Regularised Deconvolution of Multiple Images of the Same Object

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Abstract

In recent papers the problem of deconvolving multiple images of the same object has been considered and the fact that this problem can be well-posed in the sense of distributions has been stressed. In this paper we point out that the problem is still ill-posed in spaces of square integrable functions and we discuss in what cases the use of multiple images can be really advantageous with respect to the use of a single one.

Keywords: single and multiple deconvolution; linear and non-linear regularisation.

1 Introduction

Images produced by a space-invariant imaging system can be described by the following mathematical model:

$$g = K * f^{(\circ)} + w \tag{1}$$

where g is the blurred and noisy image which has been detected, K is the point spread function (PSF) of the imaging system, w is the term due to the instrumental noise and $f^{(\circ)}$ is the perfect image which should be obtained in the absence of blurring and noise. Then the problem of *image restoration* consists in obtaining an estimate f of $f^{(\circ)}$.

If the signal-to-noise ratio is sufficiently large, the noise term w can be neglected and equation (1) becomes a convolution equation

$$g = K * f \tag{2}$$

whose solution is an ill-posed problem. This problem has been widely investigated and many methods have been proposed for its stable and approximate solution[1,2,3].

In recent papers[4,5], the case of multiple images of the same object has been considered and the practical relevance of some mathematical results, which apply to this case, has been stressed. In such a case, the mathematical model (1) is replaced by the following one:

$$g_i = K_i * f^{(\circ)} + w_i \quad ; \quad i = 1, \ldots, m \tag{3}$$

and, if the noise terms w_i are neglected, equation (2) is replaced by the following system of convolution equations:

$$g_i = K_i * f \quad ; \quad i = 1, \ldots, m \quad . \tag{4}$$

The problem is to understand how one can take advantage from the redundancy of the data. We would like to note that, from a practical point of view, equation (4) simply means that the images $g_1, \ldots, g_m$ of the object f are obtained by the use of different imaging devices which are characterised by different PSF $K_1, \ldots, K_m$ and so, in particular, by different resolving powers.

In the above mentioned papers only the case of compactly supported PSF is considered. Now, since a compactly supported and integrable function defines a convolution operator which is continuous in $D'(\mathbb{R}^N)$, the space of distributions (or generalised functions) on $\mathbb{R}^N$, the multiple convolution operators (4) define a continuous operator from $D'(\mathbb{R}^N)$ into $[D'(\mathbb{R}^N)]^m = D'(\mathbb{R}^N) \oplus \ldots \oplus D'(\mathbb{R}^N)$ (m times). Then the main result, already proved and discussed in several papers[4,5], is that, *if the PSF K_i satisfy some suitable conditions*, then there exists a set of compactly supported distributions $H_1, \ldots, H_m$ such that

$$\sum_{i=1}^{m} K_i * H_i = \delta \quad . \tag{5}$$

This beautiful mathematical result implies that the operator defined by the multiple convolution operators (4) has a continuous inverse from $[D'(\mathbb{R}^N)]^m$ onto $D'(\mathbb{R}^N)$ and, therefore, that the problem (4) is well-posed in the sense of distributions. This continuity, however, is too weak for practical applications where a stronger continuity is required, for istance that of $L^2(\mathbb{R}^N)$. In fact the deconvolvers $H_1, H_2, \ldots, H_m$ do not define continuous operator in $L^2(\mathbb{R}^N)$ and therefore regularisation methods must be used also in the case of multiple images deconvolution. A first step in this direction has been already performed by the application of the Wiener filter method[6] to an example indicated in a previous paper[4].

In the present paper we continue this analysis by considering two regularisation methods: the first is the well-known Tikhonov regularisation method, the second is a constrained iterative algorithm. We apply these techniques to the example considered in the previously mentioned papers[4,6]. In particular, we compare the restorations obtained by the use of two images with the restorations obtained by the use of only one. Our result is that, while in the case of Tikhonov regularisation two images provide better results than a single one, in the case of the iterative method with the additional constraints of positivity or of compactness of the solution support, one or two images provide essentially the same results. More precisely, single deconvolution performed by the use of this constrained iterative algorithm gives reconstructions which are significantly more accurate than the ones obtained by double deconvolution with Tikhonov method.

An explanation of this fact is attempted and an example where the use of two images provides substantial improvement with respect to the case of only one independently of the method adopted is provided. This example does not correspond to compactly supported convolution kernels but this is not a difficulty because we believe that the use of multiple images is a topic of great practical relevance beyond to the cases where precise mathematical results can be obtained.

