: **E** → w**Mlc** of the w-exact category **E**. This procedure (treat first the "categories of relations") appears to be more clear and effective than the opposite one (treat first the "exact categories") which we followed in the above references.

The transfer functor allows us to study "lattice properties" (7.5-6) of RW-categories, among which normality and distributivity, and to construct "expansions" (7.6-8) which satisfy them.

7.1. The transfer RW-functor. Every RW-category **A** has an associated RW-functor

(1) $\mathrm{Rst}_{\mathbf{A}}$: **A** → w**Mlr**,

which will be called the *transfer functor* of **A**, since it describes the covariant and contravariant transfer of its restrictions; it will be shown to be Rst-faithful, Rst-full and closed (7.3).

Namely, for every object A, Rst(A) is the wm-lattice of restrictions of A (1.7). For every morphism $a: A' \to A''$ in **A**

$$(2) \quad \mathrm{Rst}(a) = (a_R, a^R): \mathrm{Rst}(A') \to \mathrm{Rst}(A''),$$

is the wm-relation whose components are defined in 1.3.1. Indeed, the characterisation 6.6b is satisfied: a_R and a^R preserve $\leq$ by 1.3.2, the relation $\lhd$ by 1.3.2 and, for $x \in \mathrm{Rst}(A')$

(3) $a^R a_R(x) = (a^\# a)_R(x) = \underline{n}((a^\# a)x(a^\# a))$ (by 1.3)

$= (\underline{n}(a^\# a) \wedge x) \vee \underline{d}(a^\# a) = (x \wedge a^R(1)) \vee a^R(\omega)$ (by 1.6 and 1.4).

Finally $Rst_{\mathbf{A}}$ is an RW-functor: it preserves the composition by 1.3, the involution by 1.3.1, the relation $\leq$ by (RW.3b) and the projections ω since

(4) $\omega_R(x) = \underline{n}(\omega x \omega) = \underline{n}(\omega) = \omega$.

Thus (6.1), for all $x, x' \in Rst(A')$ and $y, y' \in Rst(A'')$

(5) $a^R((a_R x \wedge y) \vee y') = (x \wedge a^R y) \vee a^R y'$ (for $y' \lhd y$),

(6) $a_R((a^R y \wedge x) \vee x') = (y \wedge a_R x) \vee a_R x'$ (for $x' \lhd x$).

Finally, the functorial aspect (I.7.2) extends easily to the RW-case: every RW-functor $F: \mathbf{A} \to \mathbf{B}$ determines a *horizontal transformation of vertical functors*, with values in the double category $w\mathbb{M}lhr$ of wm-homo-morphisms and wm-relations (6.8)

(7) $Rst_F: Rst_A \to Rst_B.F: \mathbf{A} \to w\mathbb{M}lhr$,

$Rst_F A: Rst_{\mathbf{A}}(A) \to Rst_{\mathbf{B}}(FA)$, $x \mapsto Fx$,

associating to each object A the *wm-homomorphism* $x \mapsto Fx$ (2.1). Also here, Rst_F is the *unique* horizontal transformation from $Rst_{\mathbf{A}}$ to $Rst_{\mathbf{B}}.F$.

7.2. wm-lattices can be realised. The transfer functor of w**Mlr** itself is canonically isomorphic to the identity functor, via

(1) $\iota: Rst \to 1: w\mathbf{Mlr} \to w\mathbf{Mlr}$,

(2) $\iota X: Rst(X) \to X$, $(\iota X)_\bullet(r) = r_\bullet(1)$, $(\iota X)^\bullet(x) = x \wedge -$,

as it follows from the characterisation of the restrictions of X in 6.4.5.

This isomorphism shows that *every wm-lattice* X *can be realised as a w-lattice of restrictions for some object* (X itself) *in a* (fixed) *RW-category* (w**Mlr**). This also shows that no proper replete subcategory of w**Mlr** can suffice to treat the transfer of restrictions for RW-categories.

7.3. Local properties of transfer functors. The transfer functor $Rst_{\mathbf{A}}: \mathbf{A} \to w\mathbf{Mlr}$ is *Rst-faithful, Rst-full and closed*. Let us check the last property, the proof of the other two being analogous. Assume that, for A in $\mathbf{A}$ and $X = Rst_{\mathbf{A}}(A)$

(1) $y \leq x$ in X, $Rst(y) \lhd Rst(x)$ in $Rst(X)$,

and apply the isomorphism 7.2.2

(2) $(\iota X)_\bullet(Rst(y)) = (Rst(y))_\bullet(1) = y_R(1) = y$;

analogously $(\iota X)_\bullet(Rst(x)) = x$, hence $y \lhd x$ in $X = Rst_{\mathbf{A}}(A)$.

7.4. Transfer RW-categories. The RW-category $\mathbf{A}$ is said to be *transfer* if its transfer functor $Rst_{\mathbf{A}}: \mathbf{A} \to w\mathbf{Mlr}$ is faithful. By 7.2, w**Mlr** is so.

Every RW-category $\mathbf{A}$ has an *associated transfer RW-category* $Trn(\mathbf{A})$ determined by the closed RW-factorisation $Rst_{\mathbf{A}} = R_2 R_1$ (3.6)

(1) $\mathbf{A} \xrightarrow{R_1} \mathrm{Trn}(\mathbf{A}) \xrightarrow{R_1} \mathrm{w}\mathbf{Mlr}$

coinciding with the *ordinary* factorisation (3.2) since $\mathrm{Rst}_{\mathbf{A}}$ is closed (7.3): Trn(**A**) is the *strict quotient* of **A** which identifies two parallel maps a, b of **A** when

(2) $a_R = b_R$, $a^R = b^R$.

In particular Trn(**A**) is projection complete when **A** is so (3.2).

7.5. Lattice properties for RW-categories. Many properties of RW-categories we want to consider *concern their transfer functor* (e.g., the fact of being a *transfer* RW-category just considered).

More particularly, let **M** be a full replete subcategory of w**Mlr** (hence an RW-subcategory, by 2.5). We say that the RW-category **A** is **M**-*latticed* if its transfer functor takes values in **M**

(1) $\mathrm{Rst}_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{M}$,

and refer generally to such a condition on **A** as a "lattice property" (dropping the prefix "w" for simplicity).

Thus, the subcategory **M** = **Mlr** of modular lattices (i.e. *normal* wm-lattices, 11.3) yields the *normal* RW-categories, which coincide with RE-categories (1.8); similarly, *subnormality* (1.8) is another lattice property.

We say now that **A** is *Rst-finite* when all its wm-lattices $\mathrm{Rst}_{\mathbf{A}}(A)$ are finite: here $\mathbf{M} = \mathrm{w}\mathbf{Mlr}^f$, the full subcategory of finite wm-lattices. Note that the latter is clearly hom-finite: it follows that *every transfer Rst-finite RW-category is hom-finite.*

Last, we say that **A** is *w-distributive* when all its wm-lattices $\mathrm{Rst}_{\mathbf{A}}(A)$ are so (11.2); here **M** = w**Dlr** (6.7).

Let $F\colon \mathbf{A} \to \mathbf{B}$ be an RW-functor. Then

(a) if F is a strict quotient and **A** is Rst-finite, or normal, or subnormal, or w-distributive, so is **B** (for Rst-finiteness, it suffices F Rst-full),

(b) if F is Rst-faithful and **B** is Rst-finite or w-distributive, so is **A**,

(c) if F is Rst-faithful and closed and **B** is normal or subnormal, so is **A**.

Distributivity, the main lattice property, will be studied in the next chapter and given various characterisations (8.1). We associate now, to each RW-category, an **M**-latticed RW-category.

7.6. Theorem (M-expansions). Let **M** be a full subcategory of w**Mlr**, *stable for closed wm-homomorphic images*: if X belongs to **M** and $h\colon X \to Y$ is a $\triangleleft$-*closed surjective homomorphism of wm-lattices* (hence in w**Mlh**, not in w**Mlr**, cf. 11.5, 7) then Y belongs to **M**.

