

A simple method using Morozov's discrepancy principle for solving inverse scattering problems[†]

David Colton[§], Michele Piana[§] and Roland Potthast[‡]

[§] Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA

[‡] Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany

Abstract. This paper is a continuation of [?] in which a simple inversion scheme was given for inverse scattering problems in the resonance region which is easy to implement and is relatively independent of the geometry and physical properties of the scatterer. The purpose of the paper is to give new and improved theorems establishing the mathematical basis of this method and to show how noisy data can be treated using Morozov's discrepancy principle where the regularization parameter is a function of an auxiliary parameter appearing in the inversion scheme.

1. Introduction

In a recent paper in this journal [?], Colton and Kirsch introduced a new method for solving inverse scattering problems in the resonance region. The attractive features of this method are that no low frequency or high frequency approximations are made, its implementation only requires the solution of a linear integral equation, and it is not required to know whether or not the medium is penetrable or impenetrable. Furthermore, if the medium is impenetrable it is not required to know the type of boundary condition the total field satisfies on the boundary whereas if the medium is penetrable the method determines the support of the medium rather than the actual values of the index of refraction. In many cases in medical imaging and non-destructive testing the latter information is sufficient and the method of [?] thus avoid the need to solve a time consuming nonlinear optimization problem for the index of refraction (c.f. [?]).

[†] This research was supported in part by grants from the Air Force Office of Scientific Research, the Consiglio Nazionale delle Ricerche and the Deutsche Forschungsgemeinschaft.

To describe the method of Colton and Kirsch, consider the two dimensional scattering problem of determining u from the equation

$$\Delta_2 u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus D \quad (1.1)$$

$$u(x) = e^{ikx \cdot d} + u^s(x) \quad (1.2)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \quad (1.3)$$

where $k > 0$ is the wave number, $x \in \mathbb{R}^2$, d is a unit vector and $r = |x|$. D is assumed to be a bounded domain with smooth boundary ∂D and unit outward normal ν . If D is a penetrable medium, $u \in C^2(\mathbb{R}^2)$ and

$$\Delta_2 u + k^2 n(x)u = 0 \quad \text{in } \mathbb{R}^2 \quad (1.4)$$

where the piecewise smooth function n is the index of refraction, whereas if D is an impenetrable obstacle u continuously assumes one of the following boundary condition on ∂D :

$$u = 0 \quad \text{on } \partial D \quad (1.5)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \quad (1.6)$$

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D \quad (1.7)$$

where λ is a positive constant. Under these conditions we have that [?]

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}; d) + O(r^{-3/2}) \quad (1.8)$$

as r tends to infinity where $\hat{x} = x/|x|$ and u_∞ is the *far field pattern* of the scattered wave u^s .

The inverse problem we are interested in is to determine D from only a knowledge of $u_\infty(\hat{x}; d)$ for $\hat{x}, d \in \Omega := \{x \in \mathbb{R}^2, |x| = 1\}$. In order to do this, in [?] the *far field equation*

$$\int_{\Omega} u_\infty(\hat{x}; d) g(d) ds(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot y} \quad (1.9)$$

was introduced and it was shown that there is a function $g = g(\cdot; y)$ that satisfies (??) arbitrarily closely in $L^2(\Omega)$ such that if v_g is the *Herglotz wave function*

$$v_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d) \quad (1.10)$$

with kernel g then $\|g\|_{L^2(\Omega)}$ and $\|v_g\|_{L^2(D)}$ become unbounded as y tends to ∂D . Numerical examples were then given showing that this g could be determined from (??) and $\|g\|_{L^2(\Omega)}$ indeed became large as y tended to ∂D . However, there were a number of gaps and omissions in this presentation, in particular the following:

- (i) For the case when D is an impenetrable obstacle the fact that $\|g\|_{L^2(\Omega)}$ becomes unbounded as y tends to ∂D was only shown for the case of a simply connected obstacle although numerical examples were given for disconnected obstacles.
- (ii) For the case when D is a penetrable medium, there was a technical gap in the proof that $\|g\|_{L^2(\Omega)}$ becomes unbounded as y tends to ∂D (if v, w is a solution of the interior transmission problem discussed in [?], then it only follows from [?] that $v - w$ depends continuously on the boundary data, not v and w separately).
- (iii) The numerical examples in [?] were computed without adding any noise to the far field pattern u_∞ and simply discretizing the far field equation (??). Since $\|g\|_{L^2(\Omega)}$ becomes unbounded as y tends to ∂D , noise on u_∞ can seriously affect the computation of g from (??). In particular, in the presence of noise the regularization parameter appearing in the regularization method chosen to solve (??) must depend on y .

The purpose of the paper is to address the above three problems. In particular, we shall show that $\|g\|_{L^2(\Omega)}$ becomes unbounded as y tends to ∂D for the case of disconnected scattering obstacles where u satisfies one of the boundary conditions (??)-(??) on the boundary of each component (possibly different conditions on each boundary), provide a new proof that $\|g\|_{L^2(\Omega)}$ becomes unbounded as y tends to ∂D for the case of a penetrable, absorbing inhomogeneous medium and finally show how a regularized solution of (??) can be found in the case of noisy data. This last problem will be addressed through the use of the generalized Morozov discrepancy principle [?] for the determination of the regularization parameter $\alpha = \alpha(y)$: if $Az = f$ describes a general linear inverse problem, the generalized Morozov discrepancy principle provides an optimum value of the regularization parameter when the noise is on the operator A rather than on the right hand side f as is more typically the case (c.f. [?]). A rather surprising result of our numerical analysis of this approach is that the level curves of the regularization parameter are usually a better indication of the location of D than the level curves of the L^2 -norm of the regularized solution. Hence we have the unusual fact that in these cases the regularization parameter is the main object of interest rather than the regularized solution of the integral equation itself!