In section 2 we formulate the problem and we recall the main results previously discussed by other authors[4,5]. In section 3 we outline the regularisation methods we consider in this paper. In section 4 we discuss our numerical results in the case of the example considered in previous papers[4,6]. Finally, in section 5, we discuss a new example where the use of two images provides a substantial improvement of the restorations.

2 Formulation of the problem

In this section we wish to formulate problem (4) as the solution of a first kind operator equation and to introduce the functional spaces we will consider. To each convolution kernel K_i we can associate a linear operator A_i defined by

$$A_i f = K_i * f \quad . \qquad (6)$$

If K_i is compactly supported and integrable then A_i is a continuous operator in $D'(\mathbb{R}^N)$ and also a continuous operator in $L^2(\mathbb{R}^N)$ (for simplicity we do not use different notations for the two operators). However we will not restrict our analysis to the case of compactly supported kernels. If this condition is not satisfied then A_i, in general, is not a continuous operator in $D'(\mathbb{R}^N)$ while it is a continuous operator in $L^2(\mathbb{R}^N)$ if K_i is integrable. A necessary and sufficient condition for the continuity of A_i in $L^2(\mathbb{R}^N)$ is the boundedness a.e. of the Fourier transform $\hat{K}_i$ of the kernel.

Given m kernels $K_1, \ldots, K_m$, let us assume that they satisfy conditions such that the corresponding operators A_i are all continuous in the same space X ($D'(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$). Then the set of m operators defines a continuous operator A from X into $Y = X^m = X \oplus \ldots \oplus X$ (m times) as follows:

$$Af = \{A_1 f, \ldots, A_m f\} \quad . \qquad (7)$$

Since the m images of equation (4) define a multiple image $g = \{g_1, \ldots, g_m\} \in X^m$, equation (4) can be written in the synthetic form

$$g = Af \quad . \qquad (8)$$

We remark that the ranges of the operators A_i are never closed subspaces of $L^2(\mathbb{R}^N)$ when K_i is integrable, and therefore also the range of A is not a closed subspace of $[L^2(\mathbb{R}^N)]^m$. It follows that the inverse operator A^{-1}, when it exists, is not continuous. If A^{-1} does not exist then it is possible to define a generalised inverse[7] $A^\dagger$ of A, but also $A^\dagger$ is not continuous. In other words, problem (8) is ill-posed in L^2-spaces when the kernels K_i are integrable.

The situation is different when we consider equation (8) in a space of distributions, i.e. $X = D'(\mathbb{R}^N)$, $Y = [D'(\mathbb{R}^N)]^m$, and the kernels K_i are compactly supported. Then, as follows from the results already proved and discussed[4,5], a set of compactly supported distributions $H_1, \ldots, H_m$ such that equation (5) holds true may exist. By Fourier transforming both sides of this equation, one obtains the so called Bezout equation

$$\sum_{i=1}^{m} \hat{K}_i(\omega)\hat{H}_i(\omega) = 1 \tag{9}$$

which, thanks to analytic continuation, must hold true for any $\omega \in \mathbb{C}^N$. Therefore the existence of the set of deconvolvers $H_1, \ldots, H_m$ is related to the existence of analytic solutions of the Bezout equation. These solutions may not exist for arbitrary kernels K_i, because, for istance, the functions $\hat{K}_i(\omega)$ have a common zero. A rigorous discussion of this problem has been the object of several papers all based, in some way, on a theorem by Hörmander[8] about the ring structure of the Paley-Wiener space. Here we only remind the necessary and sufficient condition which must be satisfied by the kernels K_i in order to ensure the existence of compactly supported deconvolvers H_i satisfying equation (9): there exist positive constants A, B and a positive integer N such that

$$(\sum_{i=1}^{m} |\hat{K}_i(\omega)|^2)^{\frac{1}{2}} \geq A \; exp(-B|Im\omega|) \; (1+|\omega|)^{-N} \quad , \quad \omega \in \mathbb{C}^N \quad . \tag{10}$$

Among other things, this condition excludes the case where all the kernels K_i are in C_0^∞. A set of convolvers satisfying condition (10) is referred to as *strongly coprime*.

In section 4 we will consider a well-known[4,6] example of a strongly coprime set of convolvers. Since the explicit expressions of the corresponding deconvolvers contain derivatives of the delta function[4], they do not define continuous operators in L^2, as it should be obvious from our analysis above.

