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The simple method for inverse scattering problems]The simple method for solving the electromagnetic inverse scattering problem: the case of TE polarized waves †

David Colton and Michele Piana

Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA

Abstract. In two previous papers in this journal we presented a simple method for determining the support of a scattering object from noisy far field data for transverse magnetic polarized electromagnetic waves. This method was based on the solution of a linear integral equation of the first kind with the mathematical analysis being based on an investigation of an interior transmission problem. In this paper we consider the case of transverse electric polarized electromagnetic waves and again obtain a linear integral equation whose solution yields the support of the scattering object. The mathematical analysis in this case is based on an interior transmission problem different from the one previously considered. Numerical examples are given in the limiting case of a perfect conductor and limited aperture data.

1. Introduction

This is the third of a series of papers published in this journal and devoted to the formulation of a "simple method" for the solution of two-dimensional electromagnetic inverse scattering problems (initial steps in extending these ideas to the three-dimensional case can be found in [6]). In the first paper of this series [1] the inversion scheme was presented for the cases of an impenetrable obstacle and an inhomogeneous medium when the incident wave is polarized perpendicular to the axis of the cylinder representing the scatterer and the electric field only has a component in the direction of the axis of the cylinder. This is called the transverse magnetic or TM mode in scattering theory. It was assumed that for fixed frequency the far field pattern $u_\infty(\hat{x}, d)$ of the scattered electric field corresponding to observation direction $\hat{x}$ and incident direction

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d was known for all unit vectors $\hat{x}$ and d. It was then shown that the support of the scatterer can be obtained by solving the *far field equation*

$$\int_\Omega u_\infty(\hat{x}, d) g_y(d) ds(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot y}, \quad \hat{x} \in \Omega \tag{1.1}$$

where $\Omega = \{x \in IR^2 \ : \ |x| = 1\}$, $k > 0$ is the wave number, y is a point in the interior of the scattering object and $g_y = g(y, \cdot) \in L^2(\Omega)$. In particular it was established that $\|g\|_{L^2(\Omega)}$ becomes unbounded as y approaches the boundary of the scatterer and hence the support can be identified by plotting the map $y \to \|g\|_{L^2(\Omega)}$. In this paper some numerical examples were presented in order to show the applicability of the algorithm and its main attractive features were pointed out: the method does not require any low-frequency or high-frequency approximations, it is very fast (it only requires the solution of a linear integral equation) and it is relatively independent of the geometry and physical properties of the obstacles (it is not necessary to *a priori* know the number of scatterers, whether or not the medium is penetrable and, if it is not penetrable, what type of boundary condition the total field satisfies).

The study initiated in [1] was continued in [5]. In particular, it was now assumed that the data u_∞ is noisy. Since the problem of solving the integral equation (1.1) is severely ill-posed, the presence of noise on the far field pattern implies a significant numerical instability in the determination of the solution. Approximate stable solutions of equation (1.1) were obtained by using Tikhonov's regularization method. In particular, in order to fix the Tikhonov regularization parameter $\alpha = \alpha(y)$, Morozov's discrepancy principle was applied for the case when the noise is on the kernel of the integral operator, as is the case for equation (1.1). It was then shown that the regularization parameter $\alpha = \alpha(y)$ can be used as a reliable indicator of the support of the scattering object.

In the present paper our analysis is directed to the case when the incident field is polarized parallel to the axis of the scatterer and the magnetic field has only one component in the direction of the axis of the cylinder. This is referred to as the transverse electric or TE mode in scattering theory. The difference in this case from the transverse magnetic mode studied in [1] and [5] is that the unknown magnetic field has a discontinuity across the boundary of the scattering object. This means that in order to provide a mathematical justification of the "simple method" for solving the inverse scattering problem we must investigate a different class of interior transmission problems than those investigated in [1] and [5] and the analysis is fundamentally changed.

If the cylinder representing the scatterer is a perfect conductor, the direct scattering problem corresponding to TE mode is an exterior Neumann problem. The applicability of the simple method to obstacle problems of this type was considered in [1, 5]. However, in these papers the method was numerically tested only for the full aperture problem and nothing was said about its efficiency when the directions of the incident field and the directions of observation of the scattered field cover only a limited angle (smaller than 2π). This situation is often important for practical applications and in the present paper we will provide some numerical examples showing how the simple method behaves when the incident and observation angles are decreased.

We now want to describe the contents of this paper. Consider an electromagnetic wave that is scattered by an infinitely long inhomogeneous cylinder. If the wave is time-harmonic, the scattering is described by the (normalized) time-harmonic Maxwell equations (c.f. [3])

$$\nabla \times E_0 - ikH_0 = 0 \qquad \nabla \times H_0 + ikE_0 = 0 \qquad x \in I\!R^2 \setminus \overline{D} \tag{1.2}$$

$$\nabla \times E - ikH = 0 \qquad \nabla \times H + ikn(x)E = 0 \qquad x \in D \tag{1.3}$$

with the continuity conditions of the tangential components of the fields

$$\nu \times H_0 = \nu \times H \qquad \nu \times E_0 = \nu \times E \qquad x \in \partial D \quad . \tag{1.4}$$

Here (E_0, H_0) and (E, H) are the electromagnetic fields respectively outside and inside the cylinder, D is the cross-section of the cylinder with twice continuously differentiable boundary ∂D and outward unit normal ν, $n(x)$ is the index of refraction defined by

$$n(x) = \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right) \tag{1.5}$$

where ϵ_0 is the constant permittivity outside the scatterer, $\epsilon(x)$ and $\sigma(x)$ are respectively the permittivity and the conductivity of the cylinder and ω is the frequency. The following conditions on $n(x)$ are assumed:

(i) $n(x) \in C^2(I\!R^2)$ and $n(x) = n_0$ for $x \in I\!R^2 \setminus D$, where $\mathrm{Im} n_0 = 0$ and $n_0 \neq 1$.

(ii) $\mathrm{Im} n(x) \geq 0$ and the set $D_0 = \{x \in D \ , \ \mathrm{Im} n(x) > 0\}$ is non empty.