2. Analysis of the far field equation: the case of obstacle scattering

In this section we will address the first problem stated at the end of the previous section. Our analysis is based on the fact that due to the special nature of the far field equation (??) its solvability can be reduced to the study of a boundary value problem for a partial differential equation. In particular, assume u satisfies one of the boundary

conditions (??)-(??) on ∂D . Then, since the right hand side of the far field equation is the far field pattern for the fundamental solution Φ defined by

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \quad (2.1)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, it follows from Rellich's lemma [?] that if g satisfies the far field equation (??) then

$$\int_{\Omega} u^s(x; d) g(d) ds(d) = \Phi(x, y) \quad , \quad x \in \mathbb{R}^2 \setminus D \quad . \quad (2.2)$$

From the boundary conditions (??), (??), (??) we can now conclude that if v_g is the Herglotz wave function with kernel g defined by (??) then $v = v_g$ satisfies one of the following boundary conditions on ∂D :

$$v(x) + \Phi(x, y) = 0 \quad \text{on} \quad \partial D \quad (2.3)$$

$$\frac{\partial}{\partial \nu}(v(x) + \Phi(x, y)) = 0 \quad \text{on} \quad \partial D \quad (2.4)$$

$$\left(\frac{\partial}{\partial \nu} + ik\lambda\right)(v(x) + \Phi(x, y)) = 0 \quad \text{on} \quad \partial D \quad . \quad (2.5)$$

In particular, the far field equation (??) is solvable in $L^2(\Omega)$ if and only if the solution v of the *interior* problem

$$\Delta_2 v + k^2 v = 0 \quad \text{in} \quad D \quad (2.6)$$

$$v \text{ satisfies (2.3), (2.4) or (2.5) \quad on } \partial D \quad (2.7)$$

is a Herglotz wave function $v = v_g$.

Only in very exceptional circumstances is the solution of (??), (??) a Herglotz wave function. However, assume that D is a finite number of disconnected domains D_1, D_2, \dots, D_N . Then, as pointed out in [?], in each D_n , $n = 1, \dots, N$, we can approximate the solution v of (??), (??) in $C^1(\overline{D}_n)$ by a Herglotz wave function v_{g_n} with kernel g_n , assuming that k^2 is not a Dirichlet or Neumann eigenvalue for D_n . Furthermore, since u_∞ depends continuously on the boundary data, an approximate solution of the far field equation in $L^2(\Omega)$ can be found if we can find a Herglotz wave function v_g with kernel g such that v_g approximates v_{g_n} in $C^1(\overline{D}_n)$ for $n = 1, \dots, N$. We are now in a position to prove the following theorem which generalizes the result of [?] for simply connected domains to the case of a finite number of disconnected obstacles where u satisfies one of the boundary conditions (??)-(??) on each obstacle (Note that the fact that v_g blows up as y tends to ∂D shows that the blowing up of g as y tends to ∂D is not only due to increasingly wild oscillations of the high frequency components of g as y tends to ∂D).

Theorem 2.1: Assume that D consists of a finite number of disconnected domains D_1, \dots, D_N , k^2 is neither a Dirichlet or Neumann eigenvalue for D_n , $n = 1, \dots, N$, and that on ∂D_n the solution u of (??)-(??) satisfies one of the boundary conditions (??)-(??). Then for every $\epsilon > 0$ and $y \in D$ there exists a solution $g = g(\cdot, y) \in L^2(\Omega)$ of the inequality

$$\left\| \int_{\Omega} u_{\infty}(\hat{x}, d)g(d)ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot y} \right\|_{L^2(\Omega)} < \epsilon$$

such that

$$\lim_{y \rightarrow \partial D} \|g\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{y \rightarrow \partial D} \|v_g\|_{L^2(D)} = \infty$$

where v_g is the Herglotz wave function with kernel g .

Proof. Note that if we can find a v_g as described in the remarks preceding the theorem, then, since the solution v of (??), (??) becomes unbounded in $C^1(\overline{D})$ as y tends to ∂D , so must v_g and hence $\|g\|_{L^2(\Omega)}$. Hence all we must do is to show the existence of v_g . To this end, let D_n^* , $n = 1, \dots, N$, be disjoint domains such that $D_n^* \supset D_n$, ∂D_n^* is smooth and k^2 is not a Dirichlet eigenvalue for D_n^* , $n = 1, \dots, N$. This is possible to do since D_n , $n = 1, \dots, N$, are disjoint and the Dirichlet eigenvalues are monotonically dependent on the domain. By using Green's function to represent v_{g_n} in D_n^* and recalling the regularity properties of solutions to elliptic partial differential equations, it now suffices to show that each v_{g_n} can be approximated in $L^2(\partial D_n^*)$ by a Herglotz wave function with kernel g . Finally, from the Jacobi-Anger expansion (c.f. [?]), it is enough to show that the set

$$u_p(r, \theta) := J_p(kr)e^{ip\theta} \quad p = 0, \pm 1, \pm 2, \dots$$

is complete in $L^2(\partial D^*)$ where $D^* = D_1^* \cup D_2^* \dots \cup D_N^*$ and J_p is a Bessel function of order p .