3 Regularised solutions

As we remarked in section 2, equation (8) is always ill-posed when data and solution belong to L^2-spaces. Since this equation has the general form of a first kind operator equation , the application of regularisation methods to this problem is straightforward. For completeness, however, we give the main formula.

We do not restrict our analysis to the case of compactly supported kernels but we require that the operator A is continuous. If we introduce the function

$$\hat{K}(\omega) = (\sum_{i=1}^{m} |\hat{K}_i(\omega)|^2)^{\frac{1}{2}} \quad , \tag{11}$$

then A is continuous from $L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)^m$ if and only if $\hat{K}(\omega)$ is bounded a.e., in what case

$$\|A\| = \sup_{\omega \in \mathbb{R}^N} \{\hat{K}(\omega)\} \quad . \tag{12}$$

It is easy to verify that the adjoint operator A^* is given by

$$A^* g = \sum_{i=1}^{m} K_i^* * g_i \tag{13}$$

where $K_i^*(x) = \overline{K}_i(-x)$ (the bar denotes complex conjugation). Finally the generalised inverse[7] $A^\dagger$ of A is given by

$$(A^\dagger g)(x) = \frac{1}{(2\pi)^N} \sum_{i=1}^{m} \int_{\mathcal{B}} \frac{\overline{\hat{K}_i(\omega)}}{|\hat{K}(\omega)|^2} \; \hat{g}_i(\omega) \; exp(ix\omega) \; d\omega \quad , \tag{14}$$

where $\mathcal{B}$ is the support of $\hat{K}(\omega)$ in $\mathbb{R}^N$ (band of the imaging system). This equation defines also the domain of the operator $A^\dagger$, i.e. the set of all the elements $g \in [L^2(\mathbb{R}^N)]^m$ such that $A^\dagger g \in L^2(\mathbb{R}^N)$. The necessary and sufficient condition is that all the functions

$$\hat{\psi}_i(\omega) = \frac{\overline{\hat{K}_i(\omega)} \hat{g}_i(\omega)}{|\hat{K}(\omega)|^2} \tag{15}$$

must be square integrable. If $\mathcal{B} = \mathbb{R}^N$, then the inverse operator A^{-1} exists and it is given again by equation (14), the integration being now extended to $\mathbb{R}^N$.

We notice that the condition $\mathcal{B} = I\!\!R^N$ is satisfied if the strongly coprime condition (10) is satisfied. In such a case, the inverse operator A^{-1} can also be written in terms of the compactly supported deconvolvers H_i:

$$(A^{-1}g)(x) = \frac{1}{(2\pi)^N} \sum_{i=1}^{m} \int_{I\!\!R^N} \hat{H}_i(\omega)\ \hat{g}_i(\omega)\ exp(ix\omega)\ d\omega \quad . \tag{16}$$

Obviously, equation (14) and (16) must be equivalent because the inverse operator is uniquely defined. The reason of this ambiguity is that the solution of the Bezout equation is not unique. In fact this equation has an infinity of analytic solutions[5] (when the strongly coprime condition is satisfied) and also non-analytic solutions, such as, for instance $\hat{H}_i'(\omega) = \overline{\hat{K}_i(\omega)}/|\hat{K}(\omega)|^2$, whose corresponding deconvolvers H_i' are not compactly supported (by the Paley-Wiener-Schwartz theorem[9]). These various solutions provide equivalent representations of the inverse operator A^{-1}.

We consider two methods for obtaining regularised solutions of equation (8): Tikhonov regularisation and constrained successive approximations.

3.1 Tikhonov regularisation

As it is well known, this method consists in minimising the functional[10]

$$\Phi_\lambda[f] = \|Af - g\|_Y^2 + \lambda \|f\|_X^2 \quad , \tag{17}$$

where λ is a positive number called *regularisation parameter*. If we denote by f_λ the function which minimises this functional, then by means of straightforward computations one finds that its Fourier transform is given by

$$\hat{f}_\lambda(\omega) = \sum_{i=1}^{m} \frac{\overline{\hat{K}_i(\omega)}}{|\hat{K}(\omega)|^2 + \lambda} \hat{g}_i(\omega) \quad . \tag{18}$$

If we neglect the noise term in equation (3) so that g_i is given by equation (4) with $f = f^{(o)}$, we find the following relationship between $\hat{f}_\lambda(\omega)$ and $\hat{f}^{(o)}(\omega)$:

$$\hat{f}_\lambda(\omega) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda} \hat{f}^{(o)}(\omega) \quad . \tag{19}$$