If the electromagnetic wave is polarized in the TE mode, it is possible to introduce the scalar fields u_0 and u such that $H_0 = (0, 0, u_0)$ and $H = (0, 0, u)$. From the Maxwell equations (1.2) and (1.3) and the conditions (1.4) it follows that u_0 and u satisfy the transmission problem

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x \in I\!R^2 \setminus \overline{D} \tag{1.6}$$

$$\nabla \cdot (\frac{1}{n}\nabla u) + k^2 u = 0 \qquad x \in D \tag{1.7}$$

$$u_0 = u \qquad \frac{\partial u_0}{\partial \nu} = \frac{1}{n_0}\frac{\partial u}{\partial \nu} \qquad x \in \partial D \quad . \tag{1.8}$$

We also require that the exterior field u_0 can be written in the form

$$u_0(x) = \mathrm{e}^{ikx \cdot d} + u^s(x) \tag{1.9}$$

where $d \in \Omega$ and $u^s(x)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial u^s}{\partial r} - iku^s) = 0 \quad . \tag{1.10}$$

From Green's formula we have that the radiating field has the asymptotic behaviour

$$u^s(x) = u_\infty(\hat{x}, d)\frac{e^{ikr}}{\sqrt{r}} + O(r^{-3/2}) \tag{1.11}$$

as $r = |x|$ tends to infinity [3].

In the next section of this paper we prove the existence of a unique solution to the boundary value problem (1.6)-(1.10). We then proceed to the formulation of the "simple method" for the solution of the inverse problem, i.e. determining the support of $n(x)-n_0$ from the knowledge of the far field pattern. In particular, we show that the formulation of the inversion algorithm requires the analysis of an interior transmission problem which is different than the one considered in [1, 5] for the case of a TM polarized wave. Finally, in the last section of our paper, we deal with the limited aperture case in the limiting situation of a perfectly conducting cylinder. This involves the application of the method to an exterior Neumann problem when the incident fields come from directions contained in an angle smaller than 2π and the observation directions lie within the same angle. We provide some numerical reconstructions showing how the method works in this underdetermined situation.

2. Existence and uniqueness for the direct scattering problem

We are interested in the following direct scattering problem: for $f \in C^{1,\alpha}(\partial D)$ and $g \in C^{0,\alpha}(\partial D)$, find solutions $u_0 \in C^2(I\!R^2 \setminus \overline{D}) \cap C^1(I\!R^2 \setminus D)$ and $u \in C^2(D) \cap C^1(\overline{D})$ of

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x \in I\!R^2 \setminus \overline{D} \tag{2.1}$$

$$\nabla \cdot (\frac{1}{n}\nabla u) + k^2 u = 0 \qquad x \in D \tag{2.2}$$

subject to the transmission conditions

$$u_0 - u = f \qquad x \in \partial D \tag{2.3}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0}\frac{\partial u}{\partial \nu} = g \qquad x \in \partial D \quad , \tag{2.4}$$

where n and n_0 are as defined in the previous section and u_0 satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial u_0}{\partial r} - iku_0) = 0 \quad . \tag{2.5}$$

Theorem 2.1. *The transmission problem* (2.1)-(2.5) *has at most one solution.*
Proof. We assume $f = g = 0$ and let u_0, u be the solution of the homogeneous transmission problem. From the radiation condition (2.5) we have that (c.f. Theorem 2.4 of [3])

$$\lim_{r\to\infty}\int_{|x|=r}\left\{\left|\frac{\partial u_0}{\partial\nu}\right|^2 + k^2|u_0|^2\right\}ds = -2k\mathrm{Im}\int_{\partial D} u_0\frac{\partial\overline{u}_0}{\partial\nu}ds \quad . \tag{2.6}$$

We now apply the divergence theorem to the function $u(1/\overline{n}\nabla\overline{u})$ and obtain

$$\int_D[-k^2|u|^2 + \frac{1}{\overline{n}}|\nabla u|^2]dy = \int_{\partial D} u\left(\frac{1}{n_0}\frac{\partial\overline{u}}{\partial\nu}\right)ds(y) \quad . \tag{2.7}$$

From this equation, the fact that n_0 is real and the (homogeneous) boundary conditions (2.3),(2.4) we now obtain

$$\mathrm{Im}\int_{\partial D} u_0\frac{\partial\overline{u}_0}{\partial\nu}ds \geq 0 \tag{2.8}$$

and this implies, from Rellich's lemma [3], that $u_0 = \partial u_0/\partial\nu = 0$ in $I\!R^2\setminus\overline{D}$. Again from the homogeneous boundary conditions we have $u = \partial u/\partial\nu = 0$ on ∂D and the fact that $u = 0$ in D now follows from the unique continuation principle [7]. □

In order to show the existence of the solution of problem (2.1)-(2.5) we apply Riesz's theory for compact operators [2]. We first rewrite the (2.2) so that we can apply potential theory. In particular, by the definition $u(x) = \sqrt{n(x)}v(x)$, the exterior transmission problem (2.1)-(2.5) becomes

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x\in I\!R^2\setminus\overline{D} \tag{2.9}$$

$$\triangle_2 v + (k^2 n + q)v = 0 \qquad x\in D \tag{2.10}$$

$$u_0 - \sqrt{n_0}v = f \qquad x\in\partial D \tag{2.11}$$

$$\frac{\partial u_0}{\partial\nu} - \frac{1}{\sqrt{n_0}}\frac{\partial v}{\partial\nu} = g \qquad x\in\partial D \tag{2.12}$$

where u_0 satisfies the radiation condition (2.5) and

$$q(x) = -\sqrt{n(x)}\triangle_2\frac{1}{\sqrt{n(x)}} \quad . \tag{2.13}$$

Then for ψ, $\phi\in C(\partial D)$ and $\psi'\in C(D)$, we introduce the two functions

$$u_0(x) = \int_{\partial D}\left\{\frac{\partial\Phi_0(x,y)}{\partial\nu(y)}\psi(y) + \Phi_0(x,y)\phi(y)\right\}ds(y) \qquad x\in I\!R^2\setminus\partial D \tag{2.14}$$

$$v(x) = \int_{\partial D} \left\{ \sqrt{n_0} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) + \Phi(x,y)\phi(y) \right\} ds(y) + \int_D \Phi(x,y) p(y) \psi'(y) dy \qquad x \in IR^2 \setminus \partial D \quad (2.15)$$

where

$$p(x) := k^2 n_0 - [k^2 n(x) + q(x)] \qquad (2.16)$$

and $\Phi_0(x,y)$ and $\Phi(x,y)$ are the fundamental solutions of the Helmholtz equations

$$\triangle_2 u + k^2 u = 0 \qquad (2.17)$$

and

$$\triangle_2 u + k^2 n_0 u = 0 \quad . \qquad (2.18)$$

Finally, we define the following set of integral operators:

$K : C(\partial D) \to C(\partial D)$ such that

$$(K\psi)(x) = 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) \qquad x \in \partial D \quad ; \qquad (2.19)$$

$S : C(\partial D) \to C(\partial D)$ such that

$$(S\phi)(x) = 2 \int_{\partial D} \Phi(x,y) \phi(y) ds(y) \qquad x \in \partial D \quad ; \qquad (2.20)$$

$T : C(\partial D) \to C(\partial D)$ such that

$$(T\psi)(x) = 2 \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) \qquad x \in \partial D \quad ; \qquad (2.21)$$

$K' : C(\partial D) \to C(\partial D)$ such that

$$(K'\phi)(x) = 2 \frac{\partial}{\partial \nu(x)} \int_{\partial D} \Phi(x,y) \phi(y) ds(y) \qquad x \in \partial D \quad ; \qquad (2.22)$$

$\tilde{K} : C(\partial D) \to C(D)$ such that

$$(\tilde{K}\psi)(x) = 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) \qquad x \in D \quad ; \qquad (2.23)$$

$\tilde{S} : C(\partial D) \to C(D)$ such that

$$(\tilde{S}\phi)(x) = 2 \int_{\partial D} \Phi(x,y) \phi(y) ds(y) \qquad x \in D \quad ; \qquad (2.24)$$

$S_p : C(D) \to C(\partial D)$ such that

$$(S_p \psi')(x) = 2 \int_D \Phi(x,y) p(y) \psi'(y) y \qquad x \in \partial D \quad ; \qquad (2.25)$$

$K'_p : C(D) \to C(\partial D)$ such that

$$(K'_p\psi')(x) = 2\frac{\partial}{\partial\nu(x)}\int_D \Phi(x,y)p(y)\psi'(y)y \qquad x \in \partial D \quad ; \tag{2.26}$$

$\tilde{S}_p : C(D) \to C(D)$ such that

$$(\tilde{S}_p\psi')(x) = 2\int_D \Phi(x,y)p(y)\psi'(y)y \qquad x \in D \quad . \tag{2.27}$$

and let K_0, S_0, T_0 and K'_0 denote the operators corresponding to K, S, T and K' respectively with Φ replaced by Φ_0.

Theorem 2.2. *There exists a unique solution of the transmission problem* (2.1)-(2.5).
Proof. It suffices to consider the transmission problem (2.9)-(2.13). From the jump relations of potential theory [3] and the definitions (2.19)-(2.27) we have (c.f. Theorem 3.2 of [4]) that the restrictions of u_0 in (2.14) to $I\!R^2 \setminus \overline{D}$ and of v in (2.15) to D solve the transmission problem provided ψ, $\phi \in C(\partial D)$ and $\psi' \in C(D)$ are solutions on ∂D of the equations

$$(K_0 - n_0K)\psi + (1+n_0)\psi + (S_0 - \sqrt{n_0}S)\phi - \sqrt{n_0}S_p\psi' = 2f \quad , \tag{2.28}$$

$$(T_0 - T)\psi + (K'_0 - \frac{1}{\sqrt{n_0}}K')\phi - (1 + \frac{1}{\sqrt{n_0}})\phi - \frac{1}{\sqrt{n_0}}K'_p\psi' = 2g \tag{2.29}$$

and in D of the equation

$$\sqrt{n_0}\tilde{K}\psi + \tilde{S}\phi + \tilde{S}_p\psi' - 2\psi' = 0 \quad . \tag{2.30}$$

Equations (2.28)-(2.30) can be written in operator notation as

$$(E + A)\begin{pmatrix} \psi \\ \phi \\ \psi' \end{pmatrix} = \begin{pmatrix} 2f \\ 2g \\ 0 \end{pmatrix} \tag{2.31}$$

where

$$E = \begin{pmatrix} 1+n_0 & 0 & 0 \\ 0 & -(1+1/\sqrt{n_0}) & 0 \\ 0 & 0 & -2I \end{pmatrix} \tag{2.32}$$

and

$$A = \begin{pmatrix} K_0 - n_0K & S_0 - \sqrt{n_0}S & -\sqrt{n_0}S_p \\ T_0 - T & K'_0 - 1/\sqrt{n_0}K' & -1/\sqrt{n_0}K'_p \\ \sqrt{n_0}\tilde{K} & \tilde{S} & \tilde{S}_p \end{pmatrix} \quad . \tag{2.33}$$

We note that E has a bounded inverse and A is compact in $C(\partial D) \times C(\partial D) \times C(D)$. Hence from Riesz's theory [2] it follows that the problem (2.31) has a solution continuously depending on the data provided $E + A$ is injective. From the uniqueness Theorem 2.1, if $f = g = 0$ and u_0 and v are defined by (2.14) and (2.15) respectively, then $u_0 = 0$ in $I\!R^2 \setminus \overline{D}$ and $v = 0$ in D. In (2.30) the term $\sqrt{n_0}\tilde{K}\psi + \tilde{S}\phi + \tilde{S}_p\psi'$ is v and so $\psi' = 0$. It now follows from the jump relations of potential theory that $\phi = \psi = 0$ (c.f. Theorem 3.3 of [4]). Thus $E + A$ is injective and the theorem follows.

