

The linear sampling method in inverse electromagnetic scattering theory*

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Abstract. We survey the linear sampling method for solving the inverse scattering problem for time-harmonic electromagnetic waves at fixed frequency. We consider scattering by an obstacle as well as scattering by an inhomogeneous medium both in \mathbb{R}^2 and \mathbb{R}^3 . Included in our discussion is the use of regularization methods for ill-posed problems and numerical examples in both two and three dimensions.

1. Introduction

Electromagnetic scattering is a physical phenomenon whereby, in the presence of an inhomogeneity, an electromagnetic incident wave is scattered and the total field at any point of the space can be written as the sum of the original incident field and the scattered field. The electromagnetic direct scattering problem is the problem of determining the scattered field when the geometrical and physical properties of the scatterer are known. The inverse scattering problem is the problem of inferring information on the inhomogeneity from the knowledge of the far-field pattern, i.e. the scattered wave at large distances from the scatterer. Classical electromagnetic inverse scattering theory provides a rich supply of challenging mathematical problems. First of all, this is due to the fact that the unknown physical conditions where the scattering takes place create difficulties in trying to construct the solution procedure. Secondly, all the inverse scattering problems significant in applications belong to the class of so-called ill-posed problems in the sense of Hadamard [25] and therefore any reliable approach to their solution must face at some stage questions of uniqueness and numerical stability. In

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particular, it is well-established that any numerical implementation of a method for the solution of an inverse scattering problem must, at some step, incorporate a regularization procedure in order to eliminate the artificial oscillations due to the ill-posedness of the problem.

In recent years, the interest of the scientific community in inverse scattering problems has grown significantly, together with the development of numerous new techniques in remote sensing and non-invasive investigation. Important electromagnetic inverse scattering problems arise, in particular, in

- military applications, for example when it is desired to identify hostile objects by means of radar;
- medical imaging, for example in the use of microwaves to detect leukemia or in hyperthermia treatment;
- non destructive testing, such as the case when small cracks are looked for inside metallic or plastic structures.

A common feature of these applications is the requirement that any inversion approach for the solution of the inverse scattering problem involved must be fast, computationally effective and provide reliable reconstructions of the scatterer without detailed *a priori* information on the physical or geometrical properties of the inhomogeneity. Unfortunately, most classical inversion methods do not fulfill this requirement at all or in part. Traditional approaches for the solution of electromagnetic inverse scattering problems can be essentially divided into two families:

- non-linear optimization schemes, where the restoration is performed iteratively from an initial guess of the position and shape of the scatterer;
- weak scattering approximation methods, such as physical optics and Born approximation, where a linear inverse problem is obtained by means of low- or high-frequency approximations.

The main practical difficulty with iterative optimization algorithms is that they typically require long (and sometimes notably long) reconstruction times. Furthermore, although these methods may provide extremely accurate approximate solutions, they typically require an accurate initial guess and for many applications such as in medical imaging the *a priori* information available on the scatterer profile are insufficient to allow one such a guess. On the other hand, both the physical optics approximation, where waves of very small wavelength scatter from large conductors, and the Born approximation, where the penetrable object has a small contrast with respect to the background and is smaller than the wavelength, ultimately require the solution of an ill-posed linear inverse problem which can be addressed, with low computational effort, with standard techniques of regularization theory for linear inverse problems. Nevertheless, weak scattering approximation methods typically require *a priori* knowledge of what

type of scattering generated the far-field pattern, i.e. whether the object has been penetrated by the wave and, if not, what kind of boundary condition the total field satisfies on the boundary of the obstacle. Finally, it must be pointed out that it can be hazardous to apply weak scattering approximation methods when weak scattering conditions do not occur. By instance, this is the case in the use of microwave tomography for medical imaging applications, whereby the wavelength of the incident field is typically of the same size of the investigated biological tissue thus leading to a situation of genuine non-linearity.

In the linear sampling method approach the above difficulties are avoided. This method, indeed, involves the solution of a linear Fredholm equation of the first kind but here the linearity is exact in the sense that it does not come from an approximation based on particular physical conditions but from an equivalence between the non-linear inverse scattering problem and the linear integral equation. In particular, a linear integral equation of the first kind, the far-field equation, is written for each point z inside the scatterer, the integral kernel of this equation being the far-field pattern, i.e. the measured datum, and the right hand side being an exactly known analytic function. The solution of this equation has no intrinsic physical meaning but it has the property that it becomes unbounded for z on and in the exterior of the boundary of the inhomogeneity (therefore this function plays the role of an indicator for the boundary). Hence, the shape of the inhomogeneity can be reconstructed simply by plotting the norm of an approximate solution of the far-field equation for all the points of a grid containing the scatterer.

The main advantages of the linear sampling method are:

- a notable computational speed, which can be increased by means of a careful implementation of the method, for example through the use of a clever sampling scheme. In the case of 2D problems, reconstructions from real data can be obtained in few minutes which become a couple of hours in the case of complicated 3D objects. However the method can be very naturally parallelized and its computational effectiveness increased proportionally to the number of available computers.
- The implementation is computationally simple, since it only requires the solution of a finite number of ill-conditioned linear systems. This also implies that the numerical instability due to the presence of noise on the far-field data can be easily handled by applying the classical algorithms of regularization theory for ill-conditioned linear systems.
- Very little *a priori* information on the scatterers is required. For example, it is not necessary to *a priori* know the number of the scatterers, nor if they are penetrable by the wave, or, if not, which kind of boundary condition is satisfied by the total field on the boundary.

The method of course also has some disadvantages. The main one is that, in the case of inhomogeneous scattering, it only provides a reconstruction of the support of the scatterer and it is not possible to infer information about the point values of the index of refraction. However this property reflects the mathematical fact that in many situations (e.g. anisotropic media) it is only the support that is uniquely determined and not the point values of the index of refraction.

Numerous papers have been written on the linear sampling method. Three of them are particularly important. In 1996, in a paper by Colton and Kirsch [15], the method was formulated for the first time, in the case of 2D obstacles with Dirichlet, Neumann and impedance boundary conditions and in the case of 2D inhomogeneous media. Then, in 1997, in a paper by Colton, Piana and Potthast [21], it was shown that the use of a regularization scheme is necessary to produce reliable reconstructions. Finally, in 2002, in a paper by Cakoni, Colton and Haddar [9], the problem of what happens when the sampling point z is in the exterior of the scatterer was addressed. All the other papers regarding the method provide its formulation in different scattering conditions, describe possible applications or discuss the effectiveness of different implementation procedures. More precisely, at this stage the method has been studied in the scalar case for transverse magnetic (TM) polarization in [15, 21, 4], transverse electric (TE) polarization in [20] and mixed boundary value problems in [10] whereas, for Maxwell's equations, the method has been formulated for the case of a perfect conductor in [5, 16], the case of mixed boundary value problems in [11] and the case of an inhomogeneous medium in [26, 16]. Applications of the linear sampling method to practical problems have been considered in microwave tomography, in the case of the detection of leukemia in human bone marrow [19], in impedance tomography [3], in elasticity [1, 12, 23, 47] and for the detection of buried objects [14]. Numerical tests validating the method with real data for different regularization algorithms can be found in [50]. More recently further applications have been made to problems involving cracks and screens [7, 8].

We note that since the original introduction of the linear sampling method in 1996, a variety of related inversion schemes have been introduced that are in the same spirit as the linear sampling method. In particular, we mention the *factorization method* of Kirsch [32, 33, 34, 35, 24, 3], a variety of methods introduced by Potthast and his co-workers [42, 39, 44], the *indicator sampling method* of You, Miao and Liu [55, 54, 56], as well as interesting papers by Ikehata [29, 30], Norris [41] and Xu, Mawata and Lin [53].

The formulation of the linear sampling method is based on the following logical steps:

- (i) the direct scattering problem is formulated in function spaces which guarantee the well-posedness of the problem;
- (ii) the inverse scattering problem is formulated and the uniqueness of its solution is

discussed;

- (iii) the far-field equation is written and an appropriate factorization of the far-field operator is given;
- (iv) an interior problem is associated to the exterior scattering problem and its solution is approximated by means of a Herglotz wave function;
- (v) a final theorem is given, describing the behaviour of an approximate solution of the far-field equation inside and outside the scatterer.

In the present survey paper we will follow this framework to present the linear sampling method for the case of electromagnetic inverse scattering problems. In particular, Section 2 will be devoted to the discussion of the 2D scalar case when the incident field is TM polarized and the scatterer is an impenetrable obstacle. In Section 3 we will consider the same polarization in the case of an inhomogeneous medium. Then in Section 4 Maxwell's equations will be studied. Section 5 will describe the implementation difficulties concerning the method and finally, in Section 6, some numerical applications will be considered involving both real and 3D far-field data.

2. TM polarization: obstacles

We consider an electromagnetic wave that is scattered by an infinitely long impenetrable cylinder. If the wave is time-harmonic the scattering is described by the (normalized) time-harmonic Maxwell equations

$$\nabla \times E - ikH = 0 \quad \nabla \times H + ikE = 0 \quad x \in \mathbb{R}^3 \setminus \overline{D} \quad (2.1)$$

with the boundary condition

$$\nu \times E = 0 \quad x \in \Gamma \quad (2.2)$$

where the boundary Γ is twice continuously differentiable. We assume that the polarization of the incident field is transverse magnetic (TM), i.e. the electric field vibrates parallel to the axis of the cylinder. It is then natural to introduce the scalar field u such that $E = (0, 0, u)$. From the Maxwell equations (2.1) and the boundary condition (2.2) it follows that u satisfies the exterior problem

$$\Delta_2 u(x) + k^2 u(x) = 0 \quad x \in \mathbb{R}^2 \setminus \overline{D} \quad (2.3)$$

$$u(x) = 0 \quad x \in \Gamma \quad (2.4)$$

where (with a slight abuse of notation) D is now the cross-section of the cylinder with boundary Γ . We also require that u can be written in the form

$$u(x) = e^{ikx \cdot d} + u^s(x) \quad (2.5)$$

where k is the real positive wavenumber, $d \in \Omega = \{x \in \mathbb{R}^2, |x| = 1\}$ and the scattered field $u^s(x)$ satisfies the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - ik u^s \right) = 0 \quad . \quad (2.6)$$

It is proved in [17] that the problem of determining a radiating solution $u^s \in C^2(\mathbb{R}^2 \setminus \overline{D}) \cap C(\mathbb{R}^2 \setminus D)$ satisfying equation (2.3) and the boundary condition $u^s = -e^{ikx \cdot d}$ on Γ is well-posed in the sense of Hadamard.

Green's formula implies that the radiating field $u^s(x)$ has the asymptotic behaviour

$$u^s(x) = u_\infty(\hat{x}, d) \frac{e^{ikr}}{\sqrt{r}} + O(r^{-3/2}) \quad (2.7)$$

as $r = |x|$ tends to infinity. The function of angular variables $u_\infty(\hat{x}, d)$ is known as the *far-field pattern* of the scattered field and, in the case of the inverse problem, it represents the measured data. Indeed the inverse scattering problem we are going to deal with is the one to determine the profile Γ from a complete knowledge of the far-field pattern. Uniqueness for this problem has been established in [36] (see also [17] and [31]).

The knowledge of the far-field pattern allows to define the far-field operator $F : L^2(\Omega) \rightarrow L^2(\Omega)$,

$$(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d) \quad . \quad (2.8)$$

This operator is of course linear and also compact since $u_\infty(\hat{x}, d)$ is known to be infinitely differentiable with respect to its independent variables [17]. It is now possible to introduce the far-field equation

$$(Fg_z)(\hat{x}) = \Phi_\infty(\hat{x}, z) \quad , \quad (2.9)$$

where $\Phi_\infty(\hat{x}, z)$ is the far-field pattern of the fundamental solution $\Phi(x, z)$ of the Helmholtz equation. Explicitly we have

$$\Phi(x, z) = \frac{i}{4} H_0^{(1)}(k|x - z|) \quad , \quad (2.10)$$

where $H_0^{(1)}$ denotes a Hankel function of the first kind of order zero and

$$\Phi_\infty(\hat{x}, z) = \gamma e^{-ik\hat{x} \cdot z} \quad (2.11)$$

with

$$\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \quad . \quad (2.12)$$

We note that equation (2.9) is actually a one-parameter family of equations, where the parameter is given by the point z in \mathbb{R}^2 . The analysis of this set of equations and, in

particular, the study of the behaviour of the solutions g_z for all possible positions of z is at the basis of the linear sampling method.