It follows that, in absence of noise, the combined effect of imaging and restoration can be described by a global PSF whose Fourier transform is given by

$$\hat{T}_{\lambda}(\omega) = \frac{|\hat{K}(\omega)|^2}{|\hat{K}(\omega)|^2 + \lambda} \quad . \tag{20}$$

We call this the *global transfer function* of the the imaging system with restoration. We also notice that regularised restorations of $f^{(o)}$ can be obtained by using separately the images g_i; these restorations, denoted by $f_{\lambda,i}$, are given by

$$\hat{f}_{\lambda,i}(\omega) = \frac{\overline{\hat{K}_i(\omega)}}{|\hat{K}_i(\omega)|^2 + \lambda}\hat{g}_i(\omega) \tag{21}$$

and the corresponding global transfer functions are given by

$$\hat{T}_{\lambda,i}(\omega) = \frac{|\hat{K}_i(\omega)|^2}{|\hat{K}_i(\omega)|^2 + \lambda} \quad . \tag{22}$$

The knowledge of the global transfer function allows to estimate the resolution limit provided by the imaging device. To show this, we consider the case of a single imaging system whose action is represented by equation (1) (the noise term is again taken into account). By Fourier transforming both sides of this equation, we obtain

$$\hat{g}(\omega) = \hat{K}(\omega)\hat{f}^{(o)}(\omega) + \hat{w}(\omega) \quad . \tag{23}$$

The effective band of the image $\hat{g}(\omega)$ is defined as the set of frequencies such that

$$|\hat{K}(\omega)\hat{f}^{(o)}(\omega)| \geq |\hat{w}(\omega)| \quad . \tag{24}$$

The effective band contains all the values of ω for which the image can provide information about $\hat{f}^{(o)}(\omega)$. Though this band cannot be exactly determined, nevertheless it can be approximately evaluated by assuming that the exact image and the noise are two uncorrelated white stationary processes with power spectra respectively E^2 and ϵ^2. Therefore, equation (24) can be replaced by

$$|\hat{K}(\omega)| \geq \frac{\epsilon}{E} \quad , \tag{25}$$

where E/ϵ is the signal-to-noise ratio. Now, it can be proved[11] that a good choice of the regularisation parameter is given by $\lambda = (\epsilon/E)^2$. In such a case, since the global transfer function is a monotonic function of $|\hat{K}(\omega)|$, as follows from equation (20) or (22), condition (25) is completely equivalent to the following one:

$$|\hat{T}_\lambda(\omega)| \geq \frac{1}{2} \quad . \tag{26}$$

This means that, if $|\hat{T}_\lambda(\omega)|$ is a decreasing function of ω, an estimate of the superior extreme ω_{eff} of the effective band is given by the frequency such that the global transfer function is equal to 1/2. Thanks to the Shannon sampling theorem[12], the corresponding resolution limit achievable by the instrument is approximately equal to $\delta = \pi/\omega_{\text{eff}}$. We observe that the solution of equation (26) only represents an easy way to determine the effective band of the image, but, as clearly shown by equation (24), this band depends on the PSF, the signal and the noise and is not modified by the introduction of regularisation. Nevertheless the regularisation method provides an improvement of the quality of the image and this generally allows also to improve the restoration of details to an amount of the order of the resolution limit.

3.2 Constrained successive approximations

The method of successive approximations is also known as the Landweber-Bialy method[10]. It is an iterative technique for solving least-squares problems associated with first kind operator equations. In the case of a linear operator equation as (8), the iteration scheme is as follows:

$$f_{k+1} = f_k + \tau(A^* g - A^* A f_k) \tag{27}$$

where τ is a relaxation parameter whose value must satisfy the following conditions:

$$0 < \tau < \frac{2}{\|A\|^2} \quad . \tag{28}$$

If we take $f_0 = 0$, then it is easy to show that, in the case of the operator (7), whose adjoint is given by equation (13), the result of the k-th iteration is equivalent to a filtering of the generalised solution (14), since we have:

$$\hat{f}_k(\omega) = [1 - (1 - \tau|\hat{K}(\omega)|^2)^k] \sum_{i=1}^{m} \frac{\overline{\hat{K}_i(\omega)}\hat{g}_i(\omega)}{|\hat{K}(\omega)|^2} \tag{29}$$

with $\hat{K}(\omega)$ defined by equation (11). This result is analogous to that which holds true in the case of a single convolution operator[3]. Therefore, if we neglect the noise term, for a fixed number of iterations k, the combined effect of imaging and iteration can be described by a global transfer function given by

$$\hat{T}_k(\omega) = 1 - (1 - \tau|\hat{K}(\omega)|^2)^k \quad . \tag{30}$$

This has a behaviour similar to that of $\hat{T}_\lambda(\omega)$, equation (20), since it is a filter which does not transmit the frequencies corresponding to small values of $\hat{K}(\omega)$.