Then every RW-category **A** has an associated **M**-latticed RW-category $\mathbf{A}^\#$, the ***M****-expansion of* **A**, equipped with a faithful closed RW-functor $F\colon \mathbf{A}^\# \to \mathbf{A}$ satisfying the universal property

(a) every RW-functor $G\colon \mathbf{D} \to \mathbf{A}$, where **D** is any **M**-latticed RW-category, has a unique Prj-full (i.e. Rst-full and closed (3.3)) lifting $G'\colon \mathbf{D} \to \mathbf{A}^\#$ satisfying $FG' = G$. Moreover, if **M** is *stable* *for normal intervals* (contains any normal interval $[x_0, x]$ of any X in **M**, see 6.4) and **A** is projection complete, so is $\mathbf{A}^{\#}$.

Proof. The objects of $\mathbf{A}^{\#}$ are the pairs (A, X), where A is in **A** and X is a sub-w-lattice of $\mathrm{Rst}_{\mathbf{A}}(A)$ belonging to **M**. A morphism

(1) (a; X, X'): (A, X) $\to$ (A', X'),

is given by any **A**-morphism a: A $\to$ A' such that

(2) $a_R(X) \subset X'$, $a^R(X') \subset X$;

in particular

(3) $\underline{\mathrm{def}}(a) = a^R(1) \in X$, $\underline{\mathrm{ann}}(a) \in X$, $\underline{\mathrm{val}}(a) \in X'$, $\underline{\mathrm{ind}}(a) \in X'$.

The composition, involution and order are "those of **A**"; the faithful functor

(4) F: $\mathbf{A}^{\#} \to \mathbf{A}$; (A, X) $\mapsto$ A, (a; X, X') $\mapsto$ a,

preserves involution and order: thus $\mathbf{A}^{\#}$ is a RO-category (and F a RO-functor). To prove that $\mathbf{A}^{\#}$ is an RW-category and F a closed RW-functor we have to verify the conditions 2.7a, b. By abuse of notation we often write a for (a; X, X').

Now, a projection e = (e; X, X) $\in$ Prj(A, X) of $\mathbf{A}^{\#}$ is given by any e $\in$ Prj(A) such that

(5) $\underline{n}e \in X$, $\underline{d}e \in X$,

the necessity of (5) following from (3), its sufficiency from

(6) $e_R(x) = e^R(x) = \underline{n}\,(exe) = (\underline{n}e \wedge x) \vee \underline{d}e$.

Thus (e; X, X) has numerator ($\underline{n}$e; X, X), which is in $\mathbf{A}^{\#}$ since $\underline{n}(\underline{n}e) = \underline{n}e$ and $\underline{d}(\underline{n}e) = \omega$; analogously ($\omega$; X, X) and ($\Omega$; X, X) are in $\mathbf{A}^{\#}$. Last, if $y \leq x$ in Rst(A, X), which means that $y \leq x$ in X, and $y \lhd x$ in **A**, the projection $e = x/y \in \mathrm{Prj}_{\mathbf{A}}(A)$ is also in $\mathrm{Prj}_{\mathbf{A}^{\#}}(A, X)$ by (5), and $y \lhd x$ in $\mathbf{A}^{\#}$.

The RW-category $\mathbf{A}^{\#}$ is **M**-latticed, because

(7) $\mathrm{Rst}_{\mathbf{A}^{\#}}(A, X) = \{(x; X, X) \mid x \in X\}$,

is isomorphic to X, hence belongs to **M**.

As to the universal property (a), let G: **D** $\to$ **A** be as stated and define

(8) G': $\mathbf{D} \to \mathbf{A}^{\#}$, $G'(D) = (G(D), \mathrm{Rst}_G(\mathrm{Rst}_{\mathbf{D}}(D)))$,

G'(d) = G(d): G'(D) $\to$ G'(D') (for d: D $\to$ D' in **D**),

which is permitted since $\mathrm{Rst}_G(\mathrm{Rst}_{\mathbf{D}}(D))$ is the image of the **M**-lattice $\mathrm{Rst}_{\mathbf{D}}(D)$ by the wm-homomorphism

(9) $\mathrm{Rst}_G: \mathrm{Rst}_{\mathbf{D}}(D) \to \mathrm{Rst}_{\mathbf{A}}(G(D))$,

provided with the "image" $\lhd$, hence belongs to **M** by hypothesis; moreover, for $x \in \mathrm{Rst}_{\mathbf{D}}(D)$

(10) $(G(d))_R(Gx) = G(d_R x) \in \mathrm{Rst}_G(\mathrm{Rst}_{\mathbf{D}}(D'))$.

Finally, let **M** be stable for normal intervals and **A** be projection complete. Consider a projection e of (A, X) in $\mathbf{A}^{\#}$; let $e = ss^{\#}$ where s: B → A is mono in **A**, and take

(11) $Y = \{y \in \mathrm{Rst}(B) \mid s_R(y) \in X\}$.

Since Y is isomorphic to a normal interval of X via s_R and s^R

(12) $Y \rightleftarrows [\underline{d}e, \underline{n}e] \subset X$,

Y belongs to **M**, hence (B, Y) is in $\mathbf{A}^{\#}$ and e factors as $ss^{\#}$ with s: (B, Y) → (A, X). □

7.7. Some expansions. Thus, by taking **M** to be w**Mlr** itself we get the *wm-expansion* wMe**A**; note that the latter category is *not* equivalent to **A**, and solves a problem of some interest (7.6). By considering **M** = w**Dlr**, we have the *wd-expansion* or *w-distributive expansion* wDe**A**.

Finally **M** = **Mlr** yields the *normal* (or *modular*) *expansion* Mle**A**, while **Dlr** gives the *distributive normal expansion* Dle**A**. The latter are both RE-categories. (These expansions were written Mdl**A** and Dst**A** in I.7.7, 7.10, for an RE-category **A**; actually in that paper there was no need of distinguishing between the above expansions and wMe**A**, wDe**A**.)

7.8. Distributive and normal graph-morphisms. Let F: Δ → **A** be a graph morphism with values in an RW-category: we shall say that F is w-distributive or normal when the category RW(F) is so (3.8).

By extending [G11, 1.4], with a similar proof, one shows that F is w-distributive (resp. normal) if and only if it satisfies the following equivalent conditions:

(a) F factors through some w-distributive (resp. normal) RW-category, via a closed RW-functor,

(b) there is a closed Rst-faithful RW-functor G: **A** → **B** such that GF is w-distributive (resp. normal),

(c) for every closed RW-functor G: **A** → **B**, GF is w-distributive (resp. normal)

7.9. The transfer w-exact functor. Consider now a w-exact category **E** (more generally: componentwise w-exact) and define the *transfer* w-exact functor of **E**

(1) $\mathrm{Sub}_{\mathbf{E}} = \mathrm{Prp}(\mathrm{Rst}_{\mathrm{Rel}\mathbf{E}}): \mathbf{E} \to \mathrm{w}\mathbf{Mlc}$.

According to the isomorphism 4.5.4, we identify $\mathrm{Sub}_{\mathbf{E}}$ with the following "isomorphic copy" (where u: A → A' is in **E**)

(2) $\mathrm{Sub}_{\mathbf{E}}(A)$ = the wm-lattice of **E**-subobjects of A,

$\mathrm{Sub}_{\mathbf{E}}(u) = (u_S, u^S): \mathrm{Sub}_{\mathbf{E}}(A) \to \mathrm{Sub}_{\mathbf{E}}(A')$,

where, as usual, $u_S(m)$ is obtained by the canonical factorisation of um and $u^S(n)$ through the pullback of (u, n). Indeed

(3) $u_R(mm^{\#}) = \underline{n}(u_P(mm^{\#})) = \underline{n}(umm^{\#}u^{\#}) = umm^{\#}u^{\#} = \underline{i}(um) = \underline{i}(u^S m)$,

and analogously $u^R(nn^{\#}) = \underline{i}(u_S n)$.

As the pair (u_S, u^S) is a wm-connection (6.9), $u_S \dashv u^S$. Thus u_S preserves 0 and existing unions (hence all ◅-unions) while u^S preserves 1 and intersections; further

(4) $u^S(u_S m \vee n) = m \vee u^S n$, for each $m \in \mathrm{Sub}A$ and $n \lhd 1$ in $\mathrm{Sub}A'$,

$u_S(u^S n \wedge m) = n \wedge u_S m$, for each $m \in \mathrm{Sub}A$ and $n \in \mathrm{Sub}A'$.