The task of analyzing the far-field equation (2.9) can be accomplished by means of an appropriate factorization of the far-field operator F . To this end we have to introduce the concept of a Herglotz wave function and the operator mapping boundary data onto far-field patterns. The Herglotz wave function with kernel g in $L^2(\Omega)$ is defined by

$$v_g(x) = \int_{\Omega} e^{ikx \cdot d} g(d) ds(d) \quad . \quad (2.13)$$

To define the boundary operator, we consider the general exterior Dirichlet problem

$$\Delta_2 w(x) + k^2 w(x) = 0 \quad x \in \mathbb{R}^2 \setminus \overline{D} \quad (2.14)$$

$$w = f \quad x \in \Gamma \quad (2.15)$$

where w is radiating. The boundary operator $B : H^{1/2}(\Gamma) \rightarrow L^2(\Omega)$ is now defined to be the linear operator mapping the boundary data f onto the far-field pattern w_{∞} of the radiating solution w . Since w depends continuously on the boundary data $f \in H^{1/2}(\Gamma)$, B is bounded. Moreover, in [4] it is proved that it is compact, injective and has range dense in $L^2(\Omega)$. A characterization of D is given in the following lemma:

Lemma 2.1. *If $\Phi_{\infty}(\hat{x}, z)$ is the far-field pattern of the fundamental solution of the Helmholtz equation $\Phi(x, z)$ then $\Phi_{\infty}(\hat{x}, z)$ is in the range of B if and only if $z \in D$.*

Proof. If z is in D , $\Phi(x, z)$ is a radiating solution of the Helmholtz equation outside \overline{D} with boundary data $\Phi(x, z)|_{\Gamma}$. Therefore $\Phi(x, z)|_{\Gamma}$ is the solution of $Bf = \Phi_{\infty}$. If z is outside D , then $\Phi(x, z)$ would be a radiating solution of the Helmholtz equation and this is not possible since it is not in $H_{loc}^1(\mathbb{R}^2 \setminus D)$. □

The factorization we are looking for can be now quite naturally obtained. In fact, if we consider the original scattering problem (2.3)-(2.6), we have that $f = -e^{ikx \cdot d}$ and $Bf = u_{\infty}$. Therefore, if $\mathcal{H}g$ is the trace of the Herglotz wave function over Γ , i.e. $(\mathcal{H}g)(x) = v_g|_{\Gamma}$, then

$$B\mathcal{H}g = -Fg \quad . \quad (2.16)$$

Now consider the Helmholtz equation in D ,

$$\Delta_2 u(x) + k^2 u(x) = 0 \quad x \in D \quad . \quad (2.17)$$

An important result concerning Herglotz wave functions and their relationship with the solutions of (interior) Dirichlet problems is given in [18, 22]. Here it is shown that a

solution of the Helmholtz equation in a bounded domain D with smooth boundary Γ can be approximated by a Herglotz wave function in $H^1(D)$ (for related approximation results for a broad class of elliptic equations, see [40]). From this result it follows that any function f in the trace space $H^{1/2}(\Gamma)$ can be approximated by the trace of a Herglotz wave function. Now we have all the ingredients to prove the general theorem characterizing the behaviour of an approximate solution of the far-field equation at the boundary of the obstacle:

Theorem 2.2. [4] *Let us assume that k^2 is not an eigenvalue for the negative Laplacian in D . Then, if F is the far-field operator corresponding to the scattering problem (2.3)-(2.6), we have that*

1) *if $z \in D$ then for every $\epsilon > 0$ there exists a solution $g_z \in L^2(\Omega)$ of the inequality*

$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon \quad (2.18)$$

such that

$$\lim_{z \rightarrow \Gamma} \|g_z\|_{L^2(\Omega)} = \infty \quad (2.19)$$

and

$$\lim_{z \rightarrow \Gamma} \|v_{g_z}\|_{H^1(D)} = \infty \quad (2.20)$$

where v_{g_z} is the Herglotz wave function with kernel g_z ;

2) *if $z \notin D$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_z \in L^2(\Omega)$ of the inequality*

$$\|Fg_z - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon + \delta \quad (2.21)$$

such that

$$\lim_{\delta \rightarrow 0} \|g_z\|_{L^2(\Omega)} = \infty \quad (2.22)$$

and

$$\lim_{\delta \rightarrow 0} \|v_{g_z}\|_{H^1(D)} = \infty \quad (2.23)$$

Proof. Let us first consider z in D . Then, from Lemma 2.1, there exists f_z in $H^{1/2}(\Gamma)$ such that $(Bf_z)(\hat{x}) = -\Phi_\infty(\hat{x}, z)$. From [22] it follows that, for $\epsilon > 0$, there exists a Herglotz wave function v_{g_z} with kernel g_z in $L^2(\Omega)$ such that

$$\|\mathcal{H}g_z - f_z\|_{H^{1/2}(\Gamma)} < \epsilon \quad (2.24)$$

and, from the continuity of B (throughout this paper ϵ is always “generic”, i.e. if C is a positive constant then $C\epsilon$ is replaced by ϵ),

$$\|B\mathcal{H}g_z - Bf_z\|_{L^2(\Omega)} < \epsilon \quad (2.25)$$

Now, from the factorization (2.16), $B\mathcal{H}g_z = -Fg_z$ and therefore

$$\|Fg_z - \Phi_\infty\|_{L^2(\Omega)} < \epsilon \quad . \quad (2.26)$$

Furthermore, since $f_z(x) = -\Phi(x, z)|_\Gamma$, when $z \rightarrow \Gamma$ $\|f_z\|_{H^{1/2}(\Gamma)}$ tends to infinity. But f_z is approximated by v_{g_z} in the trace space and therefore $\|v_{g_z}\|_{H^{1/2}(\Gamma)} \rightarrow \infty$. It follows that $\|v_{g_z}\|_{H^1(D)} \rightarrow \infty$ and, from Schwartz's inequality so does $\|g_z\|_{L^2(\Omega)}$.

Now assume that $z \notin D$. Then $-\Phi_\infty$ is not in the range of B . However, the range of B is dense in $L^2(\Omega)$. This allows to introduce a regularized solution f_z^α of the equation $Bf = -\Phi_\infty$ such that

$$\|Bf_z^\alpha + \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \delta \quad (2.27)$$

with δ small and $\lim_{\alpha \rightarrow 0} \|f_z^\alpha\|_{H^{1/2}(\Gamma)} = \infty$. Again from the density result in [22], f_z^α can be approximated in the trace space by the Herglotz wave function $v_{g_z^\alpha}$ with kernel g_z^α and from the continuity of B we have that

$$\|Bf_z^\alpha - B\mathcal{H}g_z^\alpha\|_{L^2(\Omega)} < \epsilon \quad (2.28)$$

with ϵ arbitrary small. From the inequalities (2.27) and (2.28), the factorization (2.16) and the triangle inequality we obtain

$$\begin{aligned} \|Fg_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} &= \|-B\mathcal{H}g_z^\alpha - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} \leq \\ &\leq \|-B\mathcal{H}g_z^\alpha + Bf_z^\alpha\|_{L^2(\Omega)} + \|Bf_z^\alpha + \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon + \delta \quad . \end{aligned} \quad (2.29)$$

The facts that $\lim_{\alpha \rightarrow 0} \|f_z^\alpha\|_{H^{1/2}(\Gamma)} = \infty$ and that f_z^α is approximated by $v_{g_z^\alpha}$ in the trace space implies

$$\lim_{\alpha \rightarrow 0} \|\mathcal{H}g_\alpha(\cdot, z)\|_{H^{1/2}(\Gamma)} = \infty \quad (2.30)$$

and therefore

$$\lim_{\alpha \rightarrow 0} \|g_\alpha(\cdot, z)\|_{L^2(\Gamma)} = \infty \quad (2.31)$$

and

$$\lim_{\alpha \rightarrow 0} \|v_{g_z^\alpha}\|_{H^1(D)} = \infty \quad . \quad (2.32)$$

The theorem follows by noting that $\lim_{\delta \rightarrow 0} \alpha(\delta) = 0$.

□

Motivated by the above theorem, the *linear sampling method* for solving the inverse scattering problem of determining D from a knowledge of the far-field pattern proceeds as follows: 1) select a grid of ‘‘sampling points’’ in a region known ‘‘a priori’’ to contain D ; 2) use a regularization method to compute an approximate solution to the far-field

equation; 3) choose a cut-off value C and assert that z is in D if and only if $\|g(\cdot; z)\|$ is less than C . This last assertion is not of course an analytical fact: the choice of C is heuristic (for possible methods for choosing C see [16]) and in Theorem 2.2 we cannot allow ϵ and δ to tend to zero, since in general no solution exists to the far-field equation. Hence the linear sampling method as described above should be viewed as an empirical numerical algorithm (with encouraging mathematical motivations) rather than a specific criteria for determining D . How to compute reliable approximations of the solution of the far-field equation will be the subject matter of Section 5. In Section 3 we will obtain a theorem analogous to Theorem 2.2 in the case of an inhomogeneous medium and TM polarization (again a 2D problem) while in Section 4 similar results will be obtained for Maxwell's equations in the case of an obstacle problem with (possibly) mixed boundary conditions and for the case of an inhomogeneous anisotropic problem.

We conclude this section by considering two particular scattering situations typical of real applications: the case when the scattering obstacle is situated in a piecewise constant background medium rather than a homogeneous background, and the case of limited aperture scattering data. In both these special cases the linear sampling method can be formulated with slight modifications.

Let us consider first the case of a piecewise constant background medium. In the simplest case, problem (2.3)-(2.6) is then replaced by the following problem

$$\Delta_2 u + k_0^2 u = 0 \quad x \in D_0 \setminus D \quad (2.33)$$

$$\Delta_2 u + k^2 u = 0 \quad x \in \mathbb{R}^2 \setminus \overline{D_0} \quad (2.34)$$

$$u(x) = e^{ikx \cdot d} + u^s(x) \quad (2.35)$$

$$u = 0 \quad x \in \partial D \quad (2.36)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - ik u^s \right) \quad (2.37)$$

where u is continuously differentiable across the smooth boundary ∂D_0 . We can re-write (2.35) as

$$u(x) = (e^{ikx \cdot d} + u_b^s(x)) + (u^s(x) - u_b^s(x)) \quad (2.38)$$

where u_b^s is the scattering due to the background medium alone, i.e. the case when $D = \emptyset$. Now let $G_\infty(\cdot, z)$ be the far-field pattern of the Green's function $G(\cdot, z)$ for the background medium and consider the modified far-field equation

$$\int_{\Omega} [u_\infty(\hat{x}, d) - u_{b,\infty}^s(\hat{x}, d)] g(d) ds(d) = G_\infty(\hat{x}, z) \quad (2.39)$$