As in the case of Tikhonov regularisation, if we deconvolve the various images separately, at the k-th iteration we obtain the following approximations

$$\hat{f}_{k,i}(\omega) = [1 - (1 - \tau|\hat{K}_i(\omega)|^2)^k] \frac{\hat{g}_i(\omega)}{\hat{K}_i(\omega)} \tag{31}$$

whose corresponding global transfer functions are

$$\hat{T}_{k,i}(\omega) = 1 - (1 - \tau|\hat{K}_i(\omega)|^2)^k \quad . \tag{32}$$

It is known that, in the absence of noise, this iterative method converges (strongly) to the generalised solution of the problem[13]; moreover, in the case of noisy data, it behaves as a regularisation algorithm, the number of iterations playing the role of a regularisation parameter[10].

However, the most interesting feature of the method is that it can be easily modified in order to take into account additional constraints on the solution. A typical one is that of positivity. In general one can assume that the unknown image $f^{(o)}$ belongs to a closed and convex set $\mathcal{C}$. In such a case it is quite natural to look for least squares solutions which belong to $\mathcal{C}$, i.e., for solutions of the problem

$$\|Af - g\| = minimum \qquad , \qquad P_{\mathcal{C}} f = f \tag{33}$$

where $P_{\mathcal{C}}$ is the (in general non-linear) projection operator onto the set $\mathcal{C}$.

Let us consider the following modification of the algorithm (27):

$$f_{k+1} = P_{\mathcal{C}}[f_k + \tau(A^*g - A^*Af_k)] \tag{34}$$

i.e., at each iteration we project the result onto $\mathcal{C}$. Then it is possible to prove, at least in the case where the operator A has a bounded inverse[14], that the algorithm (34) converges to the unique solution of problem (33). However, the problem (33) is, in general, ill-posed. In such a case the algorithm (34) (with $f_0 = 0$) has, presumably, a regularisation effect. Evidence, in this direction, is given by numerical experiments[3]. Therefore, by stopping the iterations it is possible to obtain a stable and constrained approximate solution.

In this paper we will consider two kinds of constraints:
(i) the solution of the problem is positive (non-negative). This is a quite natural requirement in image restoration or in spectroscopy. Now, the set of non-negative (a.e.) functions is a closed and convex set in $L^2(\mathbb{R}^N)$ and therefore the algorithm (34) can be used. The action of the projection operator $P_\mathcal{C}$ consists in replacing by zero the negative values of the function f.
(ii) The solution of the problem has a bounded support. This is a constraint which is widely used in the problem of out-of-band extrapolation and super-resolution[15]. The set of all the functions whose support is interior to a given and fixed set $\mathcal{D} \subset \mathbb{R}^n$ is a closed linear subspace of $L^2(\mathbb{R}^n)$. In this case the action of the projection operator $P_\mathcal{C}$ (which is linear) consists in replacing by zero the values of the function f in points exterior to the set $\mathcal{D}$.

4 An example of compactly supported kernels: numerical results

In this section we discuss an example which has been already considered in the literature[4,6]. This is the particular case where the kernels are given by the characteristic functions of two intervals, that is

$$K_1(x) = \chi_{[-r_1,r_1]}(x) \quad , \quad K_2(x) = \chi_{[-r_2,r_2]}(x) \tag{35}$$

the corresponding transfer functions being sinc functions

$$\hat{K}_1(\omega) = 2\frac{\sin(r_1\omega)}{\omega} \quad , \quad \hat{K}_2(\omega) = 2\frac{\sin(r_2\omega)}{\omega} \quad . \tag{36}$$

Both transfer functions have a countable set of zeroes respectively at the points $\omega_{1,n} = n\pi/r_1$ and $\omega_{2,n} = n\pi/r_2$ ($n = \pm1, \pm2, \ldots$). These zeroes never coincide if $r_1/r_2 = \sqrt{p}$ (p not a perfect square). This is precisely the condition which assures that these convolvers are strongly coprime. From now on we will always consider the case $r_1 = 1$ and $r_2 = \sqrt{2}$, because this is the case considered in the above mentioned papers.