In particular the transfer functor of w**Mlc** is isomorphic to the identity functor, and *each wm-lattice* X *is isomorphic to a lattice of subobjects of some object* (X itself) *in a suitable w-exact category* (w**Mlc**).

7.10. Lattice properties for w-exact categories. All the above considerations for RW-categories can be transferred to the present case.

We have thus the notion of *transfer* w-exact category, and also the transfer w-exact category Trn**E** associated to **E**: a sort of category of projectivities of **E**, as remarked in [G8].

We also have lattice properties for w-exact categories analogous to the ones considered in 7.7, which we still call *normality* (i.e. exactness), *subnormality*, *Sub-finiteness* and *distributivity*.

Last, every w-exact category **E** has a *wm-expansion* wMe**E**, a *wd-expansion* wDe**E**, a *normal* (or exact) *expansion* Mle**E**, a *distributive normal expansion* Dle**E**, a *Sub-finite expansion*, a *sub-normal expansion*.

More generally, if **M** is a full subcategory of w**Mlc** stable for closed wm-homomorphic images and normal intervals, the **M**-expansion $\mathbf{E}^{\#}$ of the w-exact category **E** satisfies, with the associated faithful, closed w-exact functor $F: \mathbf{E}^{\#} \to \mathbf{E}$, the following universal property

(a) every w-exact functor $G: \mathbf{D} \to \mathbf{E}$, where **D** is some **M**-latticed w-exact category, has a unique Sub-full closed w-exact lifting $G': \mathbf{D} \to \mathbf{E}^{\#}$ (satisfying $G'F = G$).

7.11. Bicommutative and exact squares. Last we recall the notion of bicommutative proper square in an RW-category together with the corresponding notion of exact square in a w-exact one; characterisations are given, respectively, by transfer mappings of restrictions and subobjects.

Take a commutative square diagram of *proper* morphisms in the RW-category **A**

(1) $$\begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[y]{} & D \end{array} \qquad w = vx = yu,$$

so that $xu^{\#} \leq v^{\#}v.xu^{\#} = v^{\#}y.uu^{\#} \leq v^{\#}y$. We say that (1) is a *bicommutative (proper) square* of **A** if it commutes and

(2) $xu^{\#} = v^{\#}y$.

The square (1) in the w-exact category **E** is said to be *exact* if it is bicommutative in Rel**E**: $vx = yu$ and $xu^{\#} = v^{\#}y$.

7.12. Lemma (Bicommutative squares). The commutative proper square 7.11.1 of **A** is bicommutative if and only if the equivalent conditions (1) and (2) hold

(1) $u_R(\underline{\mathrm{ann}}(x)) = \underline{\mathrm{ann}}(y)$, $v^R(\underline{\mathrm{val}}(y)) = \underline{\mathrm{val}}(x)$,

(2) $x_R(\underline{ann}(u)) = \underline{ann}(v)$, $y^R(\underline{val}(v)) = \underline{val}(u)$;

in this case, if $w = vx = yu$

(3) $\underline{ann}(w) = \underline{ann}(u) \vee \underline{ann}(x)$, in Rst(A),

(4) $\underline{val}(w) = \underline{val}(v) \wedge \underline{val}(y)$, in Rst(D).

(The conjunction of properties (3) and (4) is weaker than (1), as soon as **A** has a non-null object A = B = C = D; take $u = v = x = y = \omega\Omega: A \to A$, the proper null endomorphism of A).

Proof. If our square is bicommutative, then (1) holds

(4) $u_R(\underline{ann}(x)) = u_R(x^R(\omega)) = (xu^\#)^R(\omega) = (v^\# y)^R(\omega) = y^R v_R(\omega) = y^R(\omega) = \underline{ann}(y)$,

(5) $v^R(\underline{val}(y)) = v^R(y_R(1)) = (v^\# y)_R(1) = (xu^\#)_R(1) = x_R u^R(1) = x_R(1) = \underline{val}(x)$.

and, by symmetry, also (2) does. Now (1) ⇒ (2), since

(6) $\underline{ann}(v) = v^R(\omega) = v^R(\omega) \wedge x_R(1) = x_R x^R(v^R(\omega)) =$ by (1), 7.1.7

$= x_R u^R y^R(\omega) = x_R u^R(\underline{ann}(y)) = x_R u^R(u_R x^R(\omega))$ by (1)

$= (xu^\#)_R (xu^\#)^R(\omega) = (xu^\#)_R(\omega) = x_R u^R(\omega) = x_R(\underline{ann}(u))$.

and analogously for the right-hand relation of (2). By symmetry, (1) and (2) are equivalent. Let us assume now that (1) and (2) hold, and prove that the square 7.11.1 is bicommutative: since we already know that $xu^\# \leq v^\# y$, by 1.10.3 we just need to show that

(6) $\underline{c}(xu^\#) = \underline{c}(v^\# y)$, $\underline{i}(xu^\#) = \underline{i}(v^\# y)$.

Indeed

(7) $\underline{def}(xu^\#) = (xu^\#)^R(1) = u_R x^R(1) = u_R(1) = \underline{val}(u) = y^R(\underline{val}(v)) = y^R(v_R(1))$

$= (v^\# y)^R(1) = \underline{def}(v^\# y)$,

and similarly for the restrictions $\underline{ann}$, $\underline{val}$, $\underline{ind}$. Finally, if (1) and (2) hold

(8) $\underline{ann}(w) = (vx)^R(\omega) = x^R v^R(\omega) = x^R(\underline{ann}(v)) = x^R(x_R(\underline{ann}(u))) = \underline{ann}(u) \vee \underline{ann}(x)$. □

7.13. Lemma (Exact squares). In the w-exact category **E**, the commutative square 7.11.1 is exact if and only if the equivalent conditions (1) and (2) hold

(1) $u_S(\ker(x)) = \ker(y)$, $v^S(\mathrm{im}(y)) = \mathrm{im}(x)$,

(2) $x_S(\ker(u)) = \ker(v)$, $y^S(\mathrm{im}(v)) = \mathrm{im}(u)$;

in this case

(3) $\ker(w) = \ker(u) \vee \ker(x)$, in Sub(A),

(4) $\mathrm{im}(w) = \mathrm{im}(v) \wedge \mathrm{im}(y)$, in Sub(D).

Moreover, our square is exact, with *normal* horizontal morphisms, if and only if it satisfies the equivalent conditions (a), (b)

(a) there is in **E** a commutative diagram with exact rows

$$(5) \qquad \begin{array}{ccccccc} \bullet & \longrightarrow & A & \longrightarrow & B & \longrightarrow & \bullet \\ \downarrow\!\!\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & C & \longrightarrow & D & \longrightarrow & \bullet \end{array}$$

(b) x and y are normal morphisms, and in the commutative diagram with exact rows

$$(6) \qquad \begin{array}{ccccccc} \bullet & \xrightarrow{\ker(x)} & A & \xrightarrow{x} & B & \xrightarrow{\mathrm{cok}(x)} & \bullet \\ {\scriptstyle u'}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ \bullet & \xrightarrow[\ker(y)]{} & C & \xrightarrow[y]{} & D & \xrightarrow[\mathrm{cok}(y)]{} & \bullet \end{array}$$

u' is epi and v' is mono.

Proof. The first assertion is just a translation of the bicommutative square lemma (7.12). Therefore (a) is equivalent to (c), while the equivalence of (b) and (c) is easy to check directly. □

8. Distributive and idempotent RW-categories

The interest of the w-distributive and idempotent cases has been recalled in the introduction.