Noting that the kernel of the above integral operator is the far-field pattern corresponding to the incident field $u_b(x, d) = e^{ikx \cdot d} + u_b^s(x)$, we see from Rellich's lemma

and Holmgren's uniqueness theorem that if g is a solution of the modified far-field equation then

$$V_g := \int_{\Omega} u_b(x, d)g(d)ds(d) \quad (2.40)$$

satisfies

$$V_g(\cdot) = -G(\cdot, z) \quad x \in \partial D \quad . \quad (2.41)$$

Conversely, assume k_0^2 is not an eigenvalue for $-\Delta$ in D and for $f \in H^{1/2}(\partial D)$ let $v \in H^1(D)$ be the unique solution of

$$\Delta_2 v + k_0^2 v = 0 \quad x \in D \quad (2.42)$$

$$v = f \quad x \in \partial D \quad . \quad (2.43)$$

Then by Green's formula we can write v in the form

$$v(x) = v_k(x) - \int_D \Phi(x, y)[k^2 - k_0^2]v(y)dy \quad (2.44)$$

where $v_k \in H^1(D)$ is a solution of $\Delta_2 v + k^2 v = 0$ in D . Approximating v_k by a Herglotz wave function in $H^1(D)$ and using the invertibility of the Lippmann-Schwinger integral equation now shows that v can be approximated in $H^1(D)$ by a solution of $\Delta_2 v + k_0^2 v = 0$ of the form (2.40). If $Bf := w_{\infty}$ is the far-field pattern of

$$\Delta_2 w + k_0^2 w = 0 \quad x \in D_0 \setminus D \quad (2.45)$$

$$\Delta_2 w + k^2 w = 0 \quad x \in \mathbb{R}^2 \setminus \bar{D}_0 \quad (2.46)$$

$$w = f \quad x \in \partial D \quad (2.47)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial w}{\partial r} - ikw \right) = 0 \quad (2.48)$$

and

$$(Hg)(x) := \int_{\Omega} u_b(x, d)g(d)ds(d) \quad (2.49)$$

we can now repeat the analysis of the homogeneous background case and conclude that if the modified far-field equation is used then the linear sampling method is valid for the case of piecewise constant background medium. For numerical examples in this situation see [14, 19]

When \hat{x} and d are in a subset Ω_0 of the unit sphere Ω (limited aperture scattering), from the previous analysis it is clear that to treat this case it is enough to show that a Herglotz wave function and its first derivative can be approximated uniformly on compact subsets of a disk B_R of radius R containing D by a Herglotz wave function

with kernel g supported in Ω_0 . We can assume without loss of generality that k^2 is not a Dirichlet eigenvalue for B_R . It suffices to show that the set of functions

$$v_g(x) := \int_{\Omega} g(d) e^{ikx \cdot d} ds(d) \quad (2.50)$$

for $g \in L^2(\Omega)$ with support in $\Omega_0 \subset \Omega$ is complete in $L^2(\partial B_R)$. To this end, let $\varphi \in L^2(\partial B_R)$ and suppose that

$$\int_{\partial B_R} \varphi(x) \left[\int_{\Omega_0} \overline{g(d)} e^{-ikx \cdot d} ds(d) \right] ds(x) = 0 \quad (2.51)$$

for every $g \in L^2(\Omega_0)$. Then we want to show that $\varphi = 0$. To do this we interchange orders of integration to arrive at

$$\int_{\Omega_0} \overline{g(d)} \left[\int_{\partial B_R} \varphi(x) e^{-ikx \cdot d} ds(x) \right] ds(d) = 0 \quad (2.52)$$

for every $g \in L^2(\Omega_0)$ which implies that the far-field pattern of

$$(S\varphi)(y) := \int_{\partial B_R} \varphi(x) \Phi(x, y) ds(x) \quad (2.53)$$

vanishes for $\hat{y} \in \Omega_0$ and hence, by analytic continuation, for $\hat{y} \in \Omega$. By Rellich's lemma we have that $S\varphi = 0$ in $\mathbb{R}^2 \setminus \overline{B_R}$ and hence by continuity of $S\varphi$ across B_R we have that $S\varphi = 0$ in B_R as well, since k^2 is not an eigenvalue. The jump relations for single layer potentials now imply that $\varphi = 0$ and we are done.

3. TM polarization: inhomogeneous medium

Let us consider an infinitely long cylinder characterized by a refractive index

$$n(x) = \frac{1}{\epsilon_0} \left(\epsilon(x) + i \frac{\sigma(x)}{\omega} \right) \quad (3.1)$$

which is piecewise continuously differentiable and such that $m(x) := 1 - n(x)$ has compact support $\overline{D} \subset \mathbb{R}^2$. Here ϵ_0 is the permittivity of free space, ϵ and σ are the permittivity and conductivity of the cylinder and ω is the frequency. We assume that the complement of \overline{D} is connected and \overline{D} has smooth boundary Γ with unit outward normal ν . We further assume that $n(x)$ is smooth except for a jump discontinuity across Γ and that $\text{Im}n(x) \geq c > 0$ in D (the reason of this last assumption is that in the following we will introduce an interior transmission problem whose well-posedness can be established by using this hypothesis). We consider the direct problem of determining $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C^1(\mathbb{R}^2)$ such that

$$\Delta_2 u(x) + k^2 n(x) u(x) = 0 \quad x \in \mathbb{R}^2 \setminus \Gamma \quad (3.2)$$

$$u(x) = e^{ikx \cdot d} + u^s(x) \quad (3.3)$$

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - ik u^s \right) = 0 \quad . \quad (3.4)$$

In [17] it is proved that problem (3.1)-(3.4) is well-posed in the sense of Hadamard. The asymptotic behaviour of the scattered field can again be written in the form (2.7) where the explicit form for the far-field pattern is

$$u_\infty(\hat{x}, d) = -\gamma \int_D e^{-ik\hat{x}\cdot y} m(y) u(y) dy \quad \hat{x} \in \Omega \quad . \quad (3.5)$$

The inverse medium scattering problem we are interested in is to determine the support D of $m(x) = 1 - n(x)$ from a knowledge of the far-field pattern $u_\infty(\hat{x}, d)$. Uniqueness for this problem has been established in [49] (see also Corollary 3.1.13 of [43]). Note that it is unknown whether or not point values of the refractive index in \mathbb{R}^2 are uniquely determined from far-field measurements.

In order to study the far-field equation (2.9), we observe that the scattering problem (3.1)-(3.4) can be written in an integral equation form by means of the Lippmann-Schwinger equation [17]

$$e^{ikx\cdot d} = u(x) + k^2(Tu)(x) \quad (3.6)$$

where $T : L^2(D) \rightarrow L^2(D)$ is defined by

$$(T\varphi)(x) := \int_D \Phi(x, y) m(y) \varphi(y) dy \quad x \in D \quad . \quad (3.7)$$

From equations (3.5)-(3.7), and by exploiting the fact that $(I + k^2T)$ is an isomorphism in $L^2(D)$, we have that the far-field equation assumes the explicit form

$$-\gamma \int_D e^{-ik\hat{x}\cdot y} m(y) (I + k^2T)^{-1} v_{g_z}(y) dy = \Phi_\infty(\hat{x}, z) \quad . \quad (3.8)$$

Now let us introduce the linear operator $\mathcal{F} : L^2(D) \rightarrow L^2(\Omega)$, defined by

$$(\mathcal{F}\phi)(\hat{x}) := \gamma \int_D e^{-ik\hat{x}\cdot y} m(y) \phi(y) dy \quad (3.9)$$

so that the far-field equation is factorized in the form

$$\mathcal{F}(I + k^2T)^{-1} v_{g_z} = -e^{-ik\hat{x}\cdot z} \quad (3.10)$$

where $\mathcal{F}(I + k^2T)^{-1}$ is a bounded operator from $L^2(D)$ onto $L^2(\Omega)$.

Before proceeding further we need to study the interior transmission problem

$$\Delta_2 w + k^2 n(x) w = 0, \quad \Delta v + k^2 v = 0 \quad x \in D \quad (3.11)$$

$$w - v = \Phi, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} \Phi \quad x \in \Gamma \quad . \quad (3.12)$$

When $z \in D$, the pair (v, w) with $v \in \overline{H}$ (the closure of the set of the Herglotz wave functions in $L^2(D)$) and $w \in L^2(D)$ is said to be a weak solution of problem (3.11)-(3.12) with point source z if v and w satisfy the integral equation

$$w(x) + k^2(Tw)(x) = v(x) \quad x \in D \quad (3.13)$$

and the boundary condition

$$-k^2(Tw)(x) = \Phi(x, z) \quad x \in \partial B \quad (3.14)$$

where B is a ball centered in the origin, ∂B is its boundary and $\overline{D} \subset B$. It is proved in [17, 21, 46] that, if the technical condition $\text{Im}n \geq c > 0$ is satisfied, then the following theorem holds:

Theorem 3.1. *For every point $z \in D$, there exists a unique weak solution of the interior transmission problem (3.11)-(3.12).*

By means of this theorem, in [4] it is shown that the restriction of $\mathcal{F}(I + k^2T)^{-1}$ from \overline{H} onto $L^2(\Omega)$ is injective and has dense range. Moreover, Theorem 3.1 can be used to prove the following lemma characterizing the range of $\mathcal{F}(I + k^2T)^{-1}$:

Lemma 3.2. *The function $e^{-ik\hat{x}\cdot z}$ is in the range of $\mathcal{F}(I + k^2T)^{-1}$ if and only if z is in D .*

Proof. First, let z be in D . Since $(I + k^2T)^{-1}$ is an isomorphism in $L^2(D)$, we have to show that there exists ψ in $L^2(D)$ such that

$$\mathcal{F}\psi = e^{-ik\hat{x}\cdot z} \quad . \quad (3.15)$$

From Theorem 3.1 we know that there exists w in $L^2(D)$ such that (3.14) holds, i.e., after appropriate rescaling, such that

$$(Tw)(x) = \gamma^{-1}\Phi(x, z) \quad x \in \partial B \quad (3.16)$$

holds. Since Tw and $\gamma^{-1}\Phi(\cdot, z)$ are radiating solutions outside B , by the well-posedness of the direct problem they coincide in $\mathbb{R}^2 \setminus \overline{B}$. Therefore, from the unique continuation principle, they coincide up to ∂D . But $\mathcal{F}w$ is the far-field pattern of Tw and $e^{-ik\hat{x}\cdot z}$ is the far-field pattern of $\gamma^{-1}\Phi$ and so we are done.

If z is not in D , a solution of (3.15) does not exist since, if it did, (3.16) would be valid for $x \in \mathbb{R}^2 \setminus \overline{D}$ and this is impossible because $\Phi(\cdot, z)$ has a singularity outside D . □

After observing that, by the definition of \overline{H} , each function in \overline{H} can be approximated by a Herglotz wave function in $L^2(D)$ we can now establish the following theorem that is analogous to Theorem 2.2 for obstacle scattering:

Theorem 3.3. [4] *Let Γ be smooth, $\text{Im}n(x) \geq c > 0$ for $x \in D$ and F the far-field operator corresponding to the inhomogeneous scattering problem (3.1)-(3.4). Then*

1) if $z \in D$, it follows that for every $\epsilon > 0$ there is a solution $g_z \in L^2(\Omega)$ of the inequality

$$\|Fg_z - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \epsilon \quad (3.17)$$

such that

$$\lim_{z \rightarrow \Gamma} \|g_z\|_{L^2(\Omega)} = \infty \quad (3.18)$$

and

$$\lim_{z \rightarrow \Gamma} \|v_{g_z}\|_{L^2(D)} = \infty \quad , \quad (3.19)$$

where v_{g_z} is the Herglotz wave function with kernel g_z ;

2) if $z \notin D$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution g_z of the inequality

$$\|Fg_z - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \epsilon + \delta \quad (3.20)$$

such that

$$\lim_{\delta \rightarrow 0} \|g_z\|_{L^2(\Omega)} = \infty \quad (3.21)$$

and

$$\lim_{\delta \rightarrow 0} \|v_{g_z}\|_{L^2(D)} = \infty \quad . \quad (3.22)$$

Proof. If z is in D , then the interior transmission problem (3.11)-(3.12) has a unique solution $v \in \overline{H}(D)$ and $w \in L^2(D)$. By the definition of \overline{H} , v can be approximated by a Herglotz wave function v_{g_z} with kernel g_z , so that

$$\|v - v_{g_z}\|_{L(D)} < \epsilon \quad . \quad (3.23)$$

By the continuity of $(I + k^2T)^{-1}$, (3.13) and defining $u_{g_z} := (I + k^2T)^{-1}v_{g_z}$ we obtain

$$\|w - u_{g_z}\|_{L^2(D)} < \epsilon \quad . \quad (3.24)$$

Now we use equation (3.14) and the continuity of T to obtain

$$\|k^2Tu_{g_z} + \Phi(\cdot, z)\|_{C(\partial B)} < \epsilon \quad . \quad (3.25)$$

