

# Regularized Electron Flux Spectra in the July 23, 2002 Solar Flare

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## ABSTRACT

By inverting the RHESSI hard X-ray photon spectrum with the Tikhonov regularization algorithm, we infer the effective mean electron source spectrum for a time interval near the peak of the July 23, 2002 event. This inverse approach yields the smoothest electron flux spectrum consistent with the data while retaining real features, such as local minima, which cannot be found with forward model-fitting methods which involve only a few parameters. A significant dip in the recovered mean source electron spectrum near  $E = 55$  keV is noted and its significance briefly discussed.

*Subject headings:* Sun: flares

## 1. Introduction

As discussed by Brown, Emslie, & Kontar (2003), hard X-ray spectra  $I(\epsilon)$  in solar flares can be used, in conjunction with a suitable bremsstrahlung cross-section  $Q(\epsilon, E)$ , to determine the *effective mean electron flux spectrum*  $\bar{F}(E)$  in the source, viz.

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$$I(\epsilon) = \frac{1}{4\pi R^2} \bar{n}V \int_{\epsilon}^{\infty} \bar{F}(E) Q(\epsilon, E) dE, \quad (1)$$

where  $R$  is the distance to the observer and the mean target density  $\bar{n} = V^{-1} \int n(\mathbf{r}) dV$ . The function  $\bar{F}(E)$  represents the density-weighted mean electron spectrum in the source, and is “effective” in the sense that it is influenced by factors such as secondary (e.g., photospheric albedo) emission in the observed  $I(\epsilon)$ . It is a quantity which is obtained solely from the observed photon spectrum  $I(\epsilon)$ ; it requires no assumptions about physical conditions (e.g., thin-target vs. thick-target, uniform vs nonuniform) in the bremsstrahlung source (Brown, Emslie, & Kontar 2003).

The Volterra integral Equation (1) for  $\bar{F}(E)$  can be solved either by forward-fitting parametrized forms of  $\bar{F}(E)$  to  $I(\epsilon)$  (e.g., Holman et al 2003) or by direct inversion of the integral equation. Such an inversion may be performed by several equivalent methods, e.g., differentiation (possibly of fractional order – Brown 1971) or discretization followed by matrix inversion (Johns & Lin 1992). However, the ill-posed nature of the inverse problem (1) (e.g., Craig & Brown 1986) may result in large amplification of data noise in the recovered  $\bar{F}(E)$ . Thompson et al. (1992), Piana (1994) and Piana & Brown (1998) have therefore presented a *regularized* inversion algorithm that maintains much of the fidelity in the recovered  $\bar{F}(E)$ , while effectively suppressing much of the noise amplification through the imposition of a degree of smoothness in the recovered solution (see, e.g., Equation [2] below). The latter two of these works employ an approach based on the Tikhonov (1963) regularization algorithm, which utilizes the Singular Value Decomposition (SVD) of a linear operator for its implementation (Golub & Van Loan 1933). In this *Letter* we apply this Tikhonov regularization technique to deduce the form of  $\bar{F}(E)$  for a 20-second interval near the peak of the July 23, 2002 flare. Significant features in  $\bar{F}(E)$ , not realizable through limited-parameter forward-fitting techniques (e.g., Holman et al. 2003) are indeed revealed.

## 2. Derivation of Regularized Mean Electron Flux Spectra

The formal methodology of the Tikhonov (1963) regularization technique has been first applied to high-resolution solar hard X-ray spectra by Piana (1994) and Piana & Brown (1998). Essentially, the method consists of writing Equation (1) in the discretized matrix form  $\mathbf{I} = \mathbf{A}\bar{\mathbf{F}}$ , where, in general,  $\mathbf{I}$  is an  $M$ -vector representing the observed photon spectrum  $I(\epsilon)$ ,  $A$  is an  $M \times N$  matrix representing the cross-section  $Q(\epsilon, E)$  and  $\bar{\mathbf{F}}$  is an  $N$ -vector representing the mean electron spectrum  $\bar{F}(E)$ . We seek the solution of the minimization problem