8.1. Theorem (w-distributive RW-categories). The following conditions on the RW-category **A** and the componentwise w-exact category **E** = Prp(Fct**A**) are equivalent

(a) **A** is w-distributive, i.e. for each object A the w-lattice Rst(A) is w-distributive (7.4),

(b) for each morphism $a: A' \to A''$ the mapping a_R: Rst(A') → Rst(A'') preserves (binary) intersections and ◃-unions (generally a_R is not in w**Mlh**, as it need not preserve ω, 1),

(c) the category **A** is orthodox [G3] (As **A** is provided with a regular involution, this just means that idempotent endomorphisms are stable for composition),

(d) the category **A** is quasi-inverse ([G5], 4.6),

(e) the product e&f = efe on each set Prj(A) is associative,

(f) all the mappings a_P: Prj(A') → Prj(A'') preserve the operation &,

(g) the componentwise w-exact category **E** is *w-distributive*, i.e. all its w-lattices of subobjects are so,

(h) in **E** all the transfer mappings

(1) u_S: Sub(A') → Sub(A''), $\qquad u^S$: Sub(A'') → Sub(A'),

preserve (binary) intersections and ◃-unions,

(i) in **E** all the direct images of subobjects (u_S) and all the inverse images of quotients (u^Q) preserve (binary) intersections (these images trivially exist, by canonical factorisation),

(j) **E** satisfies the "cubic distributivity axioms" (0Q) and (0Q*) of [G6].

Proof. The equivalence between (c), (d), (i), (j) holds more generally for SW-categories **A** and W-categories **E** = Prp(Fct**A**) ([G6], thm. 1.9), the equivalence of (j) and (g) for γ-categories, i.e. w-exact ones, was proved in [CM]; the equivalence of (g) and (a) follows from the isomorphisms 4.5.4.

(d) ⇒ (f). Proved in [G5], 4.7, and easy to check.

(f) ⇒ (b). For $x, y \in \mathrm{Rst}(A)$, using the "calculus of projections" (1.6)

(2) $a_R(xy) = \underline{n}(a_P(x\&y)) = \underline{n}(a_P(x)\&a_P(y)) = (a_R(x) \wedge a_R(y)) \vee a_R(\omega) = a_R(x) \wedge a_R(y)$;

assume now that $y \lhd x$ and $t \leq x$

(3) $a_R(t \vee y) = \underline{n}(a_P((t \vee y)/y)) = \underline{n}(a_P(x/y \,\&\, t)) = \underline{n}(a_P(x/y) \,\&\, a_P(t)) =$
$= (a_R(x) \wedge a_R(t)) \vee a_R(y) = a_R(t) \vee a_R(y)$.

(b) ⇒ h). By 7.9.

(h) ⇒ (i). Follows from the following formula (easy to check in any category satisfying (A.0, 4))

(4) $\ker(u^Q(p)) = u^M(\ker(p))$,

and the anti-isomorphism (5.2.1)

(5) $\mathrm{Quo}(A) \xrightarrow{\ker} \mathrm{NrmSub}(A) \xrightarrow{\mathrm{cok}} \mathrm{Quo}(A)$

which transforms intersections of quotients into unions of normal subobjects.

(f) ⇒ (e). Let $e, f \in \mathrm{Prj}(A)$. We actually prove the equivalence of (e) and (f) in any category **A** with a regular involution

(6) $(e\&f)\&g = (efe)\, g\, (efe) = (efe)(ege)(efe) = e_P(f)\&e_P(g) = e_P(f\&g) = e(fgf)e = e\&(f\&g)$.

(e) ⇒ (f). Let $e, f \in \mathrm{Prj}(A')$ and take $g = a^{\#}a$, again in $\mathrm{Prj}(A')$

(7) $a_P(e\&f) = a(e\&f)a^{\#} = a(g(e\&f)g)a^{\#} = a_P(g\&(e\&f)) = a_P((g\&e)\&f) =$
$= a\,(a^{\#}a.e.a^{\#}a).f.(a^{\#}a.e.a^{\#}a)\,a^{\#} = (aea^{\#})(afa^{\#})(aea^{\#}) = a_P(e) \,\&\, a_P(f)$. □

8.2. Domination. Let **A** be a w-distributive RW-category: as every category provided with a regular involution and orthodox [0C.2], **A** is provided with a *canonical order*, or *domination*, $a \sqsubset b$ on parallel morphisms, characterised by the equivalent conditions

(1) $a = ab^{\#}a$,

(2) $a = aa^{\#}.b.a^{\#}a$,

(3) there exist idempotent endomorphisms e, f such that $a = fbe$,

(4) there exist projections e, f such that $a = fbe$.

This order is coherent with the composition and involution of **A**, and was shown in [G4] to yield a notion of "induced relations" coherent with the composition. The associated *canonical congruence* Φ of **A** ($a\Phi b$ iff $a \sqsubset b \sqsubset a$) is the finest congruence of **A** making **A**/Φ an *inverse category* [G2].

8.3. The inverse symmetrisation. Let $\mathbf{E}$ be a w-distributive w-exact category. By 8.1, $\mathrm{Rel}_W(\mathbf{E})$ is a w-distributive RW-category and the composed functor

$$(1) \qquad \mathbf{E} \xrightarrow{\mathrm{Sym}_W} \mathrm{Rel}_W(\mathbf{E}) \xrightarrow{P} \Theta\mathbf{E} = \mathrm{Rel}_W(\mathbf{E})/\Phi$$

is still an embedding [G6], called the *canonical inverse symmetrisation* (or *Θ-symmetrisation*) of **E**, and written

$$(2) \qquad \mathrm{Sym}_\Theta \colon \mathbf{E} \to \Theta\mathbf{E}.$$

Properties of Θ-symmetrisations are studied in [G6] (and in the following two papers of the same series), in the more general case of W-categories satisfying (0Q) and (0Q*) (8.1j).

We only recall ([G6], 1.16-18) that a functor $F\colon \mathbf{E} \to \mathbf{E}'$ between wd-exact categories has a (necessarily unique) extension to Θ-symmetrisations if and ondly if it preserves monos, conormal epis and the intersection of both

$$(3) \qquad \Theta F\colon \Theta\mathbf{E} \to \Theta\mathbf{E}'.$$

8.4. Theorem (Exactness and distributivity). (This theorem extends theorems 6.1, 6.3 of [G7] with a slightly different proof. In [G7], the notions of distributive union and partition in a semilattice are studied and applied to semilattices of projections in inverse categories.)

Let $\mathbf{E}$ be a w-distributive w-exact category and $\mathbf{K} = \Theta\mathbf{E}$ its canonical inverse symmetrisation. The short sequence

$$(1) \qquad \bullet \xrightarrow{m} A \xrightarrow{p} \bullet$$

is of order two (i.e. $pm = 0$) if and only if

$$(2) \qquad mm^\# \wedge p^\#p = 0 \ \text{ in } \ \mathrm{Prj}_\mathbf{K}(A),$$

while (1) is exact (i.e., $m \sim \ker(p)$) if and only if

$$(3) \qquad 1_A = mm^\# \vee p^\#p, \qquad \text{a } \textit{partition} \text{ in } \mathrm{Prj}_\mathbf{K}(A).$$

A functor $F\colon \mathbf{E} \to \mathbf{E}'$ between w-distributive w-exact categories is w-exact if it is a Θ-functor (8.3) and for each object A of $\mathbf{E}$ the mapping

$$(4) \qquad \mathrm{Prj}_{\Theta F}A\colon \mathrm{Prj}_{\Theta\mathbf{E}}\, \mathbf{A} \to \mathrm{Prj}_{\Theta\mathbf{E}'}(Fa), \qquad e \mapsto \Theta F(e),$$

preserves finite distributive unions (in particular, the null projection ω_A as union of the empty family).

Proof. The first assertion is trivial, since $pm = 0$ in $\mathbf{E}$ if and only if $(p^\#p)(mm^\#) = 0$ in $\mathbf{K}$, if and only if (2) holds.

Assume now that $pm = 0$. If $m \sim \ker(p)$, then $p^\#p = 1|(mm^\#)$ in $K = \Theta E$ and (3) holds because of 1.11. Conversely, assume (3) and take $h = \ker(p) \succ m$

$$(5) \qquad hh^\# = hh^\# \wedge 1 = ((hh^\#)\wedge(mm^\#))\vee((hh^\#)\wedge(p^\#p)) = mm^\# \vee \omega_A = mm^\# ;$$

thus $m \sim h = \ker(p)$.