From equation (3.3), and the Lippmann-Schwinger equation (3.5) we have that $u^s(x) = -k^2(Tu)(x)$. If we consider the inhomogeneous scattering problem with v_{g_z} as incident field, by superposition we have that $-k^2(Tu_{g_z})(x)$ is the corresponding scattered field with far-field pattern

$$-k^2(T_\infty v_{g_z})(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}, d) g_z(d) ds(d) \quad (3.26)$$

and therefore from (3.25) we have that

$$\left\| \int_{\Omega} e_\infty(\cdot, d) g_z(d) ds(d) - \Phi_\infty(\cdot, z) \right\|_{L^2(\Omega)} < \epsilon \quad . \quad (3.27)$$

Now we know that $H^{3/2}(\Gamma)$ is continuously embedded in $C(\Gamma)$. Therefore

$$\|\Phi(\cdot, z)\|_{C(\Gamma)} \leq c\|\Phi(\cdot, z)\|_{H^{3/2}(\Gamma)} = c\|k^2Tw\|_{H^{3/2}(\Gamma)} \quad (3.28)$$

with c a generic constant. The trace operator and T are continuous and therefore

$$\|k^2Tw\|_{H^{3/2}(\Gamma)} \leq c\|k^2Tw\|_{H^2(D)} \leq c\|w\|_{L^2(D)} \quad . \quad (3.29)$$

But $w = (I + k^2T)^{-1}v$ and $(I + k^2T)^{-1}$ is continuous; it follows that

$$\|w\|_{L^2(D)} \leq c\|v\|_{L^2(D)} \quad . \quad (3.30)$$

Finally, from (3.23), we have that

$$\|v\|_{L^2(D)} = \|v_{g_z} + (v - v_{g_z})\|_{L^2(D)} \leq \|v_{g_z}\|_{L^2(D)} + \epsilon \quad . \quad (3.31)$$

Now we combine the inequalities (3.28)-(3.31) to obtain

$$\|\Phi(\cdot, z)\|_{C(\Gamma)} \leq c(\|v_{g_z}\|_{L^2(D)} + \epsilon) \quad . \quad (3.32)$$

Since $\Phi(\cdot, z)$ has a logarithmic singularity when z tends to Γ , we obtain that $\|v_{g_z}\|_{L^2(D)}$ blows up and, from Schwartz's inequality, so does $\|g_z\|_{L^2(\Omega)}$.

Now let us consider the case where z is outside D . In this case $e^{-ik\hat{x}\cdot z}$ is not in the range of $\mathcal{F}(I + k^2T)^{-1}$ but since the range of this operator is dense in $L^2(\Omega)$, we can find a regularized approximate solution v^α such that

$$\|\mathcal{F}(I + k^2T)^{-1}v^\alpha + e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} < \delta \quad (3.33)$$

and

$$\lim_{\alpha \rightarrow 0} \|v^\alpha\|_{L^2(D)} = \infty \quad . \quad (3.34)$$

Since v^α is in \overline{H} we can approximate it by a Herglotz wave function $v_{g_z^\alpha}$ with kernel g_z^α so that

$$\|v^\alpha - v_{g_z^\alpha}\|_{L^2(D)} < \epsilon \quad . \quad (3.35)$$

By continuity of $\mathcal{F}(I + k^2T)^{-1}$ we have that

$$\|\mathcal{F}(I + k^2T)^{-1}v^\alpha - \mathcal{F}(I + k^2T)^{-1}v_{g_z^\alpha}\|_{L^2(D)} < \epsilon \quad (3.36)$$

and by combining the inequalities (3.33) and (3.36) and using the triangle inequality one obtains

$$\|\mathcal{F}(I + k^2T)^{-1}v_{g_z^\alpha} + e^{-ik\hat{x}\cdot z}\|_{L^2(D)} < \epsilon + \delta \quad . \quad (3.37)$$

Equation (3.34) and the inequality (3.35) now imply that

$$\lim_{\alpha \rightarrow 0} \|v_{g_z^\alpha}\|_{L^2(D)} = \infty \quad (3.38)$$

and

$$\lim_{\alpha \rightarrow 0} \|g_z^\alpha\|_{L^2(\Omega)} = \infty \quad (3.39)$$

thus proving the theorem, noting that $\lim_{\delta \rightarrow 0} \alpha(\delta) = 0$.

□

4. Maxwell's equations

The formulation of the linear sampling method for 3D electromagnetic inverse scattering problems is a very recent achievement and most of the work concerning these results is contained in papers still in press. The main difficulties in addressing 3D problems are basically two. First, the formulation of the direct scattering problem must be performed in function spaces that are more complicated than the ones used for 2D problems. Second, as a consequence of this, the mathematical techniques used to establish uniqueness and the linear sampling method become rather technical. However, the logical scheme one must follow in order to obtain the desired theorems is basically the same as that followed in the previous two sections. For these reasons, in the following we will provide a description of the main ideas required to appropriately deal with Maxwell's equations but for the proofs of the theorems we will refer to the original papers.

We first consider the electromagnetic inverse scattering problem for a (possibly) partially coated Lipschitz domain. We assume that D is a bounded region with boundary Γ such that $D_e := \mathbb{R}^3 \setminus \overline{D}$ is connected and each simply connected piece of D is a Lipschitz curvilinear polyhedron. Furthermore we assume that the boundary Γ is split into two disjoint parts Γ_D and Γ_I having Π as their possible common boundary in Γ and that each part Γ_D and Γ_I can be written as the union of a finite number of open smooth faces. Finally, let ν denote the unit outward normal defined almost everywhere on Γ . The direct scattering problem we are interested in is to determine an electromagnetic field (E^t, H^t) such that

$$\operatorname{curl} E^t - ikH^t = 0 \quad (4.1)$$

$$\operatorname{curl} H^t + ikE^t = 0 \quad (4.2)$$

for $x \in \mathbb{R}^3 \setminus \overline{D}$ and

$$\nu \times E^t = 0 \quad x \in \Gamma_D \quad (4.3)$$

$$\nu \times \operatorname{curl} E^t - i\lambda(\nu \times E^t) \times \nu = 0 \quad x \in \Gamma_I \quad (4.4)$$

where $\lambda > 0$ is the surface impedance which is assumed to be a possibly different constant on each connected set of Γ_I (note that the case of a perfect conductor corresponds to the case when $\Gamma_I = \emptyset$). We introduce the incident fields

$$E^i(x) := \frac{i}{k} \operatorname{curl} \operatorname{curl} p e^{ikx \cdot d} = ik(d \times p) \times d e^{ikx \cdot d} \quad (4.5)$$

$$H^i(x) := \operatorname{curl} p e^{ikx \cdot d} = ikd \times p e^{ikx \cdot d} \quad (4.6)$$

where k is the positive wave number, d is a unit vector giving the propagation direction and p is the polarization vector. Finally the scattered field (E, H) defined by

$$E^t = E^i + E \quad (4.7)$$

$$H^t = H^i + H \quad (4.8)$$

is required to satisfy the Silver-Müller radiation condition

$$\lim_{r \rightarrow \infty} (H \times x - rE) = 0 \quad . \quad (4.9)$$

Problem (4.1)-(4.9) is a particular case of the general exterior mixed boundary value problem

$$\text{curl curl } E - k^2 E = 0 \quad x \in D_e \quad (4.10)$$

$$\nu \times E = f \quad x \in \Gamma_D \quad (4.11)$$

$$\nu \times \text{curl } E - i\lambda(\nu \times E) \times \nu = h \quad x \in \Gamma_I \quad (4.12)$$

$$\lim_{r \rightarrow \infty} (H \times x - rE) = 0 \quad (4.13)$$

with $H = (1/ik)\text{curl } E$. The first question we want to address is to determine in which function spaces the direct problem is well-posed. To this end, we define

$$X(D, \Gamma_I) := \{u \in H(\text{curl}, D) : \nu \times u|_{\Gamma_I} \in L_t^2(\Gamma_I)\} \quad (4.14)$$

equipped with the natural norm

$$\|u\|_{X(D, \Gamma_I)}^2 := \|u\|_{H(\text{curl}, D)}^2 + \|\nu \times u\|_{L^2(\Gamma_I)}^2 \quad . \quad (4.15)$$

In this definition

$$H(\text{curl}, D) := \{u \in (L^2(D))^3 : \text{curl } u \in (L^2(D))^3\} \quad (4.16)$$

and

$$L_t^2(\Gamma_I) := \{u(L^2(\Gamma_I))^3 : \nu \cdot u = 0 \text{ on } \Gamma_I\} \quad , \quad (4.17)$$

where $(L^2(D))^3$ and $(L^2(\Gamma_I))^3$ denote the product of the standard L^2 spaces over D and Γ_I . The Hilbert spaces $X(D, \Gamma_I)$ and $H(\text{curl}, D)$ can also be defined on an exterior domain D_e by introducing a ball $B_R \supset D$ of radius R and by considering functions defined on $D_e \cap B_R$ for every R . The spaces obtained in this way will be denoted by $X_{loc}(D_e, \Gamma_I)$ and $H_{loc}(\text{curl}, D_e)$. We also introduce the trace space of $X(D, \Gamma_I)$ on the complementary part Γ_D as

$$Y(\Gamma_D) := \left\{ f \in (H^{-1/2}(\Gamma_D))^3 : \exists u \in H_0(\text{curl}, B_R), \right. \\ \left. \nu \times u|_{\Gamma_I} \in L_t^2(\Gamma_I) \text{ and } f = \nu \times u|_{\Gamma_D} \right\} \quad . \quad (4.18)$$

Here $(H^{-1/2}(\Gamma_D))^3$ is the product of the standard Sobolev space defined on Γ_D and

$$H_0(\text{curl}, B_R) := \{u \in H(\text{curl}, B_R) \ , \ \nu \times u|_{\partial B_R} = 0\} \ , \quad (4.19)$$

where ∂B_R is the boundary of B_R . The trace space is equipped with the norm

$$\|f\|_{Y(\Gamma_D)}^2 := \inf\{\|u\|_{H(\text{curl}, B_R)}^2 + \|\nu \times u\|_{L^2(\Gamma_I)}^2\} \quad (4.20)$$

where the minimum is taken over all functions $u \in H_0(\text{curl}, B_R)$ such that $\nu \times u|_{\Gamma_I} \in L_t^2(\Gamma_I)$ and $f = \nu \times u|_{\Gamma_D}$. With respect to this norm, $Y(\Gamma_D)$ is a Hilbert space. The introduction of all these spaces allows us to establish the well-posedness of the direct problem [11]: the problem of determining $E \in X_{loc}(D_e, \Gamma_I)$ satisfying (4.10)-(4.13) given $f \in Y(\Gamma_D)$ and $h \in L_t^2(\Gamma_I)$ has a unique solution which depends continuously on the boundary data f and h . We point out that the spaces introduced to prove this well-posedness results will also play a crucial role in the solution of the inverse scattering problem.

From [17] it is known that the radiating solutions (E, H) to the exterior problem (4.10)-(4.13) have the asymptotic behaviour

$$\begin{aligned} E(x) &= \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right) \right\} \ , \\ H(x) &= \frac{e^{ik|x|}}{|x|} \left\{ H_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right) \right\} \end{aligned} \quad (4.21)$$

as $|x| \rightarrow \infty$, where $E_\infty(\cdot; d, p)$ and $H_\infty(\cdot; d, p)$ are defined on the unit sphere Ω and are known as the electric far-field pattern and magnetic far-field pattern respectively. We consider the inverse scattering problem of determining D from a knowledge of the electric far-field pattern $E_\infty(\hat{x}; d, p)$, without any *a priori* assumption or knowledge of Γ_D , Γ_I and λ (this is a major advantage of the approach based on the linear sampling method). The solution of this inverse scattering problem is unique. In fact, to prove this one can essentially follow the approach described in Theorem 7.1 of [17], where only the use of the well-posedness for the forward problem is required.