To show the completeness of the set $\{u_p\}$ in $L^2(\partial D^*)$, suppose $\varphi \in L^2(\partial D^*)$ satisfies

$$\int_{\partial D^*} \varphi(y)J_p(k\rho)e^{ip\varphi}ds(y) = 0$$

for $p = 0, \pm 1, \pm 2, \dots$ where $y = (\rho \cos \varphi, \rho \sin \varphi)$. Then, from the addition formula for Bessel functions, the single layer potential

$$u(x) := \int_{\partial D^*} \varphi(y)\Phi(x, y)ds(y) \quad x \in \mathbb{R}^2 \setminus \overline{D^*}$$

vanishes for $|x|$ sufficiently large and hence by unique continuation $u = 0$ in $\mathbb{R}^2 \setminus \overline{D^*}$. Arguing as in Theorem 5.5 of [?], we can now conclude that φ is continuous and u satisfies the homogeneous Dirichlet problem in D^* . Hence, since k^2 is not a Dirichlet

eigenvalue for D_n^* , $n = 1, \dots, N$, we conclude that $u = 0$ in D^* and the jump relations for single layer potentials now imply that $\varphi = 0$. This establishes the completeness of the set $\{u_p\}$ and completes the proof. \square

3. Analysis of the far field equation: the case of a penetrable medium

We now turn our attention to the second problem stated in the introduction, i.e. to show that the analogue of Theorem 2.1 holds for a penetrable medium. In particular we assume that $u \in C^2(\mathbb{R}^2)$ satisfies

$$\Delta_2 u + k^2 n(x)u = 0 \quad \text{in } \mathbb{R}^2 \quad (3.1)$$

$$u(x) = e^{ikx \cdot d} + u^s(x) \quad (3.2)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \quad (3.3)$$

and has the far field pattern u_∞ . The index of refraction is assumed to be piecewise continuously differentiable with a jump discontinuity across the smooth boundary ∂D . Our aim is to show again in this case that the far field equation (??) has an approximate solution g such that $\|g\|_{L^2(\Omega)}$ and $\|v_g\|_{L^2(D)}$ become unbounded as y tends to ∂D where now \overline{D} is the support of $m := 1 - n$. The proof of this result is more difficult than that treated in Theorem 2.1 and involves an examination of a special *interior transmission problem* (c.f. [?, ?]) instead of an interior Dirichlet, Neumann or Robin problem for the Helmholtz equation as in the case of obstacle scattering. Our analysis will be based on a projection theorem for Hilbert spaces where the inner product is replaced by a bounded sesquilinear form.

Let X be a Hilbert space with the scalar product (\cdot, \cdot) and norm $\|\cdot\|$ induced by (\cdot, \cdot) . Let $\langle \cdot, \cdot \rangle$ be a bounded sesquilinear form on X such that

$$|\langle \phi, \phi \rangle| \geq \gamma \|\phi\|^2 \quad (3.4)$$

for all $\phi \in X$ where γ is a positive constant. For a subspace $H \subset X$ we define H^\perp to be the orthogonal complement of H with respect to (\cdot, \cdot) and H^{\perp_s} to be the orthogonal complement of H with respect to $\langle \cdot, \cdot \rangle$. By the Lax-Milgram Theorem there exists a unique bounded linear operator $M : X \rightarrow X$ such that

$$\langle \phi, \psi \rangle = (M\phi, \psi) \quad (3.5)$$

for all $\phi, \psi \in X$, M is bijective and the norm of M^{-1} is bounded by γ^{-1} .

Lemma 3.1: For every closed subspace $H \subset X$ we have the decomposition

$$X = H^\perp + MH$$

where $H^\perp \cap MH = \{0\}$.

Proof: define $G := H^\perp + MH$ and let $\phi \in G^\perp$. Then $\phi \in H \cup (MH)^\perp$, i.e.

$$\langle h, \phi \rangle = (Mh, \phi) = 0$$

for all $h \in H$. Setting $h = \phi$ in this equation, we obtain $\phi = 0$ and hence $X = H^\perp + MH$. Now assume that for $g \in X$ we have $g = \psi_1 + Mh_1 = \psi_2 + Mh_2$. Then for $\psi := \psi_1 - \psi_2$ and $h := h_1 - h_2$ we have $0 = \psi + Mh$ with $\psi \in H^\perp$ and $h \in H$. Therefore

$$0 = (\psi + Mh, h) = (Mh, h) = \langle h, h \rangle$$

and hence $h = 0$. This implies $\psi = 0$ and the proof is complete. \square

Now let P_0 be the orthogonal projection operator in X onto the space H with respect to the scalar product (\cdot, \cdot) and let P_M be the projection operator onto MH as defined by Lemma 3.1. By the closed graph theorem, P_M is a bounded operator.

Lemma 3.2: For every closed subspace $H \subset X$ we have

$$M^{-1}H^\perp = (M^*H)^\perp = H^{\perp_s} \quad .$$

Proof. The first equality follows from the fact that $\phi \in (M^*H)^\perp$ if and only if $(\phi, M^*h) = (M\phi, h) = 0$ for every $h \in H$ and hence $M\phi \in H^\perp$, i.e. $\phi \in M^{-1}H^\perp$. The second equality follows from the fact that $(\phi, M^*h) = (M\phi, h) = \langle \phi, h \rangle$. \square

We are now in a position to show that every $\phi \in X$ can be uniquely written as a sum $\phi = v + w$ with $v \in H^{\perp_s}$ and $w \in H$, i.e. $X = H^{\perp_s} \oplus_s H$ where \oplus_s is the orthogonal decomposition with respect to the sesquilinear form $\langle \cdot, \cdot \rangle$.

Theorem 3.3: For every closed subspace $H \subset X$ we have the orthogonal decomposition

$$X = H^{\perp_s} \oplus_s H \quad .$$

The projection operator $P : X \rightarrow H^{\perp_s}$ defined by this decomposition is bounded in X .