$$\|\mathbf{A}\bar{\mathbf{F}} - \mathbf{I}\|^2 + \lambda\|\bar{\mathbf{F}}\|^2 = \text{minimum}, \quad (2)$$

where  $\|\cdot\|$  denotes the Euclidean norm of a vector and the regularization parameter  $\lambda$  tunes the trade-off between the fidelity of the data fit (measured by the first term on the left hand side of Equation [2]) and the smoothness in the recovered  $\bar{\mathbf{F}}$  (measured by the second term). In order to compute the solution of the minimization problem (2) we utilized the Singular Value Decomposition (SVD) of  $\mathbf{A}$ , namely the set of triples  $\{\sigma_k; \mathbf{v}_k, \mathbf{u}_k\}_{k=1}^p$ ,  $p = \min(M, N)$ , that satisfy the *shifted eigenvalue problem*

$$\mathbf{A}\mathbf{u}_k = \sigma_k \mathbf{v}_k; \quad \mathbf{A}^T \mathbf{v}_k = \sigma_k \mathbf{u}_k, \quad (3)$$

where  $\mathbf{A}^T$  is the transpose of  $\mathbf{A}$ ;  $\mathbf{u}_k$  and  $\mathbf{v}_k$  are eigenvectors of length  $N$  and  $M$ , respectively, and the eigenvalues  $\sigma_k$  are real positive numbers with  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p$ . It can be shown (Bertero et al. 1988) that the formal solution of Equation (2) is given by

$$\bar{\mathbf{F}} = \sum_{k=1}^p \frac{\sigma_k}{\sigma_k^2 + \lambda} (\mathbf{I} \cdot \mathbf{v}_k) \mathbf{u}_k. \quad (4)$$

In the present case we have set  $M = N = 286$ , corresponding to energies from 15 to 300 keV with 1 keV resolution. The regularization parameter  $\lambda$  is determined using the Morozov discrepancy principle, i.e., such that  $\|\mathbf{A}\bar{\mathbf{F}} - \mathbf{I}\|^2/N$  is equal to the mean square data noise (Tikhonov et al. 1995). Equation (4) shows that singular functions  $\mathbf{u}_k(E)$  with associated eigenvalues  $\sigma_k \ll \sqrt{\lambda}$  (which are generally associated with lack of smoothness) are effectively suppressed relative to those (smoother)  $\mathbf{u}_k(E)$  that are associated with larger values of  $\sigma_k$ ; for more details see Massone et al. (2003).

### 3. Analysis Procedure

We first obtained the RHESSI count rate spectrum for the period 00:30:00 UT through 00:30:20 UT during the July 23, 2002 flare. This time interval is near the time of peak hard X-ray flux, and corresponds to one of the intervals studied by Holman et al. (2003) in their forward-fitting approach. The corresponding photon spectrum  $I(\epsilon)$  was then derived according to the procedure discussed by Kontar et al. (2003). This resulted in a photon spectrum  $I(\epsilon)$  in 1-keV bins spanning the range 15 – 300 keV. Then, using the bremsstrahlung cross-section  $Q(\epsilon, E)$  due to Haug (1997; his Equations [4] and [5]), weighted by a factor  $Z^2 = 1.44$  to take into account bremsstrahlung on heavy elements, we calculated the corresponding

singular values  $\sigma_k$  and vectors  $\mathbf{v}_k$ ,  $\mathbf{u}_k$ , and hence, using Equation (4), the regularized mean electron spectrum  $\bar{F}(E)$ .

We note that obtaining photon spectra at a high level of accuracy depends on knowledge of several instrument-related factors (e.g., pulse pileup and the spectral dependence of grid transmission versus viewing angle) that are not currently known to this level of accuracy. This remaining few-percent uncertainty in  $I(\epsilon)$  will affect the precise values of  $\bar{F}(E)$  obtained but should not radically affect the essential conclusions of the paper. Final determination of absolute mean electron spectra  $\bar{F}(E)$  must, however, await a more thorough understanding of these instrumental effects.