□

Theorems 2.1 and 2.2 can be generalized in a straightforward manner to the three dimensional case.

3. Formulation of the "simple method" for the inverse scattering problem

The aim of this section is to formulate the "simple method" discussed in [1, 5] for the solution of the inverse scattering problem associated with (1.6)-(1.10) which consists in determining the support $\overline{D}$ of $n(x) - n_0$ from the knowledge of the far field pattern $u_\infty(\hat{x}, d)$ defined in (1.11) (assuming the frequency is fixed). More precisely we want to show that if y is a point in D, then for each $\epsilon > 0$ there exists $g_y \in L^2(\Omega)$ such that

$$\| \int_\Omega u_\infty(\hat{x}, d) g_y(d) ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot y} \|_{L^2(\Omega)} < \epsilon \tag{3.1}$$

and both $\|g_y\|_{L^2(\Omega)}$ and the Herglotz wave function with kernel g_y [3] become unbounded when y approaches the boundary ∂D. To this end we first need to analyze the following *interior* transmission problem:

$$\triangle_2 u_{0,\mathrm{n}} + k^2 u_{0,\mathrm{n}} = 0 \qquad x \in D \tag{3.2}$$

$$\nabla \cdot (\frac{1}{n} \nabla u_{\mathrm{n}}) + k^2 u_{\mathrm{n}} = 0 \qquad x \in D \tag{3.3}$$

$$u_{0,\mathrm{n}} - u_{\mathrm{n}} = -\Phi_0(\cdot, y_{\mathrm{n}}) \qquad x \in \partial D \tag{3.4}$$

$$\frac{\partial u_{0,\mathrm{n}}}{\partial \nu} - \frac{1}{n_0} \frac{\partial u_{\mathrm{n}}}{\partial \nu} = -\frac{\partial \Phi_0(\cdot, y_{\mathrm{n}})}{\partial \nu} \qquad x \in \partial D \tag{3.5}$$

where

$$y_{\mathrm{n}} = y^* - \frac{R}{\mathrm{n}} \nu(y^*) \quad , \tag{3.6}$$

$R > 0$ is sufficiently small and y^* is a point on ∂D.

Theorem 3.1. *If* $D_0 = \{x \in D \ , \ \mathrm{Im} n(x) > 0\}$ *is non empty then the problem*

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x \in D \tag{3.7}$$

$$\nabla\cdot(\frac{1}{n}\nabla u)+k^2u=0 \qquad x\in D \tag{3.8}$$

$$u_0-u=f \qquad x\in\partial D \tag{3.9}$$

$$\frac{\partial u_0}{\partial\nu}-\frac{1}{n_0}\frac{\partial u}{\partial\nu}=g \qquad x\in\partial D \tag{3.10}$$

has at most one solution.

Proof. Let us consider the homogeneous problem (i.e, $f=g=0$). The application of divergence theorem to $\overline{u}(1/n\nabla u)$ and the (homogeneous) transmission conditions (3.9),(3.10) lead to

$$\int_D[\frac{1}{n}|\nabla u|^2-k^2|u|^2]dy=\int_{\partial D}\overline{u}_0\frac{\partial u_0}{\partial\nu}ds(y) \quad . \tag{3.11}$$

From this equation, Green's theorem and equation (3.7) we have

$$\mathrm{Im}\int_D\frac{1}{\overline{n}}|\nabla u|^2dy=0 \quad . \tag{3.12}$$

Since $\mathrm{Im}1/\overline{n}>0$ in D_0 and u satisfies (3.8), it follows from the unique continuation principle that $u=0$ in D and from the boundary conditions and the Helmholtz representation (c.f. equation (2.5) in [3]) u_0 also vanishes in D. □

The existence of a solution to (3.7)-(3.10) can now be proven in a way similar to the proof of Theorem 2.2. The previous change of variables $u(x)=\sqrt{n(x)}v(x)$ allows to rewrite the interior problem in the form

$$\triangle_2u_0+k^2u_0=0 \qquad x\in D \tag{3.13}$$

$$\triangle_2v+(k^2n+q)v=0 \qquad x\in D \tag{3.14}$$

$$u_0-\sqrt{n_0}v=f \qquad x\in\partial D \tag{3.15}$$

$$\frac{\partial u_0}{\partial\nu}-\frac{1}{\sqrt{n_0}}\frac{\partial v}{\partial\nu}=g \qquad x\in D \quad , \tag{3.16}$$

where f and g are defined by the right hand side of (3.4) and (3.5) respectively and q is defined by (2.13). We now define the two functions

$$u_0(x)=\int_{\partial D}\left\{\frac{\partial\Phi_0(x,y)}{\partial\nu(y)}\psi(y)+\Phi_0(x,y)\phi(y)\right\}ds(y) \qquad x\in IR^2\setminus\partial D \tag{3.17}$$

$$v(x) = \int_{\partial D} \left\{ \sqrt{n_0} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) - \Phi(x,y)\phi(y) \right\} ds(y) + \int_D \Phi(x,y) p(y) \psi'(y) dy \qquad x \in IR^2 \setminus \partial D \quad (3.18)$$

where Φ_0, Φ and p are defined as in the previous section and again ψ, $\phi \in C(\partial D)$, $\psi' \in C(D)$. From the jump relations of potential theory and the boundary conditions (again see the proof of Theorem 3.2 of [4]) we have that the restrictions of $u(x)$ and $v(x)$ to D solve problem (3.13)-(3.16) provided ψ, ϕ and ψ' are solutions on ∂D of the integral equations

$$(K_0 - n_0 K)\psi - (1 - n_0)\psi + (S_0 + \sqrt{n_0} S)\phi - \sqrt{n_0} S_p \psi' = 2f \quad , \qquad (3.19)$$

$$(T_0 - T)\psi + (K_0' + \frac{1}{\sqrt{n_0}} K')\phi + (1 + \frac{1}{\sqrt{n_0}})\phi - \frac{1}{\sqrt{n_0}} K_p' \psi' = 2g \qquad (3.20)$$

and in D of

$$\sqrt{n_0} \tilde{K} \psi - \tilde{S} \phi + \tilde{S}_p \psi' - 2\psi' = 0 \quad . \qquad (3.21)$$