The derivation of the theorem at the basis of the linear sampling method in the vector case follows the same approach as in the scalar case. In particular, we first introduce the far-field equation

$$Fg_z(\hat{x}) = E_{e,\infty}(\hat{x}, z, q) \quad (4.22)$$

for a set of sampling points $z \in \mathbb{R}^3$ and three linear independent polarizations $q \in \mathbb{R}^3$. Here $F : L_t^2(\Omega) \rightarrow L_t^2(\Omega)$ is the far-field operator defined by

$$(Fg)(\hat{x}) = \int_{\Omega} E_\infty(\hat{x}; d, g(d)) ds(d) \quad (4.23)$$

and $E_{e,\infty}(\hat{x}, z, q)$ is the electric far-field pattern of the electric dipole

$$E_e(x, z, q) := \frac{i}{k} \text{curl}_x \text{curl}_x q \Phi(x, z) \quad (4.24)$$

$$H_e(x, z, q) := \operatorname{curl}_x q \Phi(x, z) \quad (4.25)$$

with Φ being the fundamental solution of the Helmholtz equation defined by

$$\Phi(x, z) = \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|} \quad (4.26)$$

Explicitly,

$$E_{e,\infty}(\hat{x}, z, q) = \frac{ik}{4\pi} (\hat{x} \times q) \times \hat{x} e^{-ik\hat{x} \cdot z} \quad (4.27)$$

Coming back to the exterior problem (4.10)-(4.13), we introduce the linear operator $\mathcal{B} : Y(\Gamma_D) \times L_t^2(\Gamma_I) \rightarrow L_t^2(\Omega)$ mapping the boundary data (f, h) onto the electric far-field pattern $E_\infty(\hat{x}, d, p)$. It is proved in [11] that this operator is injective, compact and has dense range in $L_t^2(\Omega)$. By using \mathcal{B} it is possible to write the far-field equation as

$$\mathcal{B}(\Lambda E_{g_z})(\hat{x}) = \frac{1}{ik} E_{e,\infty}(\hat{x}, z) \quad (4.28)$$

where Λ is the trace operator corresponding to the mixed boundary condition, i.e. $\Lambda u := \nu \times u|_{\Gamma_D}$ on Γ_D and $\Lambda u := \nu \times \operatorname{curl} u - i\lambda(\nu \times u) \times \nu|_{\Gamma_I}$ on Γ_I , and E_{g_z} is the electric field of the Herglotz pair with kernel g_z defined by

$$E_{g_z}(x) = \int_{\Omega} e^{ikx \cdot d} g_z(d) ds(d) \quad , \quad H_{g_z}(x) = \frac{1}{ik} \operatorname{curl} E_{g_z}(x) \quad (4.29)$$

The linear sampling method is based on the following characterization of D [11]:

Lemma 4.1 *The electric far-field pattern of the electric dipole $E_{e,\infty}(\hat{x}, z, q)$ is in the range of \mathcal{B} if and only if $z \in D$.*

We now consider the interior problem

$$\operatorname{curl} \operatorname{curl} E - k^2 E = 0 \quad x \in D \quad (4.30)$$

$$\nu \times E = f \quad x \in \Gamma_D \quad (4.31)$$

$$\nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = h \quad x \in \Gamma_I \quad (4.32)$$

For this problem an approximation result using Herglotz wave functions is valid. In particular the following theorem is proved in [11]:

Theorem 4.2. *The electric field E of the solution of the interior mixed boundary value problem (4.30)-(4.32) can be approximated in $X(D, \Gamma_I)$ by the electric field of an electromagnetic Herglotz pair.*

The factorization (4.28) together with the approximation result described in this theorem allows us to prove the following theorem:

Theorem 4.3. [11] *Assume $\Gamma_I \neq \emptyset$. Then if F is the far-field operator corresponding to the scattering problem (4.1)-(4.9), we have that*

1) *if $z \in D$ then for every $\epsilon > 0$ there is a solution $g_z(\hat{x}) \in L_t^2(\Omega)$ satisfying the inequality*

$$\|Fg_z - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon \quad (4.33)$$

such that

$$\lim_{z \rightarrow \Gamma} \|E_{g_z}\|_{X(D, \Gamma_I)} = \infty \quad (4.34)$$

and

$$\lim_{z \rightarrow \Gamma} \|g_z\|_{L_t^2(\Omega)} = \infty \quad (4.35)$$

where E_{g_z} is the electric field of the electromagnetic Herglotz pair with kernel g_z ;

2) *If $z \notin D$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_z \in L_t^2(\Omega)$ of the inequalities*

$$\|Fg_z - E_{e,\infty}(\cdot, z, q)\|_{L_t^2(\Omega)} < \epsilon + \delta \quad (4.36)$$

such that

$$\lim_{\delta \rightarrow 0} \|E_{g_z}\|_{X(D, \Gamma_I)} = \infty \quad (4.37)$$

and

$$\lim_{\delta \rightarrow 0} \|g_z\|_{L_t^2(\Omega)} = \infty \quad (4.38)$$

This theorem holds also for the case of a perfect conductor (that is when $\Gamma_I = \emptyset$) under the condition that k is not a Maxwell eigenvalue [38]. In [5] it is shown how to avoid this problem of Maxwell eigenvalues by combining far-field patterns coming from different polarizations of the incident field. For numerical examples illustrating the implementation of Theorem 4.3 see [16, 11].

We now consider the general case of Maxwell's equations in an inhomogeneous anisotropic medium (which of course includes the isotropic medium as a special case). We assume that $D \subset \mathbb{R}^3$ is a bounded domain with connected complement such that the boundary Γ is in class C^2 with unit outward normal ν . We also introduce the 3×3 symmetric matrix N whose entries are bounded complex-valued functions in \mathbb{R}^3 and such that N is the identity matrix outside D . In particular, we will assume there exists $\gamma > 0$ such that

$$\operatorname{Re}(N\xi, \xi) \geq \gamma \|\xi\|^2 \quad \forall \xi \in C^3 \quad \text{and a.e. in } \mathbb{R}^3 \quad (4.39)$$

and

$$\operatorname{Im}(N\xi, \xi) > 0 \quad \forall \xi \in C^3 \setminus \{0\} \quad \text{and a.e. in } D \quad . \quad (4.40)$$

In order to establish the linear sampling method for this problem, we also need to make the assumption that $N - I$ is invertible and $\operatorname{Re}(N - 1)^{-1}$ is uniformly definite positive in D (partial results for the case when this is not true can be found in [43]). We now introduce the spaces needed to establish the linear sampling method. In particular, let $\mathcal{U}(D)$ be the space

$$\mathcal{U}(D) := \{u \in H(\operatorname{curl}, D) ; \operatorname{curl} u \in H(\operatorname{curl}, D)\} \quad , \quad (4.41)$$

with the scalar product

$$(u, v)_{\mathcal{U}} = (u, v)_{H(\operatorname{curl}, D)} + (\operatorname{curl} u, \operatorname{curl} v)_{H(\operatorname{curl}, D)} \quad (4.42)$$

and corresponding norm. Then $\mathcal{U}_{loc}(\mathbb{R}^3)$ is the Frechet space of functions $u \in (L^2_{loc}(\mathbb{R}^3))^3$ such that $u \in \mathcal{U}(K)$ for all compact sets $K \subset \mathbb{R}^3$. If E^i is the incident field, i.e. an entire solution to Maxwell's equations

$$\operatorname{curl} \operatorname{curl} E^i - k^2 E^i = 0 \quad x \in \mathbb{R}^3 \quad , \quad (4.43)$$

then the forward problem for the scattering of a time harmonic electromagnetic wave by an anisotropic inhomogeneous medium D with refractive index N is the problem of determining an electric field $E \in \mathcal{U}_{loc}(\mathbb{R}^3)$ such that

$$\operatorname{curl} \operatorname{curl} E - k^2 N E = 0 \quad , \quad x \in \mathbb{R}^3 \quad (4.44)$$

$$E = E^s + E^i \quad (4.45)$$

$$\lim_{r \rightarrow \infty} (\operatorname{curl} E^s \times x - ik|x|E^s) = 0 \quad (4.46)$$

uniformly in \hat{x} . In [37] it has been shown using variational methods that this problem has a unique solution.

From the Stratton-Chu formula it follows that the scattering field of problem (4.43)-(4.46) has the asymptotic behaviour

$$E^s(x) = \frac{e^{ik|x|}}{4\pi|x|} \left\{ E_{\infty}(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\} \quad (4.47)$$

uniformly in all direction \hat{x} , where E_{∞} is the electric far-field pattern. In the corresponding inverse scattering problem the electric far-field pattern is the measured data. In particular, the inverse problem we are going to discuss is the problem to determine the domain D from a knowledge of the electric far-field pattern associated with the incident field (4.5). The uniqueness of the solution of this inverse scattering problem has been recently proved in [6] by using arguments inspired by [28].

We again consider the far-field equation (4.22). We define the space

$$H_{inc}(D) := \{E_0 \in (L^2(D))^3 : \text{curlcurl}E_0 - k^2E_0 = 0, \quad x \in D\} \quad (4.48)$$

and consider the scattered field $E^s \in \mathcal{U}_{loc}(\mathbb{R}^3)$ such that for $E_0 \in H_{inc}(D)$

$$\text{curl curl } E^s - k^2NE^s = k^2(N-1)E_0 \quad x \in \mathbb{R}^3 \quad (4.49)$$

and

$$\lim_{r \rightarrow \infty} (\text{curl } E^s \times x - ik|x|E^s) = 0 \quad (4.50)$$

uniformly in \hat{x} . We introduce the scattering operator $S : H_{inc}(D) \rightarrow (L^2_{loc}(\mathbb{R}^3))^3$ such that $S(E_0) = E^s$. If one further assume that $\text{Im}(N)$ is uniformly definite positive in D , then it is proved in [26] that S is well defined and is a bounded linear operator such that for $E_0 \in H_{inc}(D)$, $S(E_0)$ satisfies (4.49) in the distributional sense. Furthermore, if \mathcal{F} is the integral operator

$$(\mathcal{F}E_0)(\hat{x}) = k^2 \int_D e^{-ik\hat{x}\cdot y} (\hat{x} \times (N-1)(SE_0 + E_0)) \times \hat{x} dy \quad (4.51)$$

then it is shown in [26] that $\mathcal{F} : H_{inc}(D) \rightarrow L^2_t(\Omega)$ is compact, injective and has dense range. The following factorization is now possible: if $\varphi \in L^2_t(\Omega)$ and

$$(\mathcal{H}\varphi)(x) = \int_{\Omega} \varphi(d) e^{+ikx\cdot d} ds(d) \quad x \in \mathbb{R}^3, \quad (4.52)$$

then the far-field equation can be written as

$$\mathcal{F}\mathcal{H}g(\hat{x}; z, q) = E_{e,\infty}(\hat{x}; z, q) \quad (4.53)$$

The linear sampling method is based on the following characterization of D :

Lemma 4.4. *The far-field $E_{e,\infty}(\hat{x}; z, q)$ is in the range of \mathcal{F} if and only if $z \in D$.*

Finally we introduce the space

$$H := \text{span}\{M_n^m, \text{curl}M_n^m : n = 1, 2, \dots, m = -n, \dots, n\} \quad (4.54)$$

where $M_n^m := \text{curl}(xu_n^m(x))$, $u_n^m(x) := j_n(k|x|)Y_n^m(\hat{x})$, Y_n^m , $n = 0, 2, \dots, m = -n, \dots, n$ is a spherical harmonic and j_n is a spherical Bessel function. Then in [26] it is shown that H is dense in $H_{inc}(D)$ with respect to the $(L^2(D))^3$ norm. This density result together with the above lemma allows us to prove the following theorem:

Theorem 4.5. [26] *Assume $N - I$ is invertible and let $\epsilon > 0$. Then*

1) *If $z \in D$, there exists $g_z \in L^2_t(\Omega)$ such that*

$$\|\mathcal{F}\mathcal{H}g_z - E_{e,\infty}(\cdot, z, q)\| < \epsilon, \quad (4.55)$$

$$\lim_{z \rightarrow \Gamma} \|\mathcal{H}g_z\|_{L^2(D)} = \infty \quad (4.56)$$

and

$$\lim_{z \rightarrow \Gamma} \|g_z\| = \infty \quad . \quad (4.57)$$

2) If $z \notin D$ and $\delta > 0$, there exists $g_z \in L^2_t(\Omega)$ such that

$$\|\mathcal{F}\mathcal{H}g_z - E_{e,\infty}(\cdot, z, q)\| < \epsilon + \delta \quad , \quad (4.58)$$

$$\lim_{\delta \rightarrow 0} \|\mathcal{H}g_z\|_{L^2(D)} = \infty \quad (4.59)$$

and

$$\lim_{\delta \rightarrow 0} \|g_z\| = \infty \quad . \quad (4.60)$$

5. Implementation

The implementation of the linear sampling method into an efficient numerical algorithm must allow for two facts. First, the solution of the far-field equation is an ill-posed problem, since in general the far-field pattern of the fundamental solution is not in the range of the far-field operator and, furthermore, the far-field operator is compact. Second, the measured far-field pattern is affected by noise arising from the measurement procedure and this implies that every possible discretization of the far-field equation is characterized by numerical instability. In particular, the condition number associated with the discretized far-field equation is in many cases up to the order of 10^{10} and this explains the wild oscillations characterizing the reconstructions provided by *naive* implementations of the linear sampling method [21]. It is known since the 1960s [51] that regularization theory for the solution of linear ill-posed inverse problems provide stable approximations of the so-called generalized solution, which is the minimum norm, least-squares solution of the linear inverse problem. It can be easily proved that in a Hilbert space setting the generalized solution exists and is unique. A systematic characterization of the relationship between the regularized solution and the approximate solution introduced in Theorems 2.2, 3.2, 4.3 and 4.5 is an open problem. Here, heuristically, we assume that, since the sought after solution only approximately satisfies the far-field equation, then the right-hand-side of the far-field equation corresponding to this solution is not exactly known (but can be arbitrary close to the far-field pattern of the fundamental solution).

As far as the application of regularization theory is concerned, the main peculiarity is that in the case of the far-field equation the measurement noise affects the linear integral operator. This fact will have a significant consequence on the criterion we will adopt in order to select the regularized solution that optimally approximates the solution

of the far-field equation. To describe the regularization procedure, let us consider the general case of a linear operator $A : X \rightarrow Y$, where X and Y are Hilbert spaces and let us assume that $u \in Y$ represents a noise-free datum. Then we consider the problem of determining $f \in X$ from

$$u_\delta = A_h f \tag{5.1}$$

where u_δ represents a noisy version of the datum and A_h an approximate version of the operator such that

$$\|u_\delta - u\| \leq \delta \tag{5.2}$$

and

$$\|A_h - A\| \leq h \tag{5.3}$$

respectively (where the norms are taken in the corresponding spaces). The solution of equation (5.1) does not exist in general, since the random components of the noise carry the datum outside the range of A_h . Furthermore in general the kernel of A_h is not empty and the compactness implies a non-continuous dependence of the solution on the data. Existence and uniqueness can be restored if one looks for the generalized solution of the problem, i.e. the function f^\dagger such that

$$\|A_h f^\dagger - u_\delta\| = \text{minimum} \tag{5.4}$$

and

$$\|f^\dagger\| = \text{minimum} \quad . \tag{5.5}$$

The aim of regularization theory is to exhibit a reliable stable approximation to f^\dagger . A linear regularization algorithm is a one-parameter family of linear operators $\{R_\alpha\}_{\alpha>0}$, $R_\alpha : Y \rightarrow X$, such that

- R_α is continuous for each α ;
- $\lim_{\alpha \rightarrow 0} \|R_\alpha u - f^\dagger\| = 0$ for each u in the range of A .

Together with this definition, a selection rule must be prescribed that is able to produce, for each given value of $\eta = (\delta, h)$, an optimal choice $\alpha_{opt} = \alpha_{opt}(\eta)$ of the regularization parameter α . More precisely, the algorithm $\{R_\alpha\}_{\alpha>0}$ is said to be regularizing if the condition

$$\lim_{\eta \rightarrow 0} R_\alpha u_\delta = f^\dagger \tag{5.6}$$

is satisfied. The Tikhonov method represents a classical regularization method. In this case the regularizing operator R_α is given by

$$R_\alpha = (A_h^* A_h + \alpha I)^{-1} A_h^* \tag{5.7}$$

where $A_h^* : Y \rightarrow X$ is the adjoint operator of A_h . It is easy to prove that in this case the regularized solution $f_\eta^\alpha := R_\alpha g_\delta$ is the element in X minimizing the functional

$$\Phi_\eta^\alpha[f] = \|A_h f - u_\delta\| + \alpha \|f\| \quad . \quad (5.8)$$

We can associate with the functional (5.8) the discrepancy function

$$\rho_\eta = \|A_h f_\eta^\alpha - u_\delta\|^2 - (\delta + h \|f_\eta^\alpha\|)^2 \quad . \quad (5.9)$$

The generalized discrepancy principle for an optimal choice of the regularization parameter is given by the following recipe [52]:

- (i) If $\|u_\delta\| \leq \delta$, let $f_\eta = 0$ be the selected approximation of the generalized solution.
- (ii) If $\|u_\delta\| > \delta$, then:
 - if there is α^* such that $\rho_\eta(\alpha^*) = 0$, then $f_\eta^{\alpha^*}$ is the selected approximation of the generalized solution;
 - if $\rho_\eta(\alpha) > 0 \quad \forall \alpha > 0$, then we directly select the generalized solution.

The following theorem, due to Morozov, proves that the generalized discrepancy principle makes the Tikhonov algorithm a regularizing algorithm.

Theorem 5.1. [52] *The regularized solution selected by the generalized discrepancy principle among the functions $f_\eta^\alpha = R_\alpha u_\delta$ when R_α is given by equation (5.7) converges to the generalized solution of problem (5.1) when $\eta \rightarrow 0$.*

Proof. In this proof we will consider the particular case where u_δ is in the range of A_h so that f^\dagger is the minimum norm solution of equation (5.1). Since the set of solutions of the minimum problem

$$\|A_h f - u_\delta\| = \text{minimum} \quad (5.10)$$

coincides with the Euler equation

$$A_h^* A_h f = A_h^* u_\delta \quad , \quad (5.11)$$

then the generalization to problem (5.4)-(5.5) is straightforward. Let α^* be such that $\rho(\alpha^*) = 0$. Assume that $\lim_{\eta \rightarrow 0} f_\eta^\alpha \neq f^\dagger$. It follows that there must exist $\epsilon > 0$ and a sequence $\{\eta_k\}$ with $\eta_k \rightarrow 0$ for $k \rightarrow \infty$ such that $\|f_{\eta_k}^{\alpha_k^*} - f^\dagger\| > \epsilon$ with $\{f_{\eta_k}^{\alpha_k^*}\}$ the sequence such that $\alpha_k^* = \alpha^*(\eta_k)$. Since $\{f_{\eta_k}^{\alpha_k^*}\}$ minimizes $\Phi_{\eta_k}^{\alpha_k^*}[f]$ we have that

$$\|A_{h_k} \{f_{\eta_k}^{\alpha_k^*}\} - u_{\delta_k}\|^2 + \alpha_k^* \|\{f_{\eta_k}^{\alpha_k^*}\}\|^2 \leq \|A_{h_k} f^\dagger - u_{\delta_k}\|^2 + \alpha_k^* \|f^\dagger\|^2 \quad . \quad (5.12)$$

From the fact that α_k^* is a zero of the discrepancy function (5.9) it follows that

$$\|A_{h_k} f_{\eta_k}^{\alpha_k^*} - u_{\delta_k}\|^2 = (\delta_k + h_k \|\{f_{\eta_k}^{\alpha_k^*}\}\|)^2 \quad (5.13)$$

and from the triangle inequality, together with (5.2) and (5.3), we have that

$$\|A_{h_k} f^\dagger - u_{\delta_k}\|^2 \leq (\|A_{h_k} f^\dagger - A f^\dagger\| + \|A f^\dagger - u_{\delta_k}\|)^2 \leq (h_k \|f^\dagger\| + \delta_k)^2. \quad (5.14)$$

Then from equations (5.12), (5.13) and (5.14) it follows that

$$(\delta_k + h_k \|f_{\eta_k}^{\alpha_k^*}\|)^2 + \alpha_k^* \|f_{\eta_k}^{\alpha_k^*}\|^2 \leq (h_k \|f^\dagger\| + \delta_k)^2 + \alpha_k^* \|f^\dagger\|^2. \quad (5.15)$$

The function $\psi : \|y\| \rightarrow (h_k \|y\| + \delta_k)^2 + \alpha_k^* \|y\|^2$ with $y \in X$ is monotonically increasing and therefore from (5.15)

$$\|f_{\eta_k}^{\alpha_k^*}\| \leq \|f^\dagger\| \quad \forall k. \quad (5.16)$$

The sequence $\{f_{\eta_k}^{\alpha_k^*}\}$ is bounded in the Hilbert space X and therefore it has a subsequence, again denoted by $\{f_{\eta_k}^{\alpha_k^*}\}$, that is weakly convergent to f^* . Furthermore, from the weak lower semicontinuity of the convex functional $y \in X \rightarrow \|y\|$ [2] we have that

$$\|f^*\| \leq \liminf_{k \rightarrow \infty} \|f_{\eta_k}^{\alpha_k^*}\| \leq \limsup_{k \rightarrow \infty} \|f_{\eta_k}^{\alpha_k^*}\| \leq \|f^\dagger\|. \quad (5.17)$$

Since also the convex functional $y \rightarrow \|Ay - Ay^\dagger\|$ is weakly lower semicontinuous we have that

$$\|Az^* - Az^\dagger\| \leq \liminf_{k \rightarrow \infty} \|A f_{\eta_k}^{\alpha_k^*} - A f^\dagger\| \leq \limsup_{k \rightarrow \infty} \|A f_{\eta_k}^{\alpha_k^*} - A f^\dagger\|. \quad (5.18)$$

From the triangle inequality we now have that

$$\begin{aligned} \|A f_{\eta_k}^{\alpha_k^*} - A f^\dagger\| &\leq \|A f_{\eta_k}^{\alpha_k^*} - A_{h_k} f_{\eta_k}^{\alpha_k^*}\| + \|A_{h_k} f_{\eta_k}^{\alpha_k^*} - u_{\delta_k}\| + \|u_{\delta_k} - u\| \leq \\ &\leq h_k \|f_{\eta_k}^{\alpha_k^*}\| + (\delta_k + h_k \|f_{\eta_k}^{\alpha_k^*}\|) + \delta_k \leq \\ &h_k \|f^\dagger\| + (\delta_k + h_k \|f^\dagger\|) + \delta_k \rightarrow 0 \end{aligned} \quad (5.19)$$

as $k \rightarrow \infty$. From this limit and from (5.18) we obtain the $Af^* = Af^\dagger$ and from the facts that $\|f^*\| \leq \|f^\dagger\|$ and that the generalized solution is the unique minimum norm solution we have that $f^* = f^\dagger$, which is a contradiction. Hence $\lim_{\eta \rightarrow 0} f_\eta^{\alpha^*} = f^\dagger$. \square

We want to point out that the generalized discrepancy principle is not necessarily the only possible selection criterion making Tikhonov's method a regularizing algorithm. However this principle provides a rather simple rule for the optimal choice of the regularization parameter in the specific case where the operator modeling the problem is not exactly known. Furthermore we observe that in the case of the far-field equation the generalized discrepancy principle is applied to the case where the noise on the datum u is zero, since in Theorems 2.2, 3.2, 4.3 and 4.5 we can always assume that $\epsilon \ll \delta$.