Any valid form of  $\bar{F}(E)$  must, of course, satisfactorily account for the observed photon spectrum. In this paper we have set up the problem as “square,” i.e., determining electron energies  $E$  at 1 keV intervals in the [15, 300] keV range using photon data at the same discrete energies. However, due to noise in the photon data, the quality of the recovered electron spectrum  $\bar{F}(E)$  above  $\sim 160$  keV was rather poor and the spectrum was therefore truncated at this value. While such a truncation does not introduce serious errors in the corresponding photon spectrum for sufficiently soft (steep) spectra, the photon spectrum near the peak of the July 23, 2002 event was actually very hard, with a spectral index  $\gamma \simeq 3.5$  (Holman et al. 2003). For such a flat spectrum, higher energy ( $E > 160$  keV) electrons *do* contribute significantly to the photon spectrum in the observed range, so that an extrapolation of the recovered electron spectrum above  $E = 160$  keV is necessary in order to correctly account for the observed photon spectrum (cf. Johns & Lin 1992).

We here extend the electron spectrum from 160 keV up to 1 MeV by a power-law of index  $\delta$ . The errors in the recovered photon spectrum resulting from the neglect of electrons of even higher energies ( $E > 1$  MeV) are negligible compared to the data noise. A full treatment of such a spectral extrapolation would calculate the photon spectrum in the range  $\epsilon = [15, 160]$  keV produced by the extrapolated part of the electron spectrum (i.e., from electrons with energies  $160 \text{ keV} < E < 1 \text{ MeV}$ ) and subtract this from the observed data to yield (cf. Johns & Lin 1992) the photon spectrum produced by the remaining electrons (i.e., those with energies in the range  $15 \text{ keV} < E < 160 \text{ keV}$ ). The electron spectrum in this range would then be obtained through the “square” regularized inversion algorithm described in §2. In this paper, we abbreviate this process somewhat by simply adding the power-law extrapolated spectrum for  $E > 160$  keV to the [15, 160] keV portion of the regularized spectrum, obtained using Equation (4). The differences in the [15, 160] keV electron spectra therefore depend on the differences between the actual electron spectrum above 160 keV and the power-law extrapolated form. Based on the fidelity of a power-law fit to the data in the range  $\epsilon > 160$  keV (Holman et al. 2003), these differences should be small.