In figure 1 we plot the functions $\hat{T}_\lambda(\omega)$, equation (20), and $\hat{T}_{\lambda,1}(\omega)$, equation (22) (in both case with $\lambda = 10^{-3}$), associated with the convolvers (35). The global transfer function corresponding to single deconvolution (figure 1(a)) has precisely the zeroes of $\hat{K}_1(\omega)$, while $\hat{T}_\lambda(\omega)$ (figure 1(b)) is never zero, even if, in correspondence with particular frequencies, it assumes very small values. This happens when the zeroes of $\hat{T}_{\lambda,1}(\omega)$ and $\hat{T}_{\lambda,2}(\omega)$ occur close to each other. As already noted[6], this is the reason why, from a general point of view, it is convenient to choose the strongly coprime PSF in such a way that the zeroes of their Fourier transforms are close to each other at the highest possible frequencies. The functions $\hat{T}_k(\omega)$, equation (30), and $\hat{T}_{k,i}(\omega)$, equation (32), have a quite similar behaviour. In fact, $\hat{T}_{k,i}(\omega) = 0$ when $\hat{K}_i(\omega) = 0$.

As already explained in section 3.1, the knowledge of the global transfer functions allows to determine the effective bands of the single and double images. In the case of $\hat{T}_{\lambda,1}(\omega)$, this function is not monotonic in ω. Nevertheless it becomes definitely smaller than 1/2 for $\omega \geq 64.4$. This suggests to assume this as the superior extreme $\omega_{\text{eff},1}$ of the effective band. The corresponding resolution limit is $\delta_1 = 0.048$. We must however observe that this effective band is not completely full because some small intervals around the zeroes of $\hat{K}_1(\omega)$ are also excluded by the condition $|\hat{T}_{\lambda,1}(\omega)| > 1/2$. In a similar way we find the effective band of the double images and the corresponding resolution limit is $\delta = 0.037$. Also in this case the effective band is not completely full, owing to the relationship between zeroes of $\hat{K}_1(\omega)$ and $\hat{K}_2(\omega)$ already observed. Anyway some improvement in resolution by the use of double images has been obtained, even if this example seems not to be particularly meaningful in order to show the improvement of resolution which double deconvolution can provide in particular cases.

The presence of zeroes in the global transfer functions corresponding to single deconvolution implies that linear filtering methods applied to the case of one image are not able to recover the Fourier transform of the object in the neighbourhoods of the zeroes of the transfer function.

As we will see, this is not true in the case of the (in general non-linear) iterative method of equation (34) which can be used when, for istance, we know that the object must be non-negative (this is a rather natural condition in spectroscopy and imaging), or it can be considered zero outside a bounded interval (see the remarks at the end of section 3). In the case of positivity, since this algorithm is non-linear, it cannot be described by means of a global transfer function as the linear methods. Moreover, in the case of compact support, the algorithm is linear but its global transfer function must be expressed in terms of the singular functions of the integral operator. Unfortunately, the computation of these functions is not easy. Therefore we will justify the efficiency of these methods in interpolating the Fourier transform by means of a numerical simulation.

The example we consider is that of a gaussian function with zero mean and standard deviation $\sigma = 0.1$, whose analytic form is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\, exp\left[-\frac{x^2}{2\sigma^2}\right] \quad . \tag{37}$$

Notice that the standard deviation of this gaussian is approximately twice the resolution distance associated with $K_1(x)$. This function and its Fourier transform are plotted in figure 2. The simulated images are obtained by convolving f with the two convolution kernels (35) (an FFT routine is used to this aim) and then by affecting the results with gaussian noise (the relative error is equal to 5%). These images and the corresponding Fourier transforms are again represented in figure 2. The behaviour of Tikhonov inversion method is described in figure 3, where the Fourier transforms of the regularised solutions are plotted in the case of the inversion of only one image g_1 (figure 3(a)) and in the case of the simultaneous inversion of g_1 and g_2 (figure 3(b)). It is evident that, owing to the linearity of the method, the reconstruction provided by equation (21) is characterised by regular "holes" corresponding to the zeroes of $\hat{K}_1(\omega)$. This lack of information can be partly recovered by adding the information contained in g_2 through equation (19). We note that the optimum value of the regularisation parameter λ is obtained by minimising the function

$$\epsilon(\lambda) = \frac{\|f - f_\lambda\|}{\|f\|} \quad . \tag{38}$$

The improvement of the restoration accuracy due to double deconvolution is measured by the value of the minimum of function (38), which is $\epsilon = 43.1\%$ for single deconvolution and $\epsilon = 16.5\%$ for double deconvolution.