The availability of a regularization scheme for solving the far-field equation allows us to formulate an effective algorithm for the implementation of the linear sampling method. In particular, for each point of a grid containing the scatterer we proceed as follows:

- construct a one-parameter family of Tikhonov regularized solutions to the far-field equation;
- apply the generalized discrepancy principle for determining an optimal value of the regularization parameter;
- plot the norm of the regularized solution determined by means of the previous two steps.

Some comments about this algorithm are necessary. First, this approach allows two different visualization procedures for the reconstruction; in fact, together with the plot of the norm of the regularized solution, one may also show the plot of the optimal value of the regularization parameter (although there is no theoretical basis for applying this last procedure). Second, it is possible to use regularization methods other than Tikhonov's one to optimally solve the regularized solution. By instance, in [50] the Truncated Singular Value Decomposition, the Landweber method and the Conjugate Gradient method are applied to the linear sampling method with encouraging results. Finally, from a computational point of view, we observe that the most time consuming operation is due to the computation of the zeros of the discrepancy function for each point of the grid. To reduce this time, it is advisable to implement the solution of the far-field equation by means of the Singular Value Decomposition of the far-field operator. Finally, we observe that a very natural parallelization of the algorithm is possible by letting different computers solve the far-field equation for different values of the sampling point z .

6. Numerical applications: 2D scatterers

As a first example of numerical applications, we exploit the measurements made available by the Electromagnetics Technology Division, Air Force Research Laboratory, Hanscom Air Force Base, Massachusetts and known by the name of The Ipswich Data. In particular we consider a data set from a measurement session performed in 1998 by using an echo-free chamber, a fixed transmitter and a receiver rotating around the scatterer with 36 incident and observation angles. The scatterer is represented by an aluminum triangle with outer circle of 6 cm radius (Ips009 at the Ipswich site). In Figure 1 we show the reconstruction of the conducting triangle obtained by means of the linear sampling method. In particular, in Fig. 1(a) the Tikhonov method and the generalized discrepancy principle have been used to obtain the regularized solution of the far-field equation for each point of the scattering region. In Fig. 1(b) the reconstruction has been performed by applying the Truncated Singular Value Decomposition method instead of Tikhonov regularization algorithm. In this case the optimal regularized solution has again been selected by means of the generalized discrepancy principle (the proof of the generalized discrepancy principle in the case of Truncated SVD is currently being

investigated). It is important to note that both reconstructions are reliable and that the Truncated SVD reconstruction is much faster (20 seconds of CPU time on a low cost PC) than the Tikhonov reconstruction (120 seconds on the same PC).

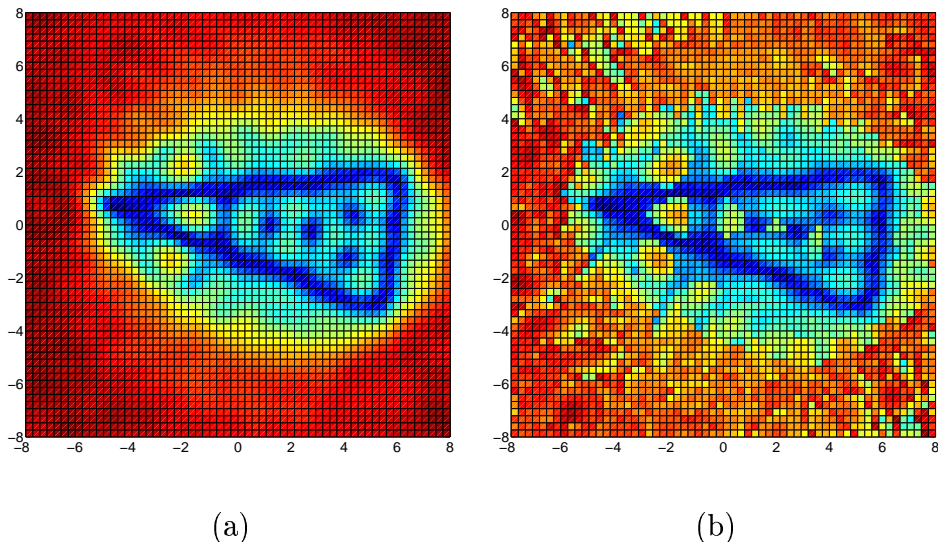


Figure 1. The Ipswich data and the linear sampling method: (a) reconstruction of an aluminum triangle (outer circle of 6 cm radius) by means of the linear sampling method when the Tikhonov algorithm is used for solving the far-field equation; (b) reconstruction of the same object obtained when the Truncated Singular Value Decomposition is applied to solve the far-field equation.

7. Numerical applications: 3D reconstructions from synthetic data

In this section we are concerned with some three dimensional reconstructions of non-smooth and non-convex objects. We choose two examples. The first one corresponds to an aircraft presented in Figure 2 and the second one corresponds to a teapot presented in Figure 6. Both are assumed to be perfectly conducting bodies. We refer to [13] for a more detailed discussion on the numerical validation of the linear sampling method for 3D electromagnetic inverse scattering problems.

7.1. Discrete far field equation

We adopt here the same discretization scheme for the far field equation (4.22) as used in [16]. We consider a uniform triangular meshing of the unit sphere Ω (or $\Omega_0 \subset \Omega$, in the case of limited aperture) containing N vertexes $(\hat{x}_i)_{1 \leq i \leq N}$. These vertexes will serve

as directions for the plane incident waves as well as degrees of freedom for the discrete solution of the far field equation.

Let \hat{p} be a unit vector in \mathbb{R}^3 satisfying $\hat{p} \times \hat{x}_i \neq 0$ for all \hat{x}_i . We define the two orthogonal polarizations

$$\hat{p}_j^\theta = (\hat{p} \times \hat{x}_j)/|\hat{p} \times \hat{x}_j| \quad \text{and} \quad \hat{p}_j^\varphi = \hat{p} \times (\hat{x}_j \times \hat{p})/|\hat{p} \times (\hat{x}_j \times \hat{p})|, \quad (7.1)$$

then set

$$F_{i,j}^\theta = E_\infty(\hat{x}_i, \hat{x}_j, \hat{p}_j^\theta) \quad \text{and} \quad F_{i,j}^\varphi = E_\infty(\hat{x}_i, \hat{x}_j, \hat{p}_j^\varphi). \quad (7.2)$$

We now construct a continuous approximation of the solution $g(\cdot; z, q)$, linear at each triangle, whose degrees of freedom are its values at the nodes $(\hat{x}_i)_{1 \leq i \leq N}$. The nodal values are denoted by $(g_j(z, q))_{1 \leq j \leq N}$. Since each vector $g_j(z, q)$ is tangent to Ω , we can set

$$g_j(z, q) = g_j^\theta(z, q) p_j^\theta + g_j^\varphi(z, q) p_j^\varphi.$$

The integral equation (4.22) can be transformed at the discrete level into the following linear system of $2N$ unknowns $(g_j^\theta, g_j^\varphi)$,

$$\sum_{j=1}^N \omega_j \left(g_j^\theta(z, q) F_{i,j}^\theta + g_j^\varphi(z, q) F_{i,j}^\varphi \right) = E_{e,\infty}(\hat{x}_i, z, q), \quad \text{for all } i, \quad (7.3)$$

where the weights ω_j are linked to the quadrature formulas used in evaluating the integrals over the mesh triangles. Let us remark that, for fixed index i , the vectorial equation (7.3) can be split into two independent scalar equations. System (7.3) is then a $2N \times 2N$ linear system that can be written into the synthetic form

$$\mathcal{F} G(z, q) = E_{e,\infty}(z, q), \quad (7.4)$$

where \mathcal{F} is a $2N \times 2N$ matrix that is independent of the parameters z and q , $G(z, q)$ is the unknown vector whose ℓ^2 norm is expected to be large when z is outside D and $E_{e,\infty}(z, q)$ represents the right hand side constructed from the far fields of electromagnetic dipoles.

7.2. Synthetic generation of the far fields

We have used CESC to create our synthetic data. CESC is a solver for electromagnetic scattering problems developed at CERFACS. The perfectly conducting case is treated by solving the Electric Field Integral Equation, whose unknown is $J = (\nu \times H)|_{\partial D}$, the electric current flowing on the surface of the scatterer. The numerical procedure is based upon a meshing of the surface with triangles and the use of Raviart-Thomas's

finite elements of lowest degree, conforming for $H^{-\frac{1}{2}}(\text{Div}, \partial D_h)$. It leads to

Find J_h in V_h such that

$$\begin{aligned} & \int_{\partial D_h} \int_{\partial D_h} k \Phi(x, y) \left(J_h(y) \cdot J_h^{test}(x) - \frac{1}{k^2} \text{Div} J_h(y) \text{Div} J_h^{test}(x) \right) ds(y) ds(x) \\ & = i \int_{\Gamma} E^i(x) \cdot J_h^{test}(x) ds(x), \quad \text{for all } J_h^{test}(x) \text{ in } V_h, \end{aligned}$$

where V_h is the space spanned by the Rao-Wilson-Glisson basis functions, [45]. The numerical computation amounts to solving a set of linear systems with a full symmetric non hermitian matrix whose size is the number of edges of the mesh. The number of second terms is the number of incident directions times two for the two polarizations. Special attention has been paid in CESC for properly taking into account the singularity of the Green kernel during the assembly process. The LU decomposition of the matrix is performed by means of a set of ScaLAPACK parallel routines. To achieve a good precision, the meshing of the scatterers have been done in such a way that the largest length of the edges is no more than a tenth of the wavelength.

Once \mathcal{F} is generated, we corrupt it with random noise. Let \mathcal{N} be a $2N \times 2N$ matrix whose entries are random numbers in the interval $[0, 1]$ and let $0 < \delta < 1$ be the noise level. We define the noisy far field matrix \mathcal{F}_δ by

$$\mathcal{F}_\delta := \mathcal{F} + \delta(\mathcal{N} + i\mathcal{N})\mathcal{F}.$$

7.3. Numerical procedure

As we mentioned before, we use Tikhonov regularization to solve the far field equation and use noisy far field data. This corresponds to solving:

$$(\alpha + \mathcal{F}_\delta^* \mathcal{F}_\delta) G(z, q) = \mathcal{F}_\delta^* E_{e,\infty}(z, q),$$

where α is the regularization parameter. If we consider a singular value decomposition of $\mathcal{F}_\delta := USV^*$, then the solution is given by

$$\{V^* G(z, q)\}_i = \frac{S_i}{\eta + S_i^2} \{U^* E_{e,\infty}(z, q)\}_i, \quad i = 1, \dots, 2N.$$

The regularization parameter is determined using the previously described Morozov's discrepancy principle by requiring that

$$\|\mathcal{F}_\delta G(z, q) - E_{e,\infty}(z, q)\| = \|\mathcal{F}_\delta - \mathcal{F}\| \|G(z, q)\|. \quad (7.5)$$

Setting $\varepsilon = \|\mathcal{F}_\delta - \mathcal{F}\|$, the regularization parameter is then the non negative solution of

$$\sum_{i=1}^{2N} \frac{\alpha^2 - \varepsilon^2 S_i^2}{(\alpha + S_i^2)^2} |\{U^* E_{e,\infty}(z, q)\}_i|^2 = 0. \quad (7.6)$$

As it has been pointed out in [16], the reconstruction is sensitive to the the choice of the polarization vector q . The numerical experiments have shown that better reconstructions are obtained by combining the values of $G(\cdot, q)$ for three linearly independent vectors q .

The following numerical results are obtained by evaluating $\mathcal{G}(z)$ defined by

$$\mathcal{G}(z) = \frac{1}{3} \left(\|G(z, q_1)\|^{-1} + \|G(z, q_2)\|^{-1} + \|G(z, q_3)\|^{-1} \right) \quad (7.7)$$

where $q_1 = (1, 0, 0)$, $q_2 = (0, 1, 0)$ and $q_3 = (0, 0, 1)$.