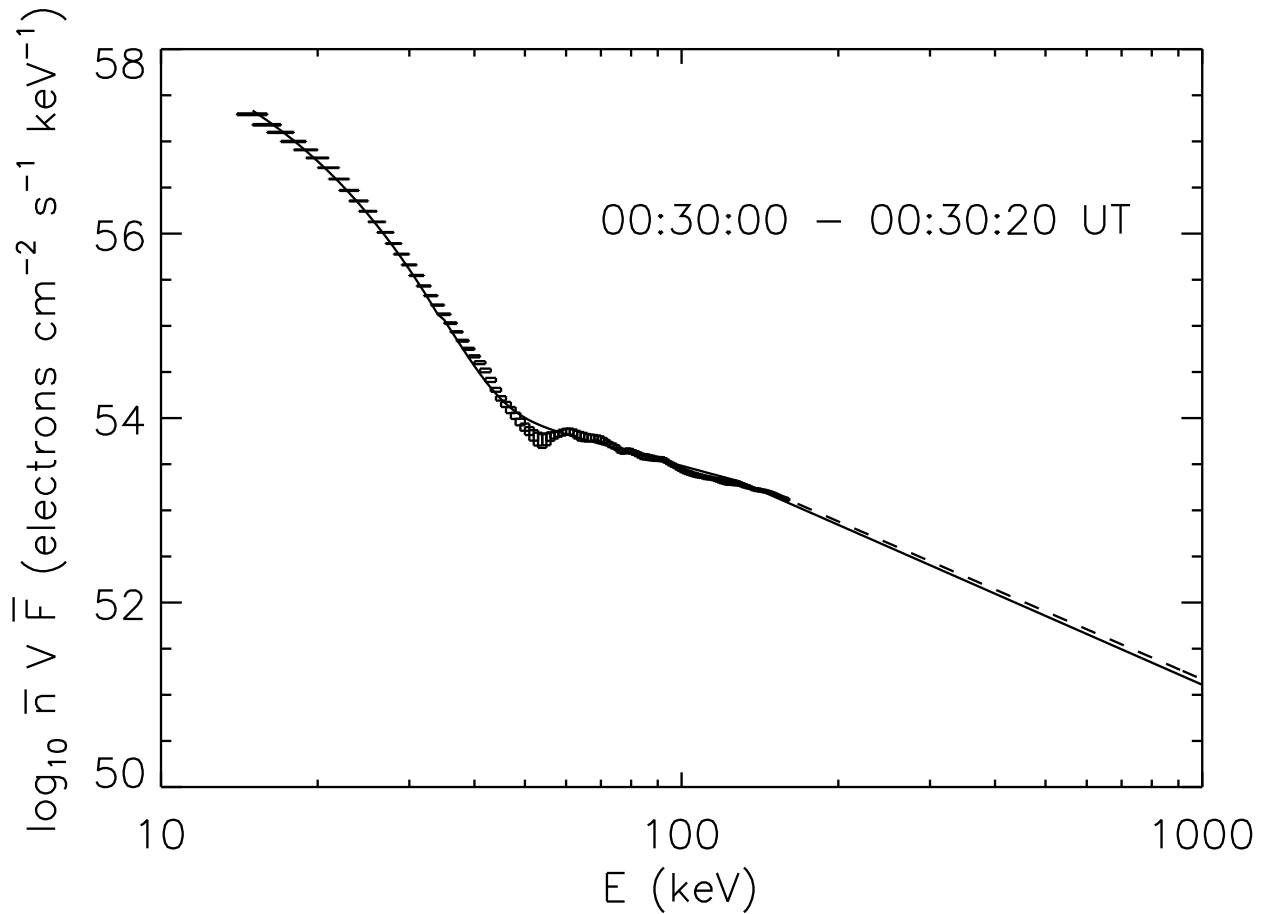


Fig. 1.— Regularized spectrum  $\bar{n}V\bar{F}$  versus  $E$  for the time interval 00:30:00 - 00:30:20 UT, obtained using the Haug (1997) cross-section. The spectrum has been extended from 160 keV to 1 MeV using a power law of index  $\delta = 2.45$  (dashed line; see text for details). The vertical size of the error boxes from  $E = 15$  keV through  $E = 160$  keV reflect the  $3\sigma$  limits caused by statistical noise in the observed photon spectrum  $I(\epsilon)$ . The forward-fitted spectrum of Holman et al. (2003), using the same cross-section, is shown as a solid line.

This extrapolation process allows us to improve the fit between the data and the reconstructed photon spectrum (determined using the electron spectrum over the entire range  $15 \text{ keV} < E < 1 \text{ MeV}$ ). The value of  $\delta$  was chosen through optimization of the fit to the observed photon spectrum (see §4); the resulting value of  $\delta = 2.45$  is very close to the value  $\delta = 2.48$  found for the energy range  $> 130 \text{ keV}$  through the forward-fitting procedure of Holman et al. (2003). However, due to unavoidable errors in the extrapolated spectrum, systematic deviations in the photon spectral fit may (and indeed do) occur.