The greater efficiency of double deconvolution is deeply related to the linearity of the algorithm which is adopted for the inversion. Things are completely different if the non-linear regularisation technique is applied. In fact, the constrained Landweber-Bialy algorithm, equation (34), allows to interpolate the values of the Fourier transform of the regularised solution also in the neighbourhoods of the zeroes of $\hat{K}_1(\omega)$, so that the performance of the method is not improved by the use of more convolvers. In figure 3(c) we give the reconstruction of the Fourier transform of function (37) obtained by the application of the Landweber-Bialy method with the constraint of the positivity of the solution to the single inversion of g_1. As it is evident, the insertion of the "a priori" information into the inversion algorithm makes the regularised solution extremely stable and allows to achieve a great restoration accuracy. As in the case of Tikhonov method, the optimum value of the number of iteration (which here plays the role of the regularisation parameter) is obtained by minimising

$$\epsilon(n) = \frac{\|f - f_n\|}{\|f\|} \quad , \tag{39}$$

and the reconstruction error is given by the minimum of this function, which is $\epsilon = 5.4\%$. As clearly shown in figure 3(d), similar results are obtained if the Landweber-Bialy method is modified by imposing that the regularised solution is zero outside a bounded interval (in this case $\epsilon = 6.5\%$). Here we have chosen the interval $[-0.5, 0.5]$, because outside this interval the function (37) is essentially zero. In fact, the compactness prescription on the support corresponds to impose the analyticity of the Fourier transform of the regularised solution. We do not report the results obtained by double deconvolution because they do not differ significantly from those obtained by a single one.

5 An example of non-compactly supported kernels

The substantial equivalence between single and double deconvolution when non-linear filtering is adopted refers mainly to the example introduced in previous papers[4,6]. In this case, the Fourier transforms of the convolution kernels, equation (36), are characterised by not superimposed zeroes but provide information about approximately the same region of the frequency spectrum, as is shown in figure 1. In other terms, the complementarity of the information provided by the two images is localised only in a neighbourhood of the zeroes of the kernels so that the application of a non-linear method, able to interpolate the Fourier transform of the regularised solution in these intervals, makes double deconvolution practically useless. On the contrary, the availability of two (or more) simultaneous images of the same object becomes advantageous, also when the constrained iterative method is applied, if the convolvers provide information about different domains of the spectrum. This is, for istance, the case of the two non-compactly supported PSF

$$K_1(x) = \frac{2}{\sqrt{2\pi}\sigma_1} exp\left[-\frac{x^2}{2\sigma_1^2}\right] \tag{40}$$

and

$$K_2(x) = \frac{4}{\sqrt{2\pi}\sigma_2}\ cos(\omega_o x)\ exp\left[-\frac{x^2}{2\sigma_2^2}\right] \quad . \tag{41}$$

As plotted in figure 4, the Fourier transform of $K_1(x)$ is again a gaussian function with standard deviation equal to $1/\sigma_1$, while the Fourier transform of $K_2(x)$ is given by two translated gaussian peaks characterised by the same standard deviation $1/\sigma_2$ and by a distance equal to $2\omega_o$. It is evident that a device with $K_1(x)$ as PSF provides images which contain information only at low frequencies while a device with $K_2(x)$ as PSF provides images which do not contain information about the low frequency region of the spectrum but contain information at higher frequencies. Therefore, the inversion of these images separately cannot give accurate reconstructions of the input functions, independently of the kind of algorithm adopted for the regularisation. On the contrary, double deconvolution of the images provided by both PSF is particularly advantageous as it allows to exploit the different kind of information contained in the two images. The usefulness of double deconvolution in this case results particularly clear if one considers the global transfer functions $\hat{T}_{\lambda,1}(\omega)$, $\hat{T}_{\lambda,2}(\omega)$, equation (22) and $\hat{T}_{\lambda}(\omega)$, equation (20), represented in figure 5 (again for $\lambda = 10^{-3}$) for the choice of the parameters: $\sigma_1 = 0.3$, $\omega_o = 14.$ and $\sigma_2 = 0.5$. In fact, $T_\lambda(\omega)$ is equal to one in correspondence with a frequency range which is approximately the union of the intervals where $T_{\lambda,1}(\omega)$ and $T_{\lambda,2}(\omega)$ are equal to one. In the case of two images the resolution distance is $\delta = 0.16$.