Proof. For $\phi \in X$ we define $\hat{\phi} := M\phi$. Then from Lemma 3.1 we have that

$$\hat{\phi} = (1 - P_M)\hat{\phi} + P_M\hat{\phi}$$

i.e.

$$M\phi = (1 - P_M)M\phi + P_MM\phi \quad .$$

Hence

$$\phi = M^{-1}(1 - P_M)M\phi + M^{-1}P_MM\phi = v + w$$

where

$$v := M^{-1}(1 - P_M)M\phi \in M^{-1}H^\perp = H^{\perp_s}$$

$$w := M^{-1}P_MM\phi \in H \quad .$$

We have thus shown that $X = H^{\perp_s} + H$. To show the uniqueness of this decomposition, suppose $v + w = 0$ with $v \in H^{\perp_s}$ and $w \in H$. Then

$$0 = |\langle v, w \rangle| = |\langle w, w \rangle| \geq \gamma \|w\|^2$$

which implies that $w = v = 0$. Finally, since from the above analysis we have that $P = M^{-1}(1 - P_M)M$ and P_M is bounded, we have that P is bounded. \square

We will now turn our attention to the problem of showing the existence of a unique weak solution v, w of the two-dimensional *interior transmission problem*

$$\Delta_2 w + k^2 n(x)w = 0 \quad , \quad \Delta_2 v + k^2 v = 0 \quad \text{in } D \tag{3.6}$$

$$w - v = \Phi(x, y) \quad , \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu_x} \Phi(x, y) \quad \text{on } D \tag{3.7}$$

where n and D are as defined above and Φ is given by (??) with $y \in D$. We will further assume that there exists a positive constant c such that

$$\text{Im } n(x) \geq c \tag{3.8}$$

for $x \in D$. To motivate the following definition of a weak solution of (??), (??) we note that if a solution $v, w \in C^2(D) \cap C^1(\overline{D})$ to (??), (??) exists, then from Green's formula and (??) we have that

$$w(x) + k^2 \int_D \Phi(x, \xi) m(\xi) w(\xi) d\xi = v(x) \quad , \quad x \in D \tag{3.9}$$

for $m := 1 - n$ and (??) will be satisfied provided

$$-k^2 \int_D \Phi(x, \xi) m(\xi) w(\xi) d\xi = \Phi(x, y) \quad , \quad x \in \partial B \tag{3.10}$$

where B is a ball centered at the origin with $\overline{D} \subset B$. This last statement follows from Rellich's lemma and the unique continuation principle.

Definition 3.4: Let H be the linear space

$$H := \{u \in C^2(\mathbb{R}^2) : \Delta_2 u + k^2 u = 0 \text{ in } \mathbb{R}^2\}$$

and \overline{H} the closure of H in $L^2(D)$. For $\phi \in L^2(D)$ define the volume potential $T_m\phi$ by

$$(T_m\phi) := k^2 \int_D \Phi(x, \xi) m(\xi) \phi(\xi) d\xi \quad x \in \mathbb{R}^2 \quad .$$

Then a pair v, w with $v \in \overline{H}$ and $w \in L^2(D)$ is said to be a weak solution of the interior transmission problem (??), (??) with the point source $y \in D$ if v and w satisfy the integral equation

$$(I + T_m)w(x) = v(x) \quad , \quad x \in D$$

and the boundary condition

$$-(T_m w)(x) = \Phi(x, y) \quad , \quad x \in \partial B \quad .$$

Before proceeding to establish the existence of a unique weak solution to the interior transmission problem (??), (??), we make a few preliminary observations. We first note that condition (??) implies that in $L^2(D)$ the sesquilinear form

$$\langle \phi, \psi \rangle := \int_D m(\xi) \phi(\xi) \overline{\psi(\xi)} d\xi \quad (3.11)$$

satisfies the assumption (??). In this case the operator M is simply the multiplication operator

$$(M\phi)(x) := m(x)\phi(x) \quad .$$

We also note that from the Jacobi-Anger expansion we have that the space of Herglotz wave functions with kernel $g \in L^2(\Omega)$ is a dense subset of \overline{H} . Finally, by the uniqueness of the solution to the exterior Dirichlet problem for the Helmholtz equation and the unique continuation principle, we see that if v, w is a weak solution of the interior transmission problem with point source $y \in D$ then $-(T_m w)(x) = \Phi(x, y)$ for all $x \in \mathbb{R}^2 \setminus D$.

Theorem 3.5: For every source point $y \in D$ there exists at most one weak solution of the interior transmission problem.

Proof. Let w and v be the difference between two weak solutions of the interior transmission problem. Then from the boundary condition $(T_m w)(x) = 0$ for $x \in \partial B$ we have

$$\int_D \Phi(x, \xi) m(\xi) w(\xi) d\xi = 0 \quad , \quad x \in \partial B \quad . \quad (3.12)$$

From the addition formula for Bessel functions we now see that

$$\langle w, h \rangle = \int_D m(\xi) w(\xi) \overline{h(\xi)} d\xi = 0 \quad (3.13)$$

for all functions h of the form $h(x) = J_p(kr)e^{ip\theta}$, $p = 0, \pm 1, \pm 2, \dots$ where $x = (r \cos \theta, r \sin \theta)$. By continuity (??) also holds for $h \in \overline{H}$, i.e. $w \in H^{\perp_s}$.