#### 4. Results

Figure 1 shows the quantity  $\bar{n}V\bar{F}(E)$  (electrons  $\text{cm}^{-2} \text{ s}^{-1} \text{ keV}^{-1}$ ) for the interval 00:30:00-00:30:20 UT. The ( $3\sigma$ ) vertical extents of the error boxes were deduced from repeated inversions using different uniform random realizations of the data set, with the uncertainties in the photon spectrum deduced from the Poisson uncertainties in the instrument count rates in each 1 keV energy channel. (This procedure for determining the uncertainties in the recovered mean source electron spectrum  $\bar{F}[E]$  does not incorporate possible systematic effects – e.g., Smith et al. 2002 – in the photon spectra used for the analysis. Therefore, features that appear may reflect, to some degree, such systematic errors rather than real features in the actual  $\bar{F}[E]$ . Differentiation between real features and those introduced by systematic effects will improve as our understanding of the latter is improved.)

The spectrum deduced by Holman et al. (2003) based on an isothermal-plus-power-law fit is shown for comparison in Figure 1. The absolute values, and general trends, of the forward-fitted and regularized inverted spectra agree very well over the whole energy range, so that “global” quantities such as energy content, etc., are equally well calculated from either spectrum. However, the regularized spectrum shows significant features (such as a local minimum near 55 keV) that cannot be detected by procedures such as parametric power-law fitting. It also shows the true form of  $\bar{F}(E)$  at low energies without any assumption about thermality or nonthermality of the electron population, far less the usual assumption of *isothermality*.

The top panel of Figure 2 shows the reconstructed photon spectrum over the photon energy range up to 300 keV, obtained by convoluting the  $\bar{n}V\bar{F}(E)$  spectrum of Figure 1 up to electron energies of 1 MeV with the Haug (1997) bremsstrahlung cross-section. The bottom panel of Figure 2 shows the residuals (data minus reconstructed spectrum, normalized to the standard deviation in the photon flux at each energy) over the same energy range.

The top panel of Figure 3 shows the distribution of the sizes of the normalized residuals;