Theorem 3.2. *The interior transmission problem* (3.7)-(3.10) *has a unique solution*
Proof. It suffices to consider the transmission problem (3.13)-(3.16). From the uniqueness Theorem 3.1, we can again conclude as in Theorem 2.2 that if $f = g = 0$ then $v = u_0 = 0$ in D where u_0 and v are defined by (3.17) and (3.18) respectively. This implies $\psi' = 0$ and the three equations (3.19)-(3.21) reduce to

$$(K_0 - n_0 K)\psi - (1 - n_0)\psi + (S_0 + \sqrt{n_0} S)\phi = 0 \qquad x \in \partial D \qquad (3.22)$$

$$(T_0 - T)\psi + (K_0' + \frac{1}{\sqrt{n_0}} K')\phi + (1 + \frac{1}{\sqrt{n_0}})\phi = 0 \qquad x \in \partial D \quad . \qquad (3.23)$$

From the jump relations of potential theory we have that

$$(u_0)_+ - (u_0)_- = \psi \quad , \quad \left(\frac{\partial u_0}{\partial \nu} \right)_+ - \left(\frac{\partial u_0}{\partial \nu} \right)_- = -\phi \qquad x \in \partial D \qquad (3.24)$$

$$(v)_+ - (v)_- = \sqrt{n_0} \psi \quad , \quad \left(\frac{\partial v}{\partial \nu} \right)_+ - \left(\frac{\partial v}{\partial \nu} \right)_- = \phi \qquad x \in \partial D \quad , \qquad (3.25)$$

where $+$ denotes the limit from outside D and $-$ denotes the limit from inside D. Since $(u_0)_- = (\partial u_0 / \partial \nu)_- = (v)_- = (\partial v / \partial \nu)_- = 0$, we have

$$(u_0)_+ - \frac{1}{\sqrt{n_0}} (v)_+ = 0 \qquad \left(\frac{\partial u_0}{\partial \nu} \right)_+ + \left(\frac{\partial v}{\partial \nu} \right)_+ = 0 \qquad x \in \partial D \quad . \qquad (3.26)$$

Using the fact that n_0 is real, this implies that

$$\mathrm{Im}\int_{\partial D}(u_0)_+\left(\frac{\partial\overline{u}_0}{\partial\nu}\right)_+ ds = -\frac{1}{\sqrt{n_0}}\mathrm{Im}\int_{\partial D}(v)_+\left(\frac{\partial\overline{v}}{\partial\nu}\right)_+ ds \tag{3.27}$$

i.e., one of the two integrals in (3.27) is non-negative. Since for $x \in I\!R^2 \setminus \overline{D}$ both u_0 and v are radiating solutions of the Helmholtz equation, Rellich's lemma implies that either u_0 or v must vanish in $I\!R^2 \setminus \overline{D}$. Then from the jump relations (3.24),(3.25) we have $\psi = \phi = 0$. The existence of a unique solution to (3.13)-(3.16) now follows from Riesz's theory of compact operators [2].

□

Theorems 3.1 and 3.2 can be generalized in a straightforward manner to the three dimensional case.

We now want to show that if $u_{0,n}$ is the solution of (3.2)-(3.6) then the sequence $\|u_{0,n}\|_{C^1(\partial D)}$ is unbounded as $n \to \infty$. To accomplish this result we need to introduce a fundamental solution $\Gamma(x,z)$ for equation (3.14). The existence of this solution is assured by the fact that if $\Gamma(x,z)$ satisfies the Lippmann-Schwinger equation

$$\Gamma(x,z) = \Phi(x,z) + \int_D \Phi(x,y)[k^2 n_0 - (k^2 n(y) + q(y))]\Gamma(y,z)dy \tag{3.28}$$

then, using the mapping properties of volume potentials (c.f. Theorem 8.2 in [3]), it is a fundamental solution for (3.14). It is proven in [3] (in $I\!R^3$ but the proof is the same in $I\!R^2$) that (3.28) is solvable if and only if the problem

$$\triangle_2 v + k^2 n' v = 0 \qquad x \in I\!R^2 \tag{3.29}$$

$$v(x) = \Phi(x,z) + v^s(x) \tag{3.30}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial v^s}{\partial r} - ikv^s\right) = 0 \tag{3.31}$$

$$n'(x) = n(x) + \frac{q(x)}{k^2} \tag{3.32}$$

has a unique solution. But (3.29)-(3.32) is equivalent to

$$\nabla\cdot(\frac{1}{n}\nabla u) + k^2 u = 0 \qquad x \in I\!R^2 \tag{3.33}$$

$$u(x) = \sqrt{n(x)}\Phi(x,z) + u^s(x) \tag{3.34}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \quad . \tag{3.35}$$

Since, under assumption (ii) of the Introduction, this new problem is easily seen to have at most one solution by using equation (2.7) (c.f. Theorem 8.7 of [3]), the original problem (3.29)-(3.32) has at most one solution and the existence of a fundamental solution for (3.14) is now guaranteed by the Fredholm alternative.

We now consider the following preparatory lemma:

Lemma 3.3. *Let D be a bounded domain with twice continuously differentiable boundary ∂D, $x^* \in \partial D$ and $B_R = \{x \in IR^2 \ : \ |x - x^*| \leq R\}$. If $u \in C^2(D) \cap C^1(\overline{D})$ solves*

$$\nabla \cdot \left(\frac{1}{n} \nabla u \right) + k^2 u = 0 \quad \text{in } D \quad , \tag{3.36}$$

then the following inequality holds:

$$\|u\|_{C(\partial D)} \leq C \left(\| \frac{\partial u}{\partial \nu} \|_{C(\partial D)} + \|u\|_{C(\partial D \setminus B_R)} \right) \tag{3.37}$$

with C a positive constant.