Now let $(v_j) \subset H$ be a sequence with $v_j \rightarrow v$ as $j \rightarrow \infty$ in $L^2(D)$ and note that $(I + T_m)^{-1}$ exists and is bounded in $L^2(D)$. Hence, for $w_j := (I + T_m)^{-1}v_j$, we have that $w_j \rightarrow w \in L^2(D)$ as $j \rightarrow \infty$. If x does not belong to D we define w_j by

$$w_j(x) := v_j(x) - (T_m w_j)(x), \quad x \in \mathbb{R}^2 \setminus D.$$

The functions v_j and w_j satisfy

$$\Delta_2 v_j + k^2 v_j = 0 \tag{3.14}$$

and

$$\Delta_2 w_j + k^2 n(x) w_j = 0 \tag{3.15}$$

in both D and $B \setminus \overline{D}$. From (??) and (??) by Green's first and second theorems applied to D and $B \setminus \overline{D}$ we have

$$\int_{\partial B} \left(w_j \frac{\partial \overline{v}_j}{\partial \nu} - \overline{v}_j \frac{\partial w_j}{\partial \nu} \right) ds = -k^2 \int_D m w_j \overline{v}_j d\xi \tag{3.16}$$

$$\operatorname{Im} \int_{\partial B} v_j \frac{\partial \overline{v}_j}{\partial \nu} ds = 0 \tag{3.17}$$

and

$$\operatorname{Im} \int_{\partial B} w_j \frac{\partial \overline{w}_j}{\partial \nu} ds = -k^2 \int_D \operatorname{Im} m |w_j|^2 d\xi. \tag{3.18}$$

We combine (??)-(??) to obtain

$$\begin{aligned} & \operatorname{Im} \int_{\partial B} (w_j - v_j) \frac{\partial}{\partial \nu} (\overline{w_j - v_j}) ds \\ &= \operatorname{Im} \int_{\partial B} \left(w_j \frac{\partial \overline{w}_j}{\partial \nu} + v_j \frac{\partial \overline{v}_j}{\partial \nu} - v_j \frac{\partial \overline{w}_j}{\partial \nu} - w_j \frac{\partial \overline{v}_j}{\partial \nu} \right) ds \\ &= -k^2 \int_D \operatorname{Im} m |w_j|^2 d\xi + k^2 \operatorname{Im} \int_D m w_j \overline{v}_j d\xi. \end{aligned} \tag{3.19}$$

We now note that from (??) and (??) we have by the Cauchy-Schwarz inequality that

$$w_j(x) - v_j(x) = -(T w_j)(x) \rightarrow -(T w)(x) = 0$$

uniformly for $x \in \partial B$ and

$$\int_D m(\xi) w_j(x) \overline{v_j(\xi)} dy \rightarrow \int_D m(\xi) w(\xi) \overline{v(\xi)} d\xi = 0$$

as $j \rightarrow \infty$. Hence, taking the limit $j \rightarrow \infty$ in (??) we obtain

$$\int_D \operatorname{Im} m |w|^2 d\xi = 0.$$

From (??) we can now conclude that $w(x) = 0$ for $x \in D$. Since $(I + T_m)w(x) = v(x)$ for $x \in D$ we can also conclude that $v(x) = 0$ for $x \in D$ and the proof is complete. \square

Theorem 3.6: For every source point $y \in D$ there exists a weak solution to the interior transmission problem.

Proof. Choosing an appropriate coordinate system, we can assume without loss of generality that $y = 0$. We consider the space

$$H_1^0 := \text{span}\{J_p(kr)e^{ip\phi} \ , \ p = \pm 1, \pm 2, \dots\}$$

and the closure H_1 of H_1^0 in $L^2(D)$. Then the space $H_1^{\perp s} \cap \overline{H}$ has an element in it which is nonzero (note that for sufficiently small ϵ a function $h \in H_1$ satisfies

$$\int_{|x| < \epsilon} h dx = 0$$

but this is not true for $h = J_0$). Let ψ be a unit vector in $H_1^{\perp s} \cap \overline{H}$ and P the projection operator from $L^2(D)$ onto $H^{\perp s}$ as defined by Theorem 3.3. Note that $\langle J_0, \psi \rangle$ is not zero. We first consider the integral equation

$$(I + PT_m)u = PT_m\psi \tag{3.20}$$

in $L^2(D)$. Since T_m is compact and P is bounded, the operator PT_m is compact in $L^2(D)$. In order to apply the Riesz theory, we will prove uniqueness for the homogeneous equation. To this end, assume that $\hat{u} \in L^2(D)$ satisfies

$$(I + PT_m)\hat{u} = 0 \quad .$$

Then $\hat{u} \in H^{\perp s}$ and

$$\begin{aligned} (I + T_m)\hat{u} &= (I + PT_m)\hat{u} + (I - P)T_m\hat{u} \\ &= (I - P)T_m\hat{u} \\ &=: \hat{v} \in \overline{H} \quad . \end{aligned}$$

Since $\hat{v} \in \overline{H}$ and $\hat{u} \in H^{\perp s}$, from Theorem 3.5 we have that $\hat{v} = \hat{u} = 0$. By the Riesz theory we now obtain the continuous invertibility of $I + PT_m$ in $L^2(D)$.