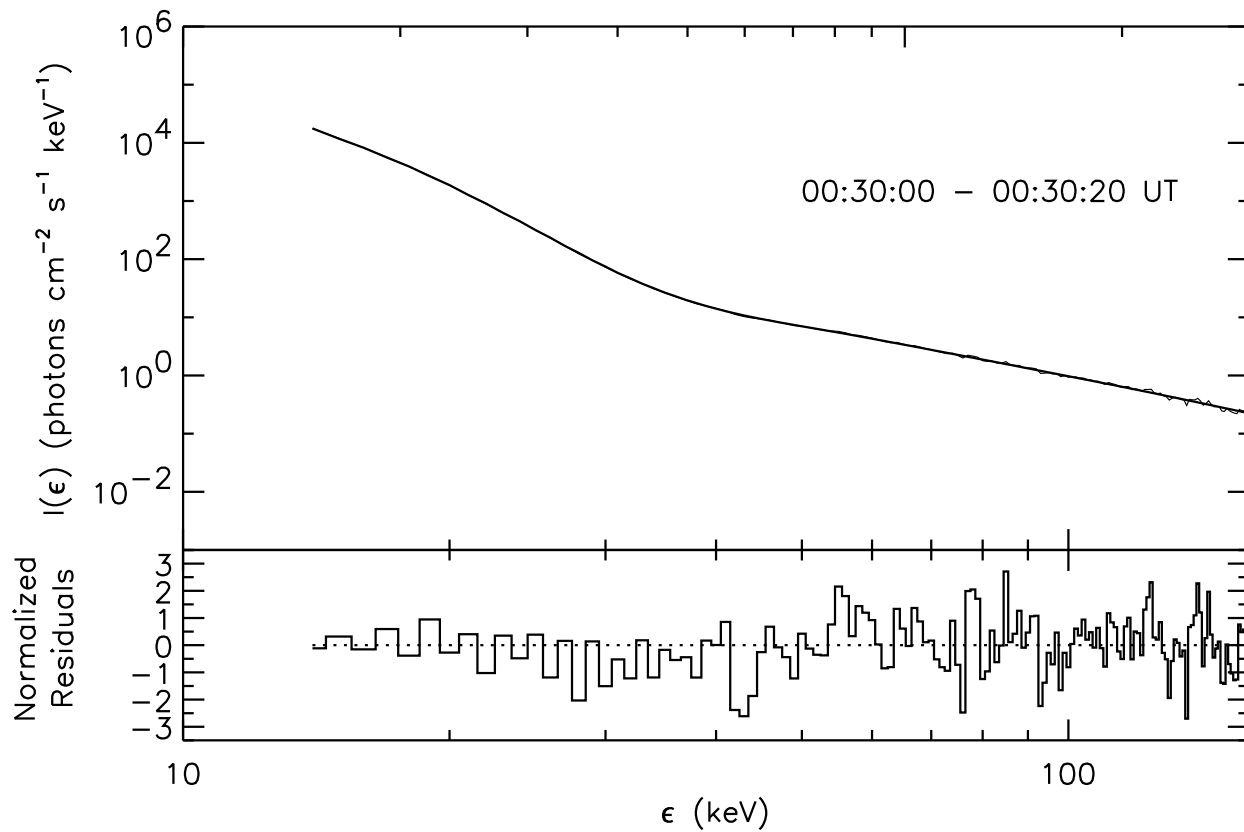


Fig. 2.— *Top panel:* Reconstructed photon spectrum  $I(\epsilon)$  for the time interval 00:30:00 - 00:30:20 UT, compared to the data. *Bottom panel:* Residuals (data - reconstructed spectrum), normalized to the standard deviation in the data at each point.

they are well-fit by a Gaussian form with a mean of  $+0.020\sigma$  and a standard deviation of  $1.034\sigma$ . With 146 points (15-160 keV in 1 keV steps) in the distribution, the standard error in the mean is  $1.034\sigma/\sqrt{146} \simeq 0.086\sigma$ ; hence the measured offset of the mean from zero is only 0.23 of a standard error and so not significant. However, the pattern of residuals shown in the lower panel of Figure 2 indicates that having a mere Gaussian distribution of residuals is not sufficient; it is also necessary to consider the distribution of residuals with respect to photon energy  $\epsilon$ , in order to assess the level of clustering of positive and/or negative residuals in certain energy ranges (e.g., the apparent tendency for the residuals to be consistently negative in the photon energy range from  $\sim 25 - 45$  keV). We therefore consider the function

$$S(\epsilon) = \frac{1}{N} \sum_{i=1}^N r_i, \quad (5)$$

where  $r_i$  is the  $i$ th normalized residual, corresponding to the photon energy  $\epsilon_o + i\Delta\epsilon$ , as shown in the bottom panel of Figure 2. The quantity  $S(\epsilon)$  is the average (normalized) residual over the photon energy interval  $[\epsilon_o, \epsilon]$ ; if the residuals were randomly distributed then  $S(\epsilon)$  would exhibit a random walk with expected deviation  $1/\sqrt{N}$ . We found that varying the spectral index  $\delta$  of the  $> 160$  keV extrapolated electron spectrum  $\bar{F}(E)$  (see §3) resulted in significantly different forms of  $S(\epsilon)$ , many of which resulted in unacceptably large values over certain ranges of  $\epsilon$ , indicating an unacceptable level of clustering of positive (or negative) residuals in such ranges. The value of  $\delta$  chosen was the one that resulted in the most acceptable behavior of  $S(\epsilon)$  (i.e., least maximum deviation from zero), i.e.,  $\delta = 2.45$ .

The lower panel of Figure 3 shows the actual variation of  $S(\epsilon)$  for the residuals shown in Figure 2.  $S(\epsilon)$  stays within the  $2\sigma$  random-walk limits over most of the energy range, so that the null hypothesis (of residuals distributed randomly with respect to photon energy) is acceptable at the 95% confidence level. Nevertheless, there is significant evidence of some clustering of residuals (e.g., the systematic decrease in  $S(\epsilon)$  from  $\sim 25 - 45$  keV), which is a subject for future investigation.