We consider a uniform mesh of the region that contains the scatterer and evaluate \mathcal{G} for every vertex z_i of this mesh. The 3-D reconstruction is obtained by plotting the surface

$$\mathcal{G}(z) = C \max_{z_i} \mathcal{G}(z_i).$$

We then vary the value of C till obtaining the “best visual” reconstruction.

7.4. Numerical results

All the following numerical results correspond to perfectly conducting scatterers and to an added random noise level $\delta = 0.01$. We shall explore first the case of full aperture experiments for different frequencies. The examples shown intend to evaluate the capability of the algorithm to handle non convex object reconstructions. They also demonstrate how the accuracy is frequency dependent. The last examples presents limited aperture reconstructions pointing out the dependence of the accuracy on the aperture location.

7.4.1. Reconstruction of an aircraft The exact geometry of the scatterer is presented in Fig. 2. We used for the reconstruction full aperture data based on 252 uniformly distributed directions (see Fig. 3).

In the first case, $k = 2\pi$ (the wavelength is twice the diameter of the aircraft), one obtains a low resolution reconstruction, see Fig. 4. For instance, one cannot distinguish small details such as engines, but larger parts such as wings are visible. In the second case (see Fig. 5) the frequency is doubled ($k = 4\pi$) and one can observe how the reconstruction resolution is highly improved.

7.4.2. Reconstruction of a teapot This is a second example of full aperture reconstruction. The exact geometry of the scatterer is presented in Figure 6.

We present some reconstructions of the teapot for different frequencies k . When the frequency increases, one needs in general to refine the mesh of the aperture (used for observation points) in order to have a good approximation of the far field. This is why

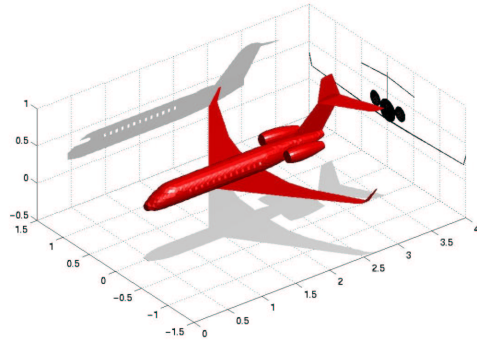


Figure 2. The aircraft exact geometry

we adapt for each frequency the specific number N of uniformly distributed nodes on the unit sphere.

Figures 7, 8, 9 and 10 show the 3-D reconstructions using respectively $(k, N) = (28, 252)$, $(k, N) = (56, 252)$, $(k, N) = (84, 492)$ and $(k, N) = (96, 492)$.

One clearly sees how an increase of the frequency improves the reconstruction. This effect is only due to the frequency and not to the number of discretization points N . For example, one gets the same reconstruction as in Fig. 7 using $(k, N) = (28, 492)$ instead of $(k, N) = (28, 252)$. We observe in Fig. 10 some non-smoothness in the reconstructed surface. We think that this is due to the lack of accuracy in the synthetic data and not to the inverse solver. For this high frequency experiment we did the forward computations of the scattered fields using only 6 points per wavelength (this limitation has been imposed by the limitation of the computer memory)!

7.4.3. Limited aperture experiments We give here the example of an aircraft (see Figure 2) using the frequency $k = 4\pi$ for the reconstruction. However we restrict the set of incident and observation points to only one quarter of the unit sphere. Figures 12 and 14 display reconstructions that respectively correspond to the lower left quarter (Fig. 11) and the lower right quarter (Fig. 13). We observe in both cases that the reconstruction is poorer than in the case of full aperture. However we still distinguish the main features of the aircraft.

8. Conclusions

The linear sampling method is a linear method for numerically determining the support of a scattering obstacle from a knowledge of the far-field pattern of the scattered

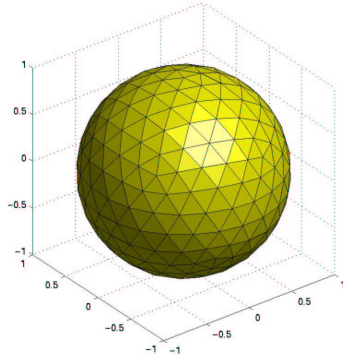


Figure 3. Mesh of the full aperture: 252 vertexes

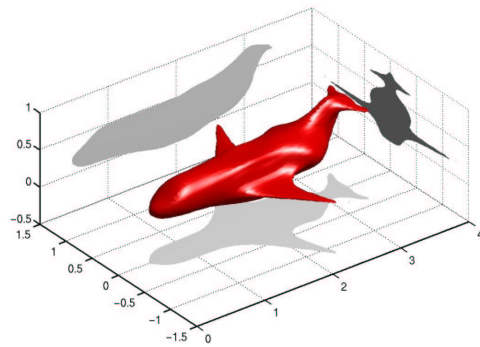


Figure 4. Reconstructed aircraft: $k = 2\pi$

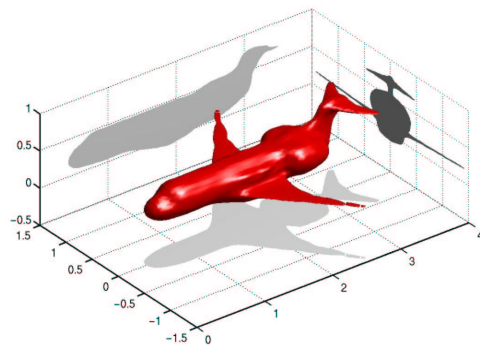


Figure 5. Reconstructed aircraft: $k = 4\pi$

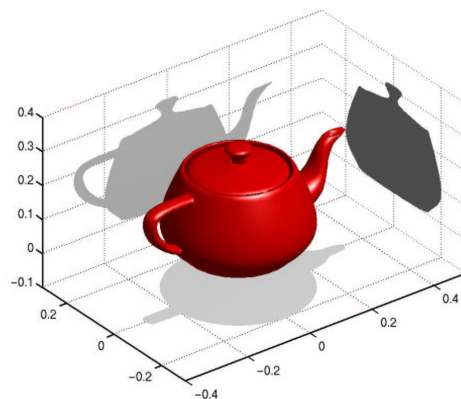


Figure 6. Exact geometry of the teapot.

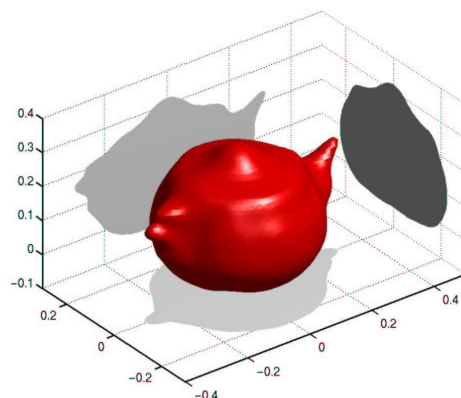


Figure 7. Reconstruction of the teapot with $k = 28$, using 252 incident directions.

wave for either a full or limited aperture. In addition to linearity, the advantages of the method are that 1) it is not based on a “weak scattering” approximation and 2) it does not depend on any “a priori” knowledge of the number of scatterers or their physical properties. The disadvantages are that, at least in the case of obstacle scattering, a relatively large amount of data is needed for its successful implementation and that, in the case of scattering by an inhomogeneous medium, it is not possible to infer information about the index of refraction. Open problems include 1) establishing the relationship between the regularized solution of the far-field equation and the function g_z in Theorems 2.2, 3.3, 4.3 and 4.5; 2) using the linear sampling method to obtain some

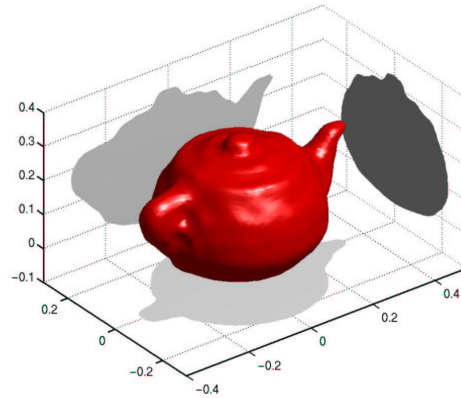


Figure 8. Reconstruction of the teapot with $k = 56$, using 252 uniformly distributed incident directions.

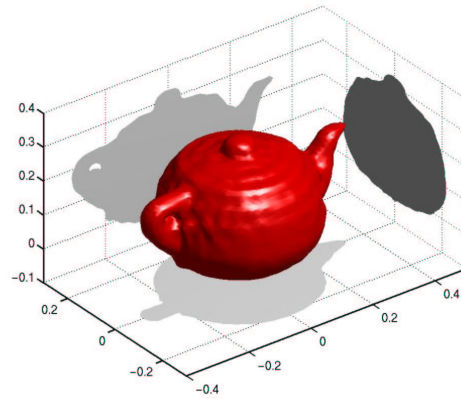


Figure 9. Reconstruction of the teapot with $k = 84$, using 492 uniformly distributed incident directions.

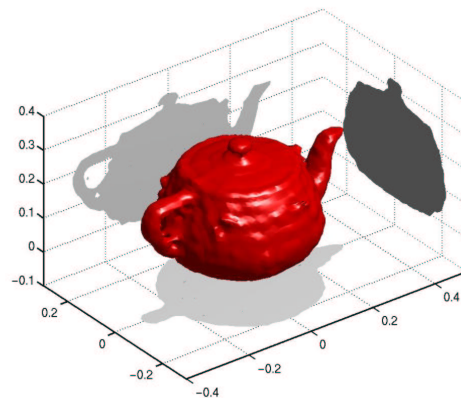


Figure 10. Reconstruction of the teapot with $k = 96$, using 492 uniformly distributed incident directions.

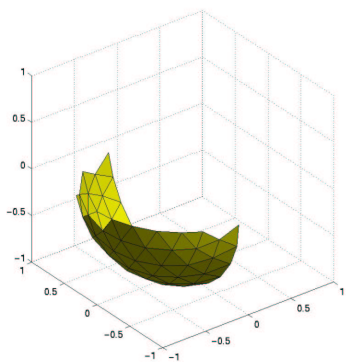


Figure 11. Mesh of the limited aperture (64 vertexes)

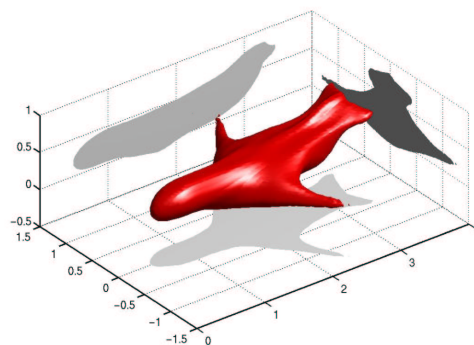


Figure 12. Reconstructed aircraft ($k = 4\pi$)

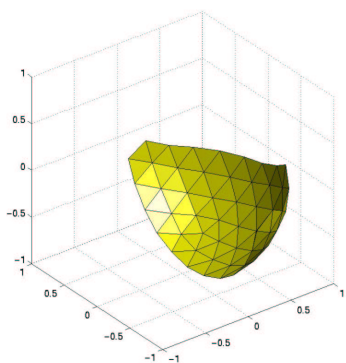


Figure 13. Mesh of the limited aperture (61 vertexes)

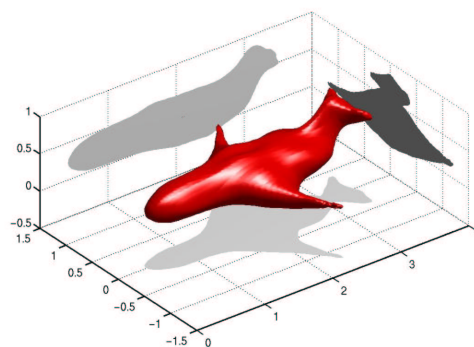


Figure 14. Reconstructed aircraft ($k = 4\pi$)

information on the physical properties of the scatterer; 3) deriving a linear sampling method for problems in the time domain and 4) improving the existing numerical techniques for implementing the linear sampling method.

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