Now let u_0 be the solution of (??) and note that $u_0 \in H^{\perp s}$. We define the constant c and function $u \in L^2(D)$ by

$$\begin{aligned} c &:= 1/(-k^2\langle J_0, \psi \rangle) \\ u &:= c(u_0 - \psi) \quad . \end{aligned}$$

Then we compute

$$\begin{aligned} (I + PT_m)u &= c(I + PT_m)u_0 - c(I + PT_m)\psi \\ &= -c\psi \end{aligned}$$

and

$$\begin{aligned}
(I + T_m)u &= (I + PT_m)u + (I - P)T_mu \\
&= -c\psi + (I - P)T_mu \\
&=: v \in \overline{H} \quad .
\end{aligned}$$

Since

$$\begin{aligned}
\langle h, u \rangle &= c\langle h, u_0 - \psi \rangle \\
&= c\langle h, u_0 \rangle - c\langle h, \psi \rangle \\
&= 0
\end{aligned}$$

for all $h \in H_1$ and

$$\begin{aligned}
\langle J_0, u \rangle &= c\langle J_0, u_0 - \psi \rangle \\
&= c\langle J_0, \psi \rangle \\
&= -1/k^2
\end{aligned}$$

we have from the Jacobi-Anger expansion that

$$\begin{aligned}
-(T_mu)(x) &= \frac{i}{4}H_0^{(1)}(kr) \\
&= \Phi(x, 0) \quad , \quad x \in \partial B
\end{aligned}$$

and the proof is complete. □

We are now in a position to show how the support \overline{D} of $m := 1 - n$ can be determined from the far field pattern u_∞ corresponding to the scattering problem (??)-(??). From our conditions on D , it suffices to determine the smooth boundary ∂D . Let $u_\infty(\hat{x}; d)$, $\hat{x}, d \in \Omega$, denote the given far field pattern and $\Phi_\infty(\hat{x}, y)$ $\hat{x} \in \Omega$, the far field pattern of the fundamental solution $\Phi(\cdot, y)$ given by the right hand side of (??). Following [?] we will solve the inverse problem of determining ∂D from u_∞ by looking for special approximate solutions of the equation

$$\int_{\Omega} u_\infty(\hat{x}; d)g(d)ds(d) = \Phi_\infty(\hat{x}, z) \quad , \quad \hat{x} \in \Omega \quad . \quad (3.21)$$

Note that (??) has a solution if and only if the interior transmission problem (??), (??) with source point $y \in D$ has a solution $v, w \in C^2(D) \cap C^1(\overline{D})$ such that $v = v_g$ is a Herglotz wave function with kernel g (c.f. Theorem 5.16 of [?]). This is true only in very special cases. However, we will show that we can always find an approximate solution of the far field equation which has the crucial property

$$\lim_{y \rightarrow \partial D} \|g(\cdot, y)\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{y \rightarrow \partial D} \|v_g\|_{L^2(D)} = \infty$$

which suffices to determine ∂D . The existence of such functions is shown in the following theorem.

Theorem 3.7: For every $\epsilon > 0$ and $y \in D$ there is a solution $g = g(\cdot, y) \in L^2(\Omega)$ of the inequality

$$\left\| \int_{\Omega} u_{\infty}(\cdot; d) g(d) ds(d) - \Phi_{\infty}(\cdot, y) \right\|_{L^2(\Omega)} \leq \epsilon \quad (3.22)$$

such that

$$\lim_{y \rightarrow \partial D} \|g\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{y \rightarrow \partial D} \|v_g\|_{L^2(D)} = \infty \quad (3.23)$$

where v_g is the Herglotz wave function with kernel g .

Proof. Let $v(\cdot, y), w(\cdot, y)$ be the unique weak solution to the interior transmission problem with source point y . From the remarks preceding Theorem 3.5 we can approximate $v(\cdot, y) \in \overline{H}$ by a Herglotz wave function v_g with kernel $g = g(\cdot, y)$, i.e. for every $\tilde{\epsilon} > 0$ and $y \in D$ there exists $g \in L^2(\Omega)$ such that

$$\|v(\cdot, y) - v_g\|_{L^2(D)} \leq \tilde{\epsilon} \quad . \quad (3.24)$$

Then by the continuity of the operator $(I + T_m)^{-1}$ we have for $u_g := (I + T_m)^{-1}v_g$ that

$$\|w(\cdot, y) - u_g\|_{L^2(D)} \leq c \tilde{\epsilon} \quad (3.25)$$

for some positive constant c and by the continuity of $T_m : L^2(D) \rightarrow C(\partial B)$ we have that

$$\|T_m u_g - \Phi(\cdot, y)\|_{C(\partial B)} \leq c' \tilde{\epsilon} \quad (3.26)$$

for some positive constant c' . We now note that the far field pattern $T_m^{\infty} u_g$ of $T_m u_g$ is given by

$$(T_m^{\infty} u_g)(\hat{x}) = \int_{\Omega} u_{\infty}(\hat{x}; d) g(d) ds(d) \quad , \quad \hat{x} \in \Omega \quad .$$

Hence, by the continuous dependence of the solution of the exterior Dirichlet problem with respect to the boundary data, we obtain from (??) the estimate

$$\|T_m^{\infty} u_g - \Phi_{\infty}(\cdot, y)\| \leq c'' \tilde{\epsilon}$$

for a positive constant c'' . Choosing $\tilde{\epsilon} = \epsilon/c''$ we get a solution of the inequality (??).

We now need to verify (??). In the following we let c denote a generic constant. We begin by noting that since the Sobolev space $H^{3/2}(\partial D)$ is continuously embedded in $C(\partial D)$ we have that

$$\|\Phi(\cdot, y)\|_{C(\partial D)} \leq c \|\Phi(\cdot, y)\|_{H^{3/2}(\partial D)} = c \|T_m w(\cdot, y)\|_{H^{3/2}(\partial D)} \quad . \quad (3.27)$$

Using the trace theorem and the boundedness of the operators $(I + T_m)^{-1} : L^2(D) \rightarrow L^2(D)$ and $T_m : L^2(D) \rightarrow H^2(D)$ we have the estimates

$$\|T_m w(\cdot, y)\|_{H^{3/2}(\partial D)} \leq c \|T_m w(\cdot, y)\|_{H^2(D)} \leq c \|w(\cdot, y)\|_{L^2(D)} \quad (3.28)$$

and

$$\|w(\cdot, y)\|_{L^2(D)} \leq c \|v(\cdot, y)\|_{L^2(D)} \quad . \quad (3.29)$$