Now, if F: **E** → **E**' satisfies our conditions, it preserves monos, conormal epis and intersection of monos; moreover, if (1) is exact in **E** then the partition (3) is transformed by ΘF into a partition of 1_{FA} in $\mathrm{Prj}_{\Theta\mathbf{E}'}(FA)$; this means that $Fm \sim \ker_{\mathbf{E}'}(Fp)$. □

8.5. Proposition (Subcategories and domination). Let **A** be a *w-distributive* RW-category.

(a) If **B** is a sub-RW-category of **A**, the closed sub-RW-category **B**' spanned by **B** (2.6) contains the same objects as **B** and the morphisms $a \in \mathbf{A}(B, B')$ satisfying the following conditions

(i) $a_R(X_B) \subset X_{B'}$, $a^R(X_{B'}) \subset X_B$,

(ii) $a \sqsubset b$, for some $b \in \mathbf{B}(B, B')$,

where $X_B = \mathrm{Rst}_{\mathbf{B}}(B)$ is the (w-distributive) w-lattice of restrictions of B in **B**.

Moreover, for all B in **B**

(1) $\mathrm{Rst}_{\mathbf{B}'}(B) = \mathrm{Rst}_{\mathbf{B}}(B) = X_B$,

(2) $\mathrm{Prj}_{\mathbf{B}'}(B) = \{e \in \mathrm{Prj}_{\mathbf{A}}(B) \mid \underline{n}e, \underline{d}e \in X_B\}$.

(b) If Δ is a subgraph of **A**, the Rst-full (hence closed) sub-RW-category **B** of **A** spanned by Δ contains the same objects as Δ and the morphisms $a \in \mathbf{A}(D, D')$ satisfying the following condition

(iii) $a \sqsubset b$, for some b in the involutive subcategory of **A** spanned by Δ.

Proof. We only verify (a), the second part being analogous. Let $\mathbf{B}_1$ be the subcategory of **A** described in our statement. $\mathbf{B}_1$ is clearly an involutive subcategory, closed with respect to numerators, ω- and Ω-projections, normality (because any projection $e \in \mathrm{Prj}_{\mathbf{A}}(B)$ is dominated by 1_B); this also proves that $\mathbf{B}_1$ satisfies properties (1) and (2).

Thus $\mathbf{B}' \subset \mathbf{B}_1$; conversely, if a: B → B' is in $\mathbf{B}_1$, by (b)

(3) $a = (aa^\#)b(a^\#a)$,

for some b in **B**; as $\mathbf{B}_1$ satisfies (2), $\underline{n}(aa^\#), \underline{d}(aa^\#) \in X_B = \mathrm{Rst}_{\mathbf{B}}(B)$: thus $aa^\#$ belongs to **B**'; analogously for $a^\#a$, which proves that $a \in \mathbf{B}'$. □

8.6. Q-morphisms. As a consequence, if F: Δ → **B** is a graph morphism with values in a *w-distributive* RW-category, then F is a q-morphism if and only if:

(a) F is bijective on the objects,

(b) F is Rst-full,

(c) for every b in **B** there is some b' in the involutive subcategory of **B** spanned by F(Δ) such that $b \sqsubset b'$.

Actually, assume that F satisfies (a) and (b) (every q-morphism does) and apply 8.5b to the subgraph F(Δ) of **B**.

In particular, if F: **A** → **B** is an RW-functor and **B** is w-distributive, then F is a weak quotient if and only if it satisfies (a), (b) and

(c') for any b in **B** there is some a in **A** such that $b \sqsubset Fa$.

8.7. Idempotent RW-categories. The RW-category **A** will be said to be *idempotent* whenever all its endomorphisms are so.

In such a case, for parallel morphisms a, b (by I.2.8)

(1) $a = b \;\Leftrightarrow\; (\underline{c}(a) = \underline{c}(b)$ and $\underline{i}(a) = \underline{i}(b))$,

$\Leftrightarrow\; (\underline{def}(a) = \underline{def}(b),\; \underline{ann}(a) = \underline{ann}(b),\; \underline{val}(a) = \underline{val}(b),\; \underline{ind}(a) = \underline{ind}(b))$.

Therefore an RW-functor $F: \mathbf{A} \to \mathbf{B}$ (where **A** is idempotent) is faithful if and only if it is Prj-faithful, if and only if it is Rst-faithful.

Every idempotent RW-category is trivially orthodox, i.e. *w-distributive* (8.1); it is also transfer by the above remark. Every idempotent Rst-finite RW-category is hom-finite, by 7.5.

Let now $F: \mathbf{A} \to \mathbf{B}$ be any RW-functor

(a) If F is a strict quotient and **A** is idempotent, so is **B**,

(b) if F is a weak quotient, **A** is idempotent and **B** is w-distributive then **B** too is idempotent,

(c) if F is faithful and **B** is idempotent so is **A**.

All these remarks are trivial, except (b) which is a consequence of 8.6: let $b: FA \to FA$ be some endomorphism of **B**; then $b \prec Fa$ for some $a \in \mathbf{A}(A, A)$, which is idempotent: thus $a \prec 1_A$ and $b \prec Fa \prec 1_{FA}$ is idempotent too.

8.8. Pre-idempotent w-exact categories. We say that the (component- wise) w-exact category **E** is *pre-idempotent* when $\mathbf{A} = \mathrm{Rel}_W(\mathbf{E})$ is idempotent. Other equivalent conditions are:

(a) for all parallel monos $s, t: H \to A$ in **A**, $t^\# s = 1_H$

(b) for every commutative diagram of **E**

(1)
$$\begin{array}{ccccc}
H & \overset{p}{\twoheadleftarrow} & \bullet & \overset{m}{\rightarrowtail} & A \\
\| & & \uparrow n' & & \| \\
 & & \bullet & & \\
\| & & \downarrow m' & & \| \\
H & \underset{q}{\twoheadleftarrow} & \bullet & \underset{n}{\rightarrowtail} & A
\end{array}$$

if (n', m') is the pullback of (m, n), then pn' and qm' are the same conormal epi.

Indeed, (a) is trivially equivalent to the idempotence of **A**. In order to prove the equivalence of (a) and (b), consider the parallel monorelations

(2) $s = mp^\#: H \to A$, $\qquad\qquad t = nq^\#: H \to A$;

form the pullback in (1) and let

(3) $u = pn'$, $\qquad v = qm'$.

Since

(4) $ts^\# = q(n^\# m)p^\# = (qm')(n^{\#\prime} p^\#) = vu^\#$,

if u = v is a conormal epi of **E**, $ts^\# = 1_H$; conversely if $ts^\# = 1$ then the proper morphisms u, v are epi in **A**, hence conormal epis of **E** and $u = 1_H u = vu^\# u \geq v$ implies u = v.

8.9. Universal distributive and idempotent categories. In order to present embedding theorems in the distributive or idempotent cases, we recall (in (a), (b), (c)) or introduce (in (d), (e), (f)) the following categories.

(a) The *distributive exact* category $\mathcal{I}$ of (small) *sets and partial bijections* (I.6.1) and its distributive RE-categories of relations, $\mathrm{Rel}(\mathcal{I})$.

(b) The distributive exact category $\mathcal{J} = \mathrm{Mle}(\mathcal{I}) = \mathrm{Dle}(\mathcal{I})$ (I.6.2); its objects are *semitopological spaces*, i.e. sets equipped with a lattice of *closed* subsets, while a map $u\colon S \to T$ is a partial homeomorphism, from some *open subspace* S_0 of S (a complement of a closed subset, provided with the induced structure) to some *closed subspace* T_0 of T.

(c) The *pre-idempotent exact* category $\mathcal{I}_0$ of sets and common parts, or partial identities (I.6.4), and its idempotent RE-category of relations, $\mathrm{Rel}(\mathcal{I}_0)$.

(d) The w-distributive w-exact category of *w-spaces and partial w-homeomorphisms*

(1) $\mathcal{J}^w = \mathrm{wMe}(\mathcal{I}) = \mathrm{wDe}(\mathcal{I})$.