We conclude that the form of  $\bar{F}(E)$  in Figure 1 yields an acceptable fit to the photon spectrum, with residuals consistent, at the 95% confidence level, with random data noise. Based on this fidelity in fitting the photon spectrum, we assert that the electron spectral features in Figure 1 are real and therefore carry important information. (While, to the eye, the nearly power-law photon spectra resulting from a single or double power-law mean electron spectrum also look very good, with a sufficiently small  $\chi^2$  – Holman et al. 2003, such apparent fits are very misleading in the case of ill-posed inverse problems [Craig & Brown 1986].) In particular, the dip in  $\bar{F}(E)$  at around 50-60 keV, the significant excess



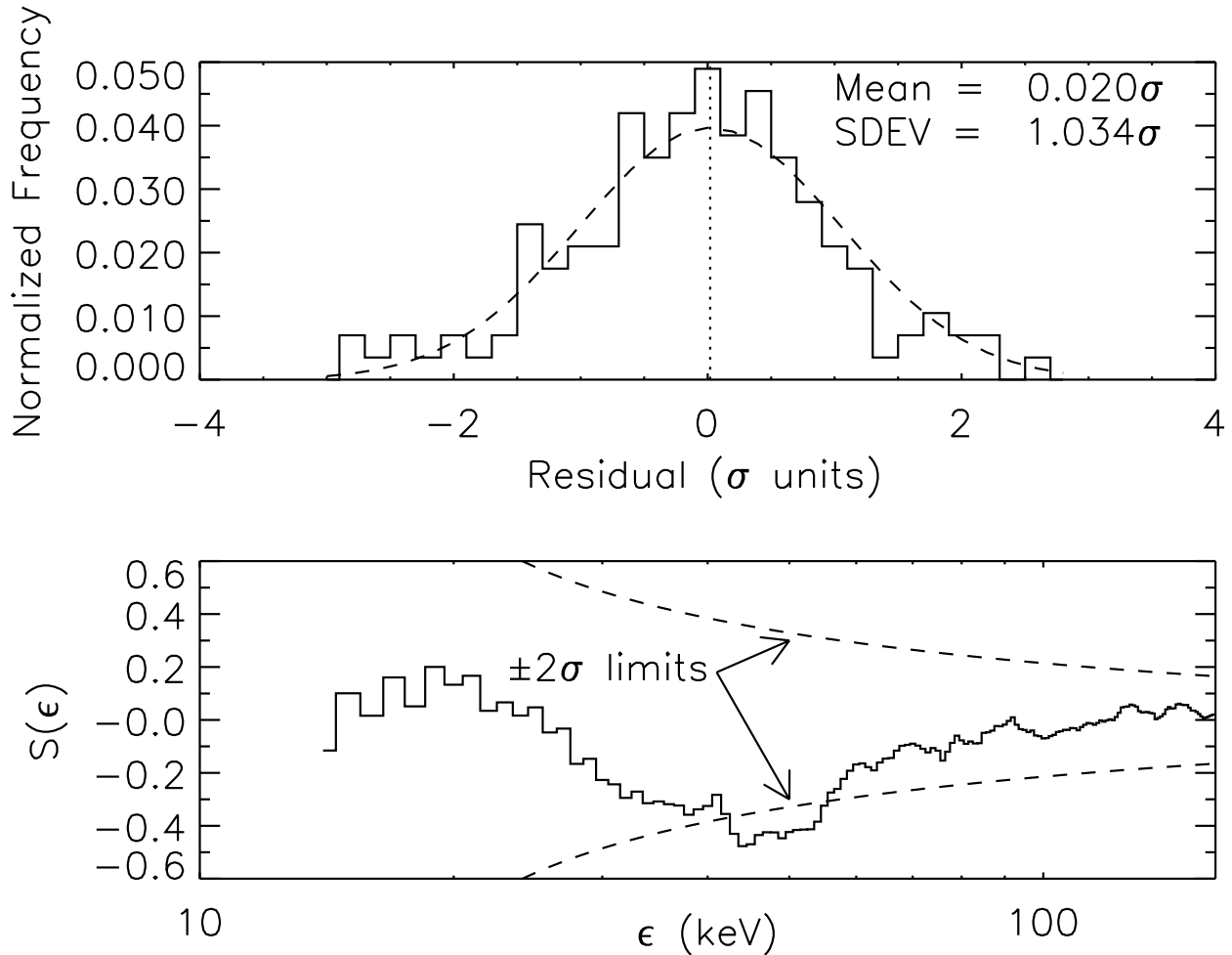


Fig. 3.— *Top panel:* Distribution function for the normalized residuals of Figure 2. A Gaussian fit, with the mean and standard deviation values indicated, has been superimposed. *Bottom panel:*  $S(\epsilon)$  (the average residual over the energy range from 15 keV to  $\epsilon$ ; see Equation [5]) as a function of  $\epsilon$ . The  $\pm 2\sigma$  limits for a random statistical process are shown for comparison.

over an isothermal form in the  $\sim 30 - 45$  keV range, and the fact that the  $\bar{F}(E)$  spectrum is *increasing* over the energy interval from  $\sim 55$  keV to  $\sim 60$  keV, are all significant features at up to the  $5\sigma$  level.

## 5. Discussion

In this *Letter* we have shown that the regularized Singular Value Decomposition inversion technique (Tikhonov 1963; Piana 1994) can be used rather productively to derive effective mean electron flux spectra ( $\bar{F}[E]$ ; Brown, Emslie & Kontar 2003) in solar flares. The spectra thus derived reveal features not accessible with the simple parametric forms usually associated with forward-fitting methods.

The feature in  $\bar{F}(E)$  around 55 keV must indicate an energy characterizing the physics either of the acceleration process or of electron propagation/energy losses. It could indicate the presence of a lower limit to the accelerated electron spectrum, with the shallowing of the  $\bar{F}(E)$  spectral slope between  $\sim 30 - 50$  keV due to the presence of thermal plasma from a source with a broad temperature distribution (see, e.g., Brown 1974; Brown & Emslie 1988), with a steadily declining emission measure over a range of temperature consistent with the range of electron energies involved, viz.  $10 \text{ keV} \lesssim kT \lesssim 50 \text{ keV}$ , much larger than the dominant purely isothermal component at  $kT \simeq 3 \text{ keV}$  found by Holman et al. (2003).