Proof. The proof follows as in Lemma 4.4 of [9]. We introduce a non-negative function $\eta \in C(\partial D)$ with support in $\partial D \setminus B_R$ and $\eta \neq 0$ and consider the impedance boundary condition

$$\frac{\partial u}{\partial \nu} - i \eta u = g \quad . \tag{3.38}$$

If $g = 0$, by applying again the divergence theorem to $\overline{u}(1/n \nabla u)$ we obtain

$$\int_D [-k^2 |u|^2 + \frac{1}{n} |\nabla u|^2] dy = \frac{i}{n_0} \int_{\partial D} \eta |u|^2 ds(y) \quad . \tag{3.39}$$

From this equation it follows that the imaginary part of the left hand side is non-positive, the imaginary part of the right hand side is non-negative and therefore $u = 0$ on $\partial D \setminus B_R$. From the (homogeneous) boundary condition (3.38) also $\partial u / \partial \nu = 0$ on $\partial D \setminus B_R$ and from the unique continuation principle $u = 0$ in D. Hence the boundary value problem (3.36),(3.38) has at most one solution. To show existence, we use again Riesz's theory. Let $\tilde{\Phi}(x,y)$ be the fundamental solution to (3.36) defined by $\tilde{\Phi}(x,y) = \sqrt{n(x)/n_0} \Gamma(x,y)$ and let the operator S, K' be defined as in the second section with Φ replaced by $\tilde{\Phi}$. Then the function

$$u(x) = \int_{\partial D} \tilde{\Phi}(x,y) \phi(y) ds(y) \quad x \in IR^2 \setminus \partial D \tag{3.40}$$

restricted to D solves problem (3.36),(3.38) provided $\phi \in C(\partial D)$ satisfies the integral equation

$$K'\phi + \phi - i\eta S\phi = 2g \quad \text{on } \partial D \quad . \tag{3.41}$$

If $g = 0$, $(u)_- = (\partial u/\partial\nu)_- = 0$ in D from the uniqueness result and from the continuity of the single-layer potential and the uniqueness of the solution of the exterior Dirichlet problem we also have that $(u)_+ = (\partial u/\partial\nu)_+ = 0$. Finally from the jump relations of potential theory we obtain $\phi = 0$. This assures the existence of the solution and its continuous dependence from the data. Thus we have:

$$\|u\|_{C(\partial D)} \le C\|g\|_{C(\partial D)} = C\|u - i\eta\frac{\partial u}{\partial\nu}\|_{C(\partial D)} \quad . \tag{3.42}$$

The theorem now follows by recalling the fact that η has support in $\partial D \setminus B_R$.

□

Theorem 3.4. *If* $u_{0,\mathrm{n}}$ and u_{n} *are the sequences of solutions of problem* (3.2)-(3.6), *then* $\lim_{\mathrm{n}\to\infty}\|u_{0,\mathrm{n}}\|_{C^1(\partial D)} = \infty$.

Proof. We assume on the contrary that there exists a positive constant c_1 such that

$$\|u_{0,\mathrm{n}}\|_{C^1(\partial D)} \le c_1 \quad . \tag{3.43}$$

We introduce the set of points in $I\!R^2 \setminus \overline{D}$ defined by

$$z_{\mathrm{n}} = y^* + \frac{R}{\mathrm{n}}\nu(y^*) \tag{3.44}$$

for $R > 0$ sufficiently small and y^* a point on ∂D and define the sequences

$$\tilde{f}_{\mathrm{n}} = \Phi_0(\cdot, y_{\mathrm{n}}) + \sqrt{n_0}\Gamma(\cdot, z_{\mathrm{n}}) \quad \text{on } \partial D \tag{3.45}$$

$$\tilde{g}_{\mathrm{n}} = \frac{\partial\Phi_0(\cdot, y_{\mathrm{n}})}{\partial\nu} + \sqrt{n_0}\frac{\partial\Gamma(\cdot, z_{\mathrm{n}})}{\partial\nu} \quad \text{on } \partial D \tag{3.46}$$

$$\tilde{u}_{\mathrm{n}} = u_{\mathrm{n}} + \sqrt{n}\Gamma(\cdot, z_{\mathrm{n}}) \quad \text{in } D \quad . \tag{3.47}$$

If B_R is the disk $B_R = \{x \in I\!R^2 \ , \ |x - y^*| \le R\}$, then, clearly,

$$\|\tilde{f}_{\mathrm{n}}\|_{C(\partial D\setminus B_R)} \le c_2 \quad , \tag{3.48}$$

where c_2 is some positive constant. Following the proof of Lemma 4.2 in [4] we obtain

$$\|\tilde{g}_{\mathrm{n}}\|_{C(\partial D)} \le c_3 \tag{3.49}$$

for some positive constant c_3. From Lemma 3.3 we also have that the estimate

$$\|\tilde{u}_{\mathrm{n}}\|_{C(\partial D)} \le c_4\left(\|\tilde{u}_{\mathrm{n}}\|_{C(\partial D\setminus B_R)} + \|\frac{\partial\tilde{u}_{\mathrm{n}}}{\partial\nu}\|_{C(\partial D)}\right) \tag{3.50}$$

holds for some positive constant c_4. Finally, from these inequalities and the boundary conditions (3.4),(3.5) we obtain that

$$\|\tilde{u}_{\rm n}\|_{C(\partial D)} \leq c_5 \tag{3.51}$$

for a positive constant c_5 and, again using (3.4), this implies that $\|\tilde{f}_{\rm n}\|_{C(\partial D)}$ is bounded, which is a contradiction.

□

We now have all the tools to formulate the "simple method" for the solution of an inverse scattering problem.

Theorem 3.5. *For every $\epsilon > 0$ there exists $g = g(\cdot, x_{\rm n}) \in L^2(\Omega)$ such that*

$$\| \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot y_{\rm n}} \|_{L^2(\Omega)} < \epsilon \tag{3.52}$$

and $\lim_{{\rm n}\to\infty} \|g\|_{L^2(\Omega)} = \infty$. *Moreover, if v_g is the* Herglotz wave function *defined by*

$$v_g(x) = \int_\Omega e^{ikx\cdot d} g(d) ds(d) \quad x \in D \quad , \tag{3.53}$$

then also $\lim_{{\rm n}\to\infty} \|v_g\|_{L^2(D)} = \infty$.