From (??) we have that

$$\|v(\cdot, y)\|_{L^2(D)} = \|v_g + (v(\cdot, y) - v_g)\|_{L^2(D)} \leq \|v_g\|_{L^2(D)} + \tilde{\epsilon} \quad . \quad (3.30)$$

If we now combine (??)-(??) together we get

$$\|\Phi(\cdot, y)\|_{C(\partial D)} \leq c(\|v_g\|_{L^2(D)} + \tilde{\epsilon}) \quad . \quad (3.31)$$

Since $\Phi(x, y)$ has a logarithmic singularity as y tends to x we obtain the second equation of (??) from (??). The first equation of (??) is now a consequence of the observation that a Herglotz wave function with bounded kernel $g \in L^2(\Omega)$ will be bounded in $L^2(D)$. The proof is now complete. \square

As in the case of the obstacle scattering, the second condition in (??) ensures that the blowing up of g as y tends to ∂D is not only due to wild oscillations of the high frequency components of g .

4. Regularization and numerical applications

As a consequence of the compactness of the far field operator, the linear inverse problem of determining the function $g(\cdot, y)$ from the far field equation is ill-posed in the sense of Hadamard [?]. Therefore the solution of the (strongly) ill-conditioned linear system representing the discretized version of the far field equation (??) becomes dramatically unstable in presence of experimental error on the far field pattern. It follows that for noisy data the fast algorithm introduced in [?] must be combined with the use of a regularization method for providing stable approximate solutions of the far field equation.

The main problems in dealing with a regularization method for solving the far field equation (??) are the following:

- (i) Whatever regularization method we use we must allow the regularization parameter α to depend on y . This is due to the fact that according to Theorems 2.1 and 3.7 we are looking for a solution $g = g(\cdot, y)$ that blows up as y approaches ∂D . In particular, if Tikhonov's method in $L^2(\Omega)$ is adopted, the regularized solution is

determined from the minimization problem

$$\begin{aligned} & \left\| \int_{-\pi}^{\pi} u_{\infty}(\hat{x}, \theta) g_{\alpha}(d, y) ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot y} \right\|_{L^2(\Omega)}^2 + \alpha \|g(\cdot, y)\|_{L^2(\Omega)}^2 \\ & = \text{minimum} \end{aligned} \quad (4.1)$$

where $\|g(\cdot, y)\|_{L^2(\Omega)}^2$ becomes unbounded as y approaches ∂D and hence the choice of α depends on y .

- (ii) The far field equation is different than the usual linear inverse problem

$$Az = f \quad (4.2)$$

where the data f on the right hand side is affected by noise and the linear operator A acting at the left hand side is known exactly. On the contrary, in our case the noisy data is the far field pattern (which is the kernel of the operator A in (??)) while the exponential function at the right hand side of equation (??) (which is the f in (??)) is known exactly.

Our idea to deal with the above problems is to regularize the far field equation using Tikhonov's method where the regularization parameter is allowed to depend on y . In particular, for each y we want to solve equation (??) using some criterium to provide an optimum value of α . The result will be a map $y \rightarrow \alpha_{\text{opt}}(y)$ and we expect that for each α_{opt} the norm of the corresponding regularized solution will increase as y approaches the boundary of the scatterer. The crucial point of the choice of the regularization parameter will be addressed by applying the generalized discrepancy principle given in [?]. This criterium represents a generalization of Morozov's discrepancy principle to linear operator equations when the operator is not known exactly.

To describe the generalized discrepancy principle, let us consider the general linear inverse problem (??) where A is a linear continuous map from the Hilbert space X to the Hilbert space Y . Suppose that f is known exactly and A is known with an error δ , that is, only the noisy data A_{δ} is available where $\|A - A_{\delta}\| \leq \delta$. As already noted, Tikhonov regularization method consists in solving the minimum problem

$$\|A_{\delta}z_{\alpha} - f\|_Y^2 + \alpha \|z_{\alpha}\|_X^2 = \text{minimum} \quad (4.3)$$

The solution of (??) gives a one parameter family of regularized solutions $\{z_{\alpha}\}_{\alpha>0}$ and the problem is to determine a value α_{opt} of the regularization parameter such that the corresponding regularized solution can be considered an optimum approximate solution of problem (??). The generalized discrepancy principle for the choice of an optimum Tikhonov regularized solution can be stated as follows: let us define the generalized discrepancy function

$$\mu(\alpha) = \|A_{\delta}z_{\alpha} - f\|_Y^2 - \delta^2 \|z_{\alpha}\|_X^2 \quad (4.4)$$

Then

- (i) If there is an $\alpha_{\text{opt}} > 0$ such that $\mu(\alpha_{\text{opt}}) = 0$, we take $z_{\alpha_{\text{opt}}}$ as an optimum approximate solution (we note that since $\mu(\alpha)$ is monotonically increasing, if a zero of $\mu(\alpha)$ exists, it is unique).
- (ii) If $\mu(\alpha) > 0 \quad \forall \alpha > 0$, we take $\lim_{\alpha \rightarrow 0} z_{\alpha}$ as the approximate solution.

We conclude that the far field equation (??) can be solved by the following algorithm:

- Solve the minimum problem (??). The result is a one-parameter family $\{g_{\alpha}\}$ of regularized solutions.
- Apply the generalized discrepancy principle to determine an optimum value α_{opt} of the regularization parameter.
- Repeat the previous steps for all the values of y on a grid containing the object. Then an image of the object can be obtained by plotting for each y the norm of the optimum regularized solution $g_{\alpha_{\text{opt}}}$ or the optimum value α_{opt} of the regularization parameter.