If the acceleration were purely stochastic in character one would expect a scale-free acceleration spectrum – i.e., a pure power law. Attributing the feature in  $\bar{F}(E)$  to the accelerator would thus require either that the stochastic process have a threshold energy or that acceleration be by large scale electric fields with a minimum associated potential drop. As far as propagation effects are concerned, it should be remarked that the energy of this feature is very close to the value of the quantity  $E_*$  found by Kontar et al. (2003) in their forward-fitting analysis of the hard X-ray photon spectrum for this event. In that analysis,  $E_*$  represents the energy required to penetrate to the weakly ionized chromospheric layers of the flare, and it is therefore not unexpected that the  $\bar{F}(E)$  spectrum would show interesting features in the vicinity of this energy (Kontar, Brown, & McArthur 2002; Brown, Emslie, & Kontar 2003).

Such features, and their significance for electron acceleration and propagation in solar flares, will be addressed in future work. The key point for now is that, unlike forward-fitting techniques that utilize only a few parameters (e.g., power-law spectral index), the regularized inversion technique utilized here is capable of revealing such interesting features and their evolution with time. In future papers the sensitivity of such features to the exact form of

the bremsstrahlung cross-section used, and the evolution of such features throughout this (and other) events will be discussed and evaluated.

This work was supported by NASA's Office of Space Science, by PPARC RHESSI Mission and Visitor Grants, and by a Collaboration Grant from the Royal Society. We thank Hugh Hudson and Bob Lin for their critical commentary throughout the evolution of this work and the entire RHESSI team for the hardware and software development that made this analysis possible.

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