In order to show the different performance of single and double deconvolution in this case, we consider the source function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \; exp \left[-\frac{(x - x_\circ)^2}{2\sigma^2} \right] + \; exp \left[-\frac{(x + x_\circ)^2}{2\sigma^2} \right] \quad , \qquad (42)$$

which is plotted in figure 6(a) for $\sigma = 0.03$, $x_\circ = 0.15$. This distance is of the order of the resolution distance previously estimated. In figure 6(b) we plot the result obtained by applying the Landweber-Bialy algorithm, with the positivity constraint, to invert the image g_1. This image has been obtained by convolving $f(x)$ with $K_1(x)$, equation (40), and by affecting the result again with 5% gaussian noise. As one can see, an imaging device with $K_1(x)$ as PSF is unable to resolve the two peaks of the input function (the reconstruction error is enormous: $\epsilon = 76.8\%$). The use of $K_2(x)$ would allow to achieve this resolution, even if the lack of information about the Fourier transform of the solution in the central part of the spectrum implies the presence of artefacts, evident in figure 6(c), and, then, again a large restoration error ($\epsilon = 74.2\%$). Finally, as clearly shown in figure 6(d), we find that a great improvement in the restoration accuracy ($\epsilon = 14.7\%$) together with the achievement of a high resolution power can be obtained only by the use of double deconvolution, as it allows to exploit the information about the source function contained in both images.

6 Concluding remarks

In this paper we consider the deconvolution of multiple images of the same object. In recent papers it has been shown that, if the convolvers are compactly supported functions, this problem can be well-posed in the sense of distributions. Nevertheless here we show that, if a stronger continuity, such as the one in L^2, is required, the problem is still ill-posed. This justifies the use of the regularisation techniques in order to improve the restoration accuracy. In our numerical applications we point out that an example of double deconvolution widely discussed in the literature, is not very useful for demonstrating the advantage of double deconvolution with respect to single one. Moreover we show that, in this case, single deconvolution performed by a constrained iterative method, which allows for "a priori" information on the solution, provides better results than double deconvolution performed by a well-known linear filter. On the contrary, we remark that multiple deconvolution is always advantageous, independently of the kind of algorithm adopted for the regularisation, when the different images contain information about different regions in the Fourier spectrum. We provide an example supporting this assertion. A potential application of this example is in confocal microscopy, if it were possible to combine images provided by a conventional confocal microscope with images provided by a 4Pi confocal microscope[16].

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Figure captions

Figure 1: (a) global transfer functions as defined in equation (22) in the case of the convolver $K_1(x)$ of equation (35) with $r_1 = 1$. The value of the regularisation parameter is $\lambda = 10^{-3}$; (b) global transfer function of equation (20) corresponding to double deconvolution of $K_1(x)$ and $K_2(x)$, with $r_1 = 1$ and $r_2 = \sqrt{2}$ respectively (equation (35)), for the same value of λ.
Figure 2: (a) input gaussian function, with standard deviation $\sigma = 0.1$; (b) Fourier transform of the input function; (c) image g_1 obtained by convolving the gaussian function in (a) with the convolver K_1, equation (35), $r_1 = 1$, and by affecting the result with 5% gaussian noise.; (d) Fourier transform of the image g_1; (e) image g_2, obtained by the use of the convolver K_2, equation (35), $r_2 = \sqrt{2}$; (f) Fourier transform of g_2.
Figure 3: inversion of the images described in figure 2: (a) regularised deconvolution of g_1 by the use of Tikhonov method, equation (21). The restoration error is $\epsilon = 43.1\%$; (b) regularised double deconvolution provided by equation (18) ($\epsilon = 16.5\%$); (c) deconvolution of g_1 by the use of successive approximations with the constraint of positivity ($\epsilon = 5.4\%$); (d) deconvolution of g_1 by the use of successive approximations with the constraint of compactness of the support of the solution ($\epsilon = 6.5\%$).
Figure 4: Fourier transforms of the kernels of equations (40) (solid) and (41) (dashes).
Figure 5: (a) global transfer functions, equation (22), in correspondence with the kernel $K_1(x)$, equation (40) (solid) and $K_2(x)$, equation (41) (dashes); (b) global transfer function in the case of the simultaneous use of $K_1(x)$ and $K_2(x)$.
Figure 6: (a) input function represented by equation (37) in the case $\sigma = 0.03$, $x_\circ = 0.15$; (b) single deconvolution in the case of the convolver $K_1(x)$, equation (40), when the Landweber-Bialy method with positivity is adopted. The noise affecting the image is again 5% gaussian noise; (c) single deconvolution in the case of the convolver $K_2(x)$, equation (41), again by the use of the constrained iterative method; (d) double deconvolution of g_1 and g_2 with the same algorithm.