According to 7.7, an object, or *w-space*, is a set S provided with a (necessarily w-distributive) sub-w-lattice X of $\mathrm{Sub}_{\mathcal{I}}(S) = \mathcal{P}(S)$ (see 11.6; note that the normality relation $\triangleleft$ of X is generally finer than the order relation $\subset$ induced by $\mathcal{P}(S)$); as in (b), the elements of X are called *closed* subsets of S. A morphism (or *partial w-homeomorphism*) $u\colon S \to T$ is a *homeomorphism* of w-spaces $u_0\colon S_0 \to T_0$ from some *open subspace* $S_0 = S \backslash S'$ of S (a complement of a closed *normal* subset $S' \triangleleft S$, provided with the induced structure) to some *closed subspace* T_0 of T.

We also introduce the *w-distributive* RW-category

(2) $\mathcal{L}^w = \mathrm{Rel}(\mathcal{J}^w) = \mathrm{wMe}(\mathrm{Rel}\mathcal{J}) = \mathrm{wDe}(\mathrm{Rel}\mathcal{J})$.

(e) The pre-idempotent w-exact category

(3) $\mathcal{J}_0{}^w = \mathrm{wMe}(\mathcal{J}_0) = \mathrm{wDe}(\mathcal{J}_0)$,

whose morphisms $L\colon S \to T$ are given by common subspaces L of S and T, where $L = S\backslash S'$ (with $S' \triangleleft S$) is closed in T. It determines the *idempotent* RW-category

(4) $\mathcal{L}_0{}^w = \mathrm{Rel}(\mathcal{J}_0{}^w) = \mathrm{wMe}(\mathrm{Rel}\mathcal{I}_0) = \mathrm{wDe}(\mathrm{Rel}\mathcal{I}_0)$.

(f) Last we consider the hom-finite (and Sub-finite or Rst-finite) full subcategories determined by *finite* sets or *finite* w-spaces: $\mathcal{I}^f$, $\mathcal{I}_0{}^f$, $\mathcal{J}^{wf}$, $\mathcal{L}^{wf}$, $\mathcal{J}_0{}^{wf}$, $\mathcal{L}_0{}^{wf}$.

8.10. Concrete representations for w-distributive w-exact categories. Extending [G9], § 4.8, by means of these results one proves that every w-distributive w-exact category **E** (resp. every w-distributive RW-category **A**) has w-exact (resp. RW-) *spectrum* functors

(1) $\mathrm{Spc}_{\mathbf{E}}\colon \mathbf{E} \to \mathcal{I}$, (resp. $\mathrm{Spc}_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{Rel}\mathcal{I}$)

(2) $\mathrm{Spc}_{\mathbf{E}}{}^\#\colon \mathbf{E} \to \mathcal{J}^w$, (resp. $\mathrm{Spc}_{\mathbf{A}}{}^\#\colon \mathbf{A} \to \mathcal{L}^w$)

the second being Sub-full and $\triangleleft$-closed (resp. Rst-full); these functors are *embeddings when* **E** *is transfer*.

Analogously, by extending [G9], 4.9, we get that every *small* wd-exact category **E** (resp. every *small* w-distributive RW-category **A**) has *extended spectrum* w-exact (resp. RW-) embeddings

(3) $Spc_{\mathbf{E}}^{\wedge}: \mathbf{E} \to \mathcal{I}$, (resp. $Spc_{\mathbf{A}}^{\wedge}: \mathbf{A} \to Rel\mathcal{I}$)

(4) $Spc_{\mathbf{E}}^{\#\wedge}: \mathbf{E} \to \mathcal{J}^{w}$, (resp. $Spc_{\mathbf{A}}^{\#\wedge}: \mathbf{A} \to \mathcal{L}^{w}$)

the second being Sub-full and $\triangleleft$-closed (resp. Rst-full). These embeddings take values in $\mathcal{I}^{f}$ (resp. $Rel(\mathcal{I}^{f})$, $\mathcal{L}^{wf}$) when **E** is finite.

Finally, by extending [G9], § 5.7, we have that every pre-idempotent w-exact category **E** has a w-exact embedding in $\mathcal{I}_0$ and a closed Sub-full one in $\mathcal{J}_0{}^{w}$, while every idempotent RW-category **A** has an RW-embedding in $Rel\mathcal{I}_0$ and a Rst-full one in $\mathcal{L}_0{}^{w} = Rel(\mathcal{J}_0{}^{w})$.

9. R$\overline{\text{W}}$-categories and $\overline{\text{w}}$-exact categories

We consider here the lattice property determined by $\overline{\text{w}}$m-lattices (11.4). The corresponding $\overline{\text{w}}$-exact categories are characterised by the stability of normal subobjects with respect to intersection.

9.1. Definition. The RW-category **A** will be said to be $\overline{\text{w}}$-*latticed*, or an R$\overline{\text{W}}$-*category*, when all its restriction-sets Rst(A) are $\overline{\text{w}}$m-lattices (11.4), i.e. satisfy a stronger variant of (wl.2)

($\overline{\text{w}}$l.2) if $y \triangleleft x$ and $t \triangleleft z$ then $yt \triangleleft xz$.

As a consequence, the set Nrm(Rst(A)) of normal restrictions of the object A is a sub-$\overline{\text{w}}$-lattice of Rst(A) and a modular lattice in its own right. Notice, however, that a_R-mappings do not preserve normal restrictions, generally: there is *no* functor NrmRst: **A** → **Mlr**.

9.2. Proposition. R$\overline{\text{W}}$-categories are also characterised by the following axiom, to be added to (RW.0-3):

(RW.4) each projection set $Prj_{\mathbf{A}}(A)$ is a $\wedge$-semilattice with respect to $\leq$.

Proof. Actually, if **A** is $\overline{\text{w}}$-latticed and $e = x/y$, $f = z/t$ are projections of **A**, by (RW3.b)

(1) $(x \wedge z)/(y \wedge t) = e \wedge f$.

Conversely, for $y \triangleleft x$ and $t \triangleleft z$ in $Rst_{\mathbf{A}}(A)$, consider $e = x/y$, $f = z/t$ and $e \wedge f = u/v$. Again by (RW.3b), $u \leq x \wedge z$ and $v \leq y \wedge t$; moreover

(2) $x \wedge z \leq e \wedge f = u/v$,

(3) $(y \wedge t)/(y \wedge t) \leq e \wedge f = u/v$,

which proves that $x \wedge z \leq u$ and $y \wedge t \leq v$.

Thus $x \wedge z = u = \underline{n}(e \wedge f)$, $y \wedge t = v = \underline{d}(e \wedge f)$ and the latter is normal in the former. □

9.3. The 2-category $\mathbf{R\overline{W}}$. Let $\mathbf{R\overline{W}}$ be the full sub-2-category of **RW** of $R\overline{W}$-categories; it is strictly 2-complete. Given an RW-functor (in the original **RW**)

(1) $H: \mathbf{C} \to \mathbf{D}$,

it is easy to see that

(a) if H is a strict quotient and **C** is $R\overline{W}$, so is **D**,

(b) if H is a closed, Rst-faithful functor and **D** is $R\overline{W}$, so is **C**.

It follows that *both* the ordinary and the canonical factorisation (Section 3) of an $R\overline{W}$-functor $F: \mathbf{A} \to \mathbf{B}$ have "central" category in $\mathbf{R\overline{W}}$, i.e. live entirely there.

A sub-$R\overline{W}$-category $\mathbf{A}_0$ of **A** is a sub-RW-category which is $\overline{w}$-latticed; equivalently, by 9.2, it has to be stable for intersection of parallel projections with respect to $\leq$.

9.4. The projection complete case. A projection complete RW-category **A** is $\overline{w}$-latticed if and only if its restriction-sets Rst(A) satisfy the following condition, generally weaker than ($\overline{w}$l.2)

(a) if $x, y \lhd 1$ then $xy \lhd 1$.

Actually, if (a) holds and $y \lhd x$, $t \lhd z$ in Rst(A), split the restriction xz

(1) $A \overset{m}{\longleftarrow\!\!<} M \overset{m}{>\!\!\longrightarrow} A \qquad\qquad xz = mm^{\#}$,

and consider in Rst(M)

(2) $y_0 = m^R(y) = m^P(y), \qquad\qquad t_0 = m^R(t) = m^P(t)$.