We note that all the operations necessary to perform the algorithm can be simplified by computing the singular value decomposition of the far field operator. In fact, if $\{\sigma_n; u_n, v_n\} \quad n = 1, 2, \dots$ is the singular system of the operator A_{δ} , equation (??) becomes

$$\mu(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^2 - \delta^2 \sigma_n^2}{(\sigma_n^2 + \alpha)^2} \left| \left(\frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot y}, v_n \right)_Y \right|^2 \quad (4.5)$$

where $(\cdot, \cdot)_Y$ denotes the scalar product in Y . In the case of the discretized version of the far field equation (??), the sum on the right hand side of equation (??) is truncated at the number of incident waves (in the applications we consider, this number coincides with the number of view points). Then the optimum value of the regularization parameter is obtained by computing the zero of the monotone increasing (truncated) function (??) for each value of the grid point y . This computation takes only a few second for each grid point. In fact, the reconstructions we will show in the following have been obtained in less than a minute for the two obstacle scattering cases with wavenumber $k = 1$ and in approximately two minutes for the two cases with wavenumber $k = 7$ (We have used a SUN Workstation).

We present now some numerical examples showing the applicability of this method. In particular, we consider problems of the scattering of plane waves by obstacles satisfying a Dirichlet boundary condition and one example of scattering of plane waves by an inhomogeneous medium. In the cases of obstacle scattering, the far field pattern is computed by using Nyström's method (c.f. [?]) while in the case of inhomogeneous scattering we use a coupled finite element and spectral method (c.f. [?]). The far field patterns we obtain are subjected pointwise to 5% Gaussian noise. For each value of y the corresponding far field equation is solved by using Tikhonov's

method and the generalized discrepancy principle for the choice of the optimum value of the regularization parameter. For each example we exhibit: (a) the object we want to restore; (b) the contour of the map $y \rightarrow \log \|g(\cdot, y)\|_{L^2(\Omega)}$ where $g(\cdot, y)$ is the solution of the far field equation without regularization; (c) the contour of the map $y \rightarrow \log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|_{L^2(\Omega)}$ where $g_{\alpha_{\text{opt}}}(\cdot, y)$ is the regularized solution obtained by combining Tikhonov's method and the generalized discrepancy principle; (d) the contour of the map $y \rightarrow \alpha_{\text{opt}}(y)$. We have chosen grids for y of 61x61 points and 50 contour lines in every case.

Figure 1 shows the case of the 'single kite' parametrized by

$$x(t) = (1.5 \sin(t), \cos(t) + 0.65 \cos(2t) - 0.65) \quad 0 \leq t \leq 2\pi \quad .$$

The value of the wave number is $k = 1$.

Figure 2 shows the case of the 'double kite' (i.e. the 'single kite' plus a second kite rotated 45° and displaced from the first kite along the vector (5, 5)) again for $k = 1$. In figure 3 we consider the same problem for $k = 7$.

We note that in the examples of obstacle scattering corresponding to the case $k = 1$ the far field pattern $u_\infty(\hat{x}, d)$ has been computed at 32 equidistantly distributed observation points and 32 uniformly distributed directions. In the case $k = 7$, owing to reasons connected to the convergence of the Nyström method, we have increased the number of points and incident waves up to 128x128.

Finally in figure 4 we describe the result for the scattering by an inhomogeneous medium. As in [?] we have considered the 'twin peaks' example

$$n(x_1, x_2) = \begin{cases} \frac{1}{2}(3 + \cos 2\pi r_1) & \text{if } r_1 < 0.5 \\ \frac{1}{2}(3 + \cos 2\pi r_2) & \text{if } r_2 < 0.5 \\ 1 & \text{otherwise} \end{cases} \quad (4.6)$$

where $r_1 = [(x_1 - \frac{1}{2})^2 + x_2^2]^{1/2}$ and $r_2 = [(x_1 + \frac{1}{2})^2 + x_2^2]^{1/2}$. We remark that this function has imaginary part equal to zero and so it does not satisfy the assumption (??). Nevertheless, as figure 4 shows, the method seems to work also in this case. The far field pattern corresponding to this example has dimension 51x51. We also note that in this paper both the solution and the regularized solution for the 'twin peaks' example have been computed by using the singular system of the far field pattern and not a least squares trigonometric best fit as in [?].

We would like to thank Andreas Kirsch and Peter Monk for providing us the codes to compute the far field patterns.

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(a)

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Figure 1. The case of the ‘single kite’ with $k = 1$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

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Figure 2. Figure 2. The case of the ‘double kite’ with $k = 1$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

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Figure 3. Figure 3. The case of the ‘double kite’ with $k = 7$: (a) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (b) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

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(a)

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(c)

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Figure 4. Figure 4. Results for ‘twin peaks’ with $k = 7$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of α_{opt} with 5% Gaussian noise.

Figure captions

Figure 1. The case of the ‘single kite’ with $k = 1$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

Figure 2. The case of the ‘double kite’ with $k = 1$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

Figure 3. The case of the ‘double kite’ with $k = 7$: (a) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (b) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\alpha_{\text{opt}}(y)$ with 5% Gaussian noise.

Figure 4. Results for ‘twin peaks’ with $k = 7$: (a) the object; (b) plot of $\log \|g(\cdot, y)\|$ with 5% Gaussian noise; (c) plot of $\log \|g_{\alpha_{\text{opt}}}(\cdot, y)\|$ with 5% Gaussian noise; (d) plot of α_{opt} with 5% Gaussian noise.