Proof. From Theorem 3.2 the interior transmission problem (3.2)-(3.6) has one solution which is not, in general, a Herglotz wave function. However it can be shown [1] that a Herglotz wave function $U_{0,{\rm n}}$ with kernel $g = g(\cdot, y_{\rm n}) \in L^2(\Omega)$ exists, which approximates $u_{0,{\rm n}}$ in $C^1(\overline{D})$. We denote by u_0 the total field which solves the original exterior transmission problem (1.6)-(1.10) and define u_0^* and u^* by $u_0^*(y) = u_0(y, -\hat{x})$and $u^*(y) = u(y, -\hat{x})$. From the reciprocity relation and the two dimensional version of the representation (2.13) in [3] of the far field pattern we obtain (c.f. [4], p. 288)

$$\int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} (u_0^* \frac{\partial U_{0,{\rm n}}}{\partial \nu} - \frac{\partial u_0^*}{\partial \nu} U_{0,{\rm n}}) ds(y) \quad . \tag{3.54}$$

Since $U_{0,{\rm n}} \approx u_{0,{\rm n}}$ in $C^1(\overline{D})$,

$$\int_{\partial D} (u_0^* \frac{\partial U_{0,{\rm n}}}{\partial \nu} - \frac{\partial u_0^*}{\partial \nu} U_{0,{\rm n}}) ds(y) \approx \int_{\partial D} (u_0^* \frac{\partial u_{0,{\rm n}}}{\partial \nu} - \frac{\partial u_0^*}{\partial \nu} u_{0,{\rm n}}) ds(y) \quad . \tag{3.55}$$

From the boundary conditions (3.4),(3.5) we have that

$$\int_{\partial D} (u_0^* \frac{\partial u_{0,{\rm n}}}{\partial \nu} - \frac{\partial u_0^*}{\partial \nu} u_{0,{\rm n}}) ds(y) = \int_{\partial D} u_0^* \left[\frac{1}{n_0} \frac{\partial u_n}{\partial \nu} - \frac{\partial \Phi_0(\cdot, y_{\rm n})}{\partial \nu} \right] ds(y) - \int_{\partial D} \frac{\partial u_0^*}{\partial \nu} [u_n - \Phi_0(\cdot, y_{\rm n})] ds(y) \quad . \tag{3.56}$$

Then we use (3.55), (3.56) and the boundary condition (1.8) to obtain

$$\int_{\partial D}\left(u_0^*\frac{\partial U_{0,\mathrm{n}}}{\partial\nu}-\frac{\partial u_0^*}{\partial\nu}U_{0,\mathrm{n}}\right)ds(y)\approx\frac{1}{n_0}\int_{\partial D}\left(u^*\frac{\partial u_\mathrm{n}}{\partial\nu}-\frac{\partial u^*}{\partial\nu}u_\mathrm{n}\right)ds(y)-$$
$$\int_{\partial D}\left(u^*\frac{\partial\Phi_0(\cdot,y_\mathrm{n})}{\partial\nu}-\frac{1}{n_0}\frac{\partial u^*}{\partial\nu}\Phi_0(\cdot,y_\mathrm{n})\right)ds(y)\quad . \tag{3.57}$$

By using the divergence theorem, it can be seen that the first term on the right hand side of (3.57) is zero while, by using again the boundary conditions (1.8) the second term is $e^{-ik\hat{x}\cdot y_\mathrm{n}}$ (see Theorem 3.1 and Theorem 3.3 in [2]). It follows that

$$\int_\Omega u_\infty(\hat{x},d)g(d)ds(d)\approx\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot y_\mathrm{n}}\quad . \tag{3.58}$$

The fact that $\|g\|_{L^2(\Omega)}$ is unbounded as $\mathrm{n}\to\infty$ is shown by assuming on the contrary that $\|g\|_{L^2(\Omega)}$ is bounded and hence $\|U_{0,\mathrm{n}}\|_{C^1(D)}$ is bounded. This implies that $\|u_{0,\mathrm{n}}\|_{C^1(\partial D)}$ is bounded and this contradicts Theorem 3.4. The theorem now follows.

□

We note that since the far field operator

$$(Fg)(\hat{x})=\int_\Omega u_\infty(\hat{x},d)g(d)ds(d) \tag{3.59}$$

is compact, the solution of the far field equation (1.1) is an ill-posed problem in the sense of Hadamard. This implies that the presence of noise on the far field pattern makes the solution of the discretized version of the equation extremely unstable. The theory of improperly posed problems [8, 10] and Theorem 3.5 suggest the following simple algorithm for the solution of the inverse scattering problem [5]:

- Solve the minimization problem

$$\left\|\int_{-\pi}^{\pi}u_\infty(\hat{x},\theta)g_\alpha(d,y)ds(d)-\frac{e^{i\pi/4}}{\sqrt{8\pi k}}e^{-ik\hat{x}\cdot y}\right\|^2_{L^2(\Omega)}+\alpha\ \|g_\alpha(\cdot,y)\|^2_{L^2(\Omega)}$$
$$=\text{minimum} \tag{3.60}$$

  where the optimum value $\alpha_{opt}=\alpha_{opt}(y)$ of the regularization parameter α is determined from the generalized discrepancy principle of Morozov [10].
- Repeat the above step for all the values of y on a grid containing the object. Then an image of the object can be obtained by plotting for each y either the norm of the optimum regularized solution or the optimum value of the regularization parameter.

The computation of the algorithm can be simplified by the use of the singular system of the far field operator (c.f. [5]).