Since

(3) $m^R(x) \geq m^R(xz) = m^R m_R(1) = 1_M$,

we have $y_0 \lhd m^R(x) = 1_M$. Analogously $t_0 \lhd m^R(z) = 1_M$, and $y_0 t_0 \lhd 1_M$. Now

(4) $m_R(y_0) = m_R m^R(y) = (y \wedge xz) \vee \omega = yz$,

(5) $m_R(y_0 t_0) = m_R(m^R(t) \wedge y_0) = (t \wedge m_R(y_0)) \vee m_R(\omega) = t \wedge yz = yt$,

and $yt \lhd m_R(1_M) = mm^{\#} = xz$.

9.5. $\overline{W}$-exact categories. Say $\overline{w}$-exact any w-exact category **E** satisfying the following properties, equivalent by 9.4 and the W-symmetrisation theorem III (5.5):

(a) $Rel_W(\mathbf{E})$ is an $R\overline{W}$-category,

(b) all w-lattices $Sub_{\mathbf{E}}(A)$ are $\overline{w}$-lattices,

(c) intersection of normal subobjects is normal.

Of course the W-symmetrisation theorem and its consequences have thus a $\overline{W}$-version.

9.6. The $\overline{w}$-expansion. Every RW-category **A** has an associated $\overline{w}$-*expansion* (7.5-6) $\overline{w}$Me**A**, satisfying the universal problem determined by the full subcategory $\overline{w}$**Mlr** of w**Mlr**, determined by

$\bar{w}$m-lattices. Similarly we have a $\bar{w}$d-expansion $\bar{w}$De**A** determined by $\bar{w}$d-lattices. If **A** is projection complete, so are these expansions (by the last point in 7.5).

Analogously for w-exact categories.

9.7. The universal distributive and idempotent RW-categories. The *exact* categories $\mathfrak{I}$ and $\mathfrak{I}_0$ (8.10), being normal, are also $\bar{w}$-latticed. They produce

(a) the *w-distributive* $\bar{w}$*-exact* category,

(1) $\mathcal{J}^{\bar{w}}$ = $\bar{w}$Me$\mathfrak{I}$ = $\bar{w}$De$\mathfrak{I}$,

of $\bar{w}$-spaces and partial open-closed $\bar{w}$-homeomorphisms, having a description similar to $\mathcal{J}^{w}$ in 8.9d, together with the w-distributive R$\bar{W}$-category

(2) $\mathcal{L}^{\bar{w}}$ = Rel$\mathcal{J}^{\bar{w}}$ = $\bar{w}$Me(Rel$\mathfrak{I}$) = $\bar{w}$De(Rel$\mathfrak{I}$);

(b) the *pre-idempotent* $\bar{w}$*-exact* category

(3) $\mathcal{J}_0{}^{\bar{w}}$ = $\bar{w}$Me($\mathfrak{I}_0$) = $\bar{w}$De($\mathfrak{I}_0$),

of $\bar{w}$*-spaces and open-closed common parts* (see 8.10d), together with the idempotent RW-category

(4) $\mathcal{L}_0{}^{\grave{w}}$ = Rel($\mathcal{J}_0{}^{\grave{w}}$) = $\bar{w}$Me(Rel$\mathfrak{I}_0$) = $\bar{w}$De(Rel$\mathfrak{I}_0$).

The embeddings considered in 8.10 have analogous ones here for R$\bar{W}$-categories and $\bar{w}$-exact categories.

10. The dual and selfdual cases

We consider here briefly the dual case of RW^0-categories and w*-exact categories. The selfdual case, of course, leads to RE-categories and Puppe-exact categories.

10.1. Order duality. For an RO-category **A** = (**A**, $^\#$, ≤) we are interested in its *order opposite* RO-category

(1) $\mathbf{A}^0$ = (**A**, $^\#$, ≥).

(Owing to its 2-categorical structure, given by the order ≤, **A** has three "opposite" RO-categories; but the other two, **A*** = (**A***, $^{\#}$*, ≤) and $\mathbf{A}^0$* = (**A***, $^{\#}$*, ≥) are less interesting since the former is RO-isomorphic to **A** and the latter to $\mathbf{A}^0$, via a ↦ $a^\#$.)

The order duality turns restrictions into corestrictions, proper morphisms into *coproper* morphisms and ω-projections into Ω-projections; moreover there is a natural isomorphism

(2) (Prp**A**)* → Prp($\mathbf{A}^0$), u ↦ $u^\#$,

showing that the order duality of RO-categories corresponds to the usual duality of categories, via the 2-functor Prp: **RO** → **CAT**.

10.2. RW^0-categories. Accordingly, we say that the triple **A** = (**A**, $^{\#}$, $\leq$) is an *RW^0-category* if $\mathbf{A}^0$ = (**A**, $^{\#}$, $\geq$) is an RW-category.

In other words **A** is to satisfy the selfdual axioms (RW.0, 2) and

(RW^0.1) for every projection e there exists exactly one corestriction $\underline{n}^0e$ (the *conumerator* of e) such that $e \prec \underline{n}^0 e \geq e$; the corestriction $\underline{d}^0 e = \underline{n}^0(e\Omega e) \prec \underline{n}^0 e$ will be called the *codenominator* of e;

(RW^0.3) for all parallel projections e, f

(a) $e \prec f \Leftrightarrow (\underline{n}^0 e \prec \underline{n}^0 f$ and $\underline{d}^0 e \succ \underline{d}^0 f)$,

(b) $e \geq f \Leftrightarrow (\underline{n}^0 e \prec \underline{n}^0 f$ and $\underline{d}^0 e \prec \underline{d}^0 f)$.

The order duality takes conumerators and codenominators of **A** into numerators and denominators of $\mathbf{A}^0$. Note that $\underline{n}^0 e$ coincides with the ^{c}-denominator recalled in 1.1.

10.3. Corestrictions and conormality. For each object A in the RW^0-category **A** the set Crs(A) of its corestrictions will be mainly ordered by $\prec$ (opposite to $\leq$).

If $x, y \in$ Crs(A) we say that $y \lhd^0 x$ (y is *conormal* in x) whenever there exists a (unique) projection e of A such that $y = \underline{d}^0 e$ and $x = \underline{n}^0 e$ (iff $y \lhd x$ in $\mathbf{A}^0$); we write $e = x/y$.

10.4. W*-exact categories. We say that the category **E** is *w*-exact* if its opposite category **E*** is *w-exact*; in particular the *canonical factorisation* in **E** is by epimorphisms and normal monos. A classical example is the category of pointed sets.

10.5. The weak adjunction. It is now straightforward to introduce the 2-complete 2-category $\mathbf{RW}^0$ of RW^0-categories, the 2-category **WE*** of w*-exact categories and dualise the equivalence and weak-adjunction relations considered in Section 5.

10.6. Transfer of corestrictions. Every RW^0-category **A** has a *transfer* RW^0-functor

(1) $\mathrm{Crs}_{\mathbf{A}} = (\mathrm{Rst}_{\mathbf{A}^0})^0 \colon \mathbf{A} \to \mathrm{w}\mathbf{Mlr}^0 = (\mathrm{w}\mathbf{Mlr}, {}^{\#}, \geq)$,

assigning to each object A the wm-lattice $(\mathrm{Crs}_{\mathbf{A}}(A), \prec, \lhd^0)$ and to each morphism $a\colon A' \to A''$ the wm-relation

(2) $\mathrm{Crs}_{\mathbf{A}}(a) = (a_C, a^C)$,

$a_C\colon \mathrm{Crs}(A') \to \mathrm{Crs}(A'')$, $\qquad x \mapsto \underline{n}^0(axa^{\#})$,

$a^C\colon \mathrm{Crs}(A'') \to \mathrm{Crs}(A')$, $\qquad y \mapsto \underline{n}^0(a^{\#}xa)$.

Note that $a \leq b$ gives $\mathrm{Crs}(a) \geq \mathrm{Crs}(b)$.

10.7. Transfer of quotients. Analogously every w*-exact category has a *transfer* w*-exact functor (preserving epis, their intersections and short exact sequences)

(1) $\mathrm{Quo}\colon \mathbf{E} \to \mathrm{w}\mathbf{Mlc}^*$.