4. The inverse scattering problem for a limited aperture

When the conductivity of the scatterer tends to infinity, the scatterer becomes impenetrable for the incident electromagnetic wave. In this limiting case the scattering problem (1.6)-(1.10) reduces to the exterior Neumann problem

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x \in IR^2 \setminus \overline{D} \tag{4.1}$$

$$\frac{\partial u_0}{\partial \nu} = 0 \qquad x \in \partial D \tag{4.2}$$

$$u_0(x) = \mathrm{e}^{ikx\cdot d} + u^s(x) \tag{4.3}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{4.4}$$

where ν is the unit outward normal to ∂D. The inverse scattering problem we are interested in now consists in determining ∂D from measurements of the far field pattern. The formulation of the simple method in the case of obstacle scattering (with Dirichlet, Neumann or impedence boundary conditions) has been dealt with in [1, 5]. In these papers the numerical implementation of the method is presented both when the far field pattern is exact [1] and in the case of noisy data [5]. All these numerical examples are characterized by a finite number of incident waves coming from directions uniformly distributed over an angle 2π. The corresponding far field pattern is computed at the same number of equidistantly distributed observation points. However, in many practical applications we must consider the scattering of incident waves from directions covering a limited angle (smaller than 2π) and of far field patterns measured at observation points distributed over a limited aperture. Here we provide some examples testing the efficiency of the simple method in this setting.

We first consider the case of a single kite parametrized by

$$x(t) = (1.5\sin(t), \cos(t) + 0.65\cos(2t) - 0.65) \qquad 0 \le t \le 2\pi \tag{4.5}$$

when the total field satisfies a Neumann boundary condition on ∂D. This kite is shown in Figure 1. By using Nyström's method, we compute the far field pattern at 128 equidistantly observation points and 128 uniformly distributed directions, for the case $k = 7$. 5% gaussian noise is added pointwise. From this far field matrix we extract sub-matrices of decreasing order corresponding to smaller incident and observation apertures. Finally we apply the simple method for the reconstruction of ∂D, i.e., for each of these matrices we exhibit the map $y \to \log \|g_{\alpha_{opt}}(\cdot, y)\|_{L^2(\Omega)}$ where α_{opt} is the optimum value of the regularization parameter chosen by the use of the generalized discrepancy principle of Morozov and y is on a 61x61 grid containing the kite. The results are represented for the full 128x128 far field pattern (Figure 2(a)), for the case of

64 incident waves and 64 view points uniformly distributed from 0 to π (Figure 2(b)), for the case of 32 incident waves and 32 view points uniformly distributed from 0 to $\pi/2$ (Figure 2(c)) and for the case of 16 incident waves and 16 view points uniformly distributed from 0 to $\pi/4$ (Figure 2(d)). In Figure 3 we plot the map $y \to \alpha_{opt}(y)$ for the corresponding apertures.

As a final application we test the behaviour of the simple method in the case of two kites satisfying the Dirichlet boundary condition when a 5% gaussian noise is added to the far field pattern. These two kites, represented in Figure 4, are obtained from the single kite plus a second kite rotated 45° and displaced from the first kite along the vector $(5,5)$. In Figures 5 and 6 we plot respectively the map $y \to \log \|g_{\alpha_{opt}}\|_{L^2(\Omega)}$ and the map $y \to \alpha_{opt}(y)$. The incident and observation angles are the same considered in Figures 2 and 3. All the reconstructions presented in this section have been obtained in less than one minute (we have used a SUN workstation). From these results it seems that for aperture significantly smaller than π the algorithm becomes less reliable. A better resolution can possibly be achieved by coupling the simple method with a regularization method exploiting *a priori* information on the solution.

In some ways the preliminary numerical results described above raise more questions than they answer. In particular, what happens for different values of k? Why is the reconstruction of the single kite so much poorer than that for the double kite? What causes the orientations evidenced in figures 2(c), 2(d) and 5(c), 5(d)? We don't have the answers to all of these questions but in the folloiwng paragraphs we will venture to make a few observations.

The value $k = 7$ for the wave number was chosen rather arbitrarily, motivated by the fact that for the figures chosen this value corresponds to a frequency in the middle of the resonance region for which our method is designed. We would expect the reconstructions to deteriorate for lower values of k and to improve for higher values of k but this has not yet been investigated thoroughly.

The quality of reconstruction for the single kite (Neumann boundary data) is clearly not as good as for the double kite (Dirichlet boundary data). A less dramatic difference between Dirichlet and Neumann data reconstructions was also noted in [1] for a single kite, no noise and full aperature data. It is not clear to us why this is so or even how much it has to do with the boundary data. However, note that we are not reconstructing our objects using a standard least squares optimization algorithm. Instead, our method is based on an auxiliarly function $\|g_y\| = \|g(y,\cdot)\|$ becoming unbounded as a parameter y approaches the boundary and it is unclear when this happens in an abrupt fashion, thus producing a good reconstruction. For example, in the case of a circle with Dirichlet boundary data, $\|g_y\|$ becomes unbounded relatively slowly thus leading to a poor reconstruction [1]. It seems as if the more complicated the scatterer the better the reconstruction with the Dirichlet case yielding somewhat better results than the

Neumann case. Clearly, more work needs to be done in examining this apparent behavior of our method.

The orientation of the reconstructins in the case of limited aperature is intriguing since for a single kite it is roughly perpendicular to the bisector of the aperature whereas for the case of two kites it seems to be parallel to this line! If the far field equation (1.1) was valid for a single incident direction d and observation angle $\hat{x} = d$ (which it is not!) then g_y and hence $\|g_y\|$ would not change in directions y perpendicular to $\hat{x}$. This might explain the orientation for the single scatterer corresponding to 2(c) and 2(d). On the other hand, in figures 5(c) and 5(d) corresponding to two scatterers near each other (see also figures 6(c) and 6(d)) this behavior seems to be dominated by $\|g_y\|$ having difficulty in becoming large in the region between the two kites. (Recall that $\|g_y\|$ is small for y in the interior of the kites). There is even some evidence of this behavior in the large aperature cases of figures 5(a) and 5(b)). However, the above comments are obviously pure speculation at this point and are meant merely as guidance for further investigation.

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