To every **E**-object A, we associate the wm-lattice of its quotients $(\mathrm{Quo}(A), \prec, \lhd^*)$, where $y \lhd^* x$ (y is conormal in x) means that there is a conormal epi p such that $px = y$; and to every morphism $u: A' \to A''$ we associate the *co-connection* (u_Q, u^Q), with $u^Q \dashv u_Q$

(2) $\mathrm{Quo}(u) = (u_Q, u^Q)$,

(3) $u_Q: \mathrm{Quo}(A') \to \mathrm{Quo}(A'')$ (constructed by push-out),

(4) $u^Q: \mathrm{Quo}(A'') \to \mathrm{Quo}(A')$ (constructed by canonical factorisation),

satisfying

(5) u_Q and u^Q preserve $\prec$ and $\lhd^*$; moreover $u_Q(1) = 1$ and $u^Q(0) = 0$,

(6) $u^Q u_Q(x) = x \wedge u^Q(1)$, for all $x \in \mathrm{Quo}(A')$,

(7) $u_Q u^Q(y) = y \vee u_Q(0)$, for all $y \in \mathrm{Quo}(A'')$.

10.8. The self-dual case. Finally, the following conditions on a RO-category **A** are equivalent:

(a) **A** is an RE-category,

(b) **A** is both an RW and an RW^0-category,

(c) **A** is a normal RW-category,

(d) **A** is a conormal RW^0-category (i.e., every corestriction $x \in \mathrm{Crs}A$ is conormal in 1_A, for all objects A).

Actually (a) is equivalent to (c) by 1.8, and to (d) by order duality; thus (a) implies (b), and the converse is trivial since (RE.1) is the conjunction of (RW.1) and (RW^0.1).

Analogously the following conditions on the category **E** are equivalent:

(a') **E** is exact,

(b') **E** is both w-exact and w*-exact,

(c') **E** is w-exact and *normal* (i.e., every subobject is normal),

(d') **E** is w*-exact and *conormal* (i.e., every quotient is conormal).

11. Appendix: w-lattices and homomorphisms

We introduce here the w-lattices and their homomorphisms.

11.1. Definition. A *w-lattice* (or *weak lattice*, or *normolattice*) is a triple $X = (X, \leq, \lhd)$ such that (for all $x, y, z, t \in X$)

(wl.0) $(X, \leq)$ is a meet-semilattice with 0 and 1;

(wl.1) $\lhd$ is a binary relation on X (called *normality*); $0 \lhd 1$; $x \lhd x$; $x \lhd y \Rightarrow x \leq y$;

(wl.2) if $x \lhd y$ then $x \wedge z \lhd y \wedge z$;

(wl.3) if $x \lhd t$ and $y \leq t$, then $x \vee y$ exists;

(wl.4) if $x \lhd t$ and $y \lhd z \leq t$, then $x \vee y \lhd x \vee z$.

Note that X, generally, is not a lattice; also when it is, the only *structural* joins (used below to define homomorphisms) are the $\lhd$-*joins* considered in (wl.3). The elements $x \lhd 1$ will be said to be *normal*; they form an ordered subset Nrm(X), which is a *join-semilattice*; note that the induced normality relation coincides with $\leq$, by (wl.2).

11.2. Weak modularity and distributivity. A w-lattice X is said to be *w-modular*, or a *wm-lattice* if it satisfies the following axioms:

(wm.1) if $y \leq z$, $x \lhd t$, $y \leq t$, then $(x \vee y) \wedge z = (x \wedge z) \vee y$,

(wm.2) if $x \leq z$, $x \lhd t$, $y \leq t$, then $(x \vee y) \wedge z = x \vee (y \wedge z)$.

It is said to be *w-distributive*, or a *wd-lattice*, if it satisfies the stronger axioms:

(wd.1) if $x \lhd t$, $y \leq t$, then $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,

(wd.2) if $x \lhd t$, $y \leq t$, $z \leq t$, then $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

To deduce (wm.2) from (wd.2), take $z' = z \wedge t \leq t$.

11.3. Normal w-lattices. Say that a w-lattice X is *normal* when every element is so, i.e. X = Nrm(X): then $(X, \leq)$ is a lattice and $\lhd$ coincides with $\leq$. Conversely every lattice X determines a normal w-lattice $W(X) = (X, \leq, \leq)$. The modular and distributive cases proceed in the same way.

11.4. $\bar{\text{w}}$-lattices. The subset Nrm(X) of the w-lattice X is not meet-stable, generally. For instance, consider the following wd-lattice (the trivial normality relations are understood: $0 \lhd x \lhd x$, for all elements x)

(1) $0 < a \wedge b < \begin{matrix} a \\ b \end{matrix} < 1$, $\qquad 0 \lhd a \wedge b \lhd \begin{matrix} a \\ b \end{matrix} \lhd 1$,

where $x \wedge y$ is not normal. We say that X is a $\bar{\text{w}}$-*lattice* if it satisfies the following condition, stronger than (wl.2)

($\bar{\text{w}}$l.2) $y \lhd x$ and $t \lhd z$ $.\Rightarrow.$ $y \wedge t \lhd x \wedge z$.

Analogously we consider $\bar{\text{w}}$m-lattices and $\bar{\text{w}}$d-lattices. If X is a $\bar{\text{w}}$- (resp. $\bar{\text{w}}$m-, $\bar{\text{w}}$d-) lattice then Nrm(X) is a lattice (resp. a modular, distributive one).

11.5. Homomorphisms. A *homomorphism* of w-lattices $h: X \to X'$ has to preserve 0, 1, (binary) intersection, the normality relation $\lhd$ and $\lhd$-unions.

Thus we have the (ordered) category w**Lth** of (small) *w-lattices and homomorphisms*. Its isomorphisms are the bijective mappings preserving and reflecting the relations $\leq$ and $\lhd$. The category w**Lth** is clearly complete, with limits preserved by the forgetful functor into **Set**.

The functor W: **Lth** $\to$ w**Lth**, defined on the objects in 11.3, embeds the category of lattices and homomorphisms as a subcategory of w**Lth**.

In the same way the full subcategories w**Mlh** and w**Dlh** of w**Lth** given by wm- and wd-lattices respectively contain, as a full reflective subcategory, the categories **Mlh** and **Dlh** of modular and distributive lattices.

11.6. Sub-w-lattices. The monos of the categories w**Lth**, w**Mlh** and w**Dlh** are their injective homomorphisms, in the usual set-theoretic sense. This follows from the existence of the free object on one generator x, which in all three cases is

(1) $\{0, x, 1\}$, $\quad 0 < x < 1, \quad 0 \lhd 0 \lhd 1 \lhd 1.$

Therefore we define a *sub-w-lattice* of X to be a subset X_0 provided with *the* induced order $\leq$ and *some* relation $\lhd_0$ so that, for all $x, y, z, t \in X_0$

(a) $0, 1, x \wedge y \in X_0$,

(b) $0 \lhd_0 1, \quad x \lhd_0 x, \quad y \lhd_0 x \Rightarrow y \lhd x$,

(c) $x \lhd_0 y \Rightarrow x \wedge z \lhd_0 y \wedge z$,

(d) $x \lhd_0 t$ and $y \leq t$ $.\Rightarrow.$ $x \vee y \in Y_0$,

(e) $x \lhd_0 t$ and $y \lhd_0 z \leq t$ $.\Rightarrow.$ $x \vee y \lhd_0 x \vee z$.

Then $X_0 = (X_0, \leq, \lhd_0)$ is a w-lattice, w-modular or w-distributive if X is so, and the embedding $X_0 \to X$ is a monomorphism of w**Lth**. However, if X is a $\bar{\text{w}}$-lattice or is normal (i.e., a lattice), X_0 need *not* be so.

11.7. Closed homomorphisms. A homomorphism of w-lattices $h: X \to X'$ is said to be *closed* (or $\lhd$-closed) whenever

(1) if $x, x' \in X$ and $h(x) \lhd h(x')$ in Y, then there exist $a \lhd a'$ in X such that $h(a) = h(x)$ and $h(a') = h(x')$.

A sub-w-lattice X_0 of X is said to be *closed* in X if its embedding is so; in other words, if its normality relation is induced by the one of X, or - equivalently - if it is a regular subobject (an equaliser).

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