

**Abstract.** Various methods are explored for obtaining regularised solutions of the severely ill-posed Laplace inversion problem involved in deriving plasma temperature ( $T$ ) structure (differential emission measure  $\xi(T)$ ) from bremsstrahlung spectra. Inversions of simulated data show that zero-order regularisation (Tikhonov regularisation in  $L^2$  space) is very unsatisfactory even with weighting, while first-order regularisation (Tikhonov regularisation in Sobolev space) yields reasonable results.

The method is applied to a high resolution hard X-ray flare spectrum observed by Lin et al. and yields a positive solution for  $\xi(T)$  showing that a purely thermal interpretation is possible for that event. The form of  $\xi(T)$  found has two broad features: one peaking at around  $10^7$  K and falling off steeply toward  $2 \times 10^8$  K; a second spread around a peak near  $4.5 \times 10^8$  K. The interpretation of such  $\xi(T)$  in terms of plasma heating and conductive flux is discussed in terms of plasma heat fluxes and heating rates. For 1-D geometry, the distribution of the plasma heating rate  $H(T)$  per unit volume is inferred from  $\xi(T)$  in the limits of classical diffusive conduction and of saturated heat flux, the former being relevant at  $T$  below around  $5 \times 10^7$  K and the latter at much higher  $T$ . We find there exists a maximum in  $H(T)$  around  $2 \times 10^8$  K, a fact which may be important for energy release theories.

**Key words:** flares: solar hard X-rays - regularisation: Laplace.

# 1. Introduction

Lin and Schwartz (1987) have presented Ge detector high resolution spectra of a solar flare allowing, for the first time, genuine deconvolution to yield the (collisional bremsstrahlung) source electron distribution as first proposed by Brown (1971). Since the source is spatially unresolved, the only property of the electron distribution which can be inferred from the photon spectrum, in a model-independent way, is the density-weighted volumetric mean electron flux energy spectrum (Brown, 1971)

$$\bar{F}(E) = \frac{1}{\bar{n}_p V} \int_V n_p(\mathbf{r}) F(E, \mathbf{r}) d\mathbf{r}, \quad (1)$$

where  $n_p(\mathbf{r})$ ,  $\bar{n}_p$  are the local and volumetric mean proton density,  $F(E, \mathbf{r})$  is the local electron flux spectrum, and  $V$  the source volume. The derivation of  $\bar{F}(E)$  from the photon data has been discussed in detail by Thompson et al. (1992), Johns and Lin (1992) and Piana (1994).

Further interpretation of  $\bar{F}(E)$  requires model dependent assumptions. For example, Johns and Lin (1992) consider a thick target model (Brown, 1971) interpretation after subtraction from the raw spectrum of a thermal component of moderate temperature, the existence of which is assumed on the basis of gradual behaviour in the light curve. Such derivation of the thick-target electron injection spectrum essentially involves a further deconvolution (differentiation) and is at the limits of the data noise. A second and physically quite different, interpretation of hard X-ray bursts (Chubb, 1970; Brown, 1974) is that the electron spectrum is locally Maxwellian but with the combined distribution of plasma density  $n_p(\mathbf{r})$  and temperature  $T(\mathbf{r})$  (in fact

the emission measure  $\xi(T)$  differential in  $T$ ) governing the photon spectrum. General conditions on the photon spectrum required for such a thermal interpretation to be physically possible have been discussed by Brown and Emslie (1988), including the relation between  $\xi(T)$  and  $\overline{F}(E)$ , and applied to the Lin and Schwartz (1987) data by Emslie, Coffey and Schwartz (1989).

In the present paper we consider the purely thermal interpretation of the Lin and Schwartz data as a deconvolution problem, i.e. we determine the  $\xi(T)$  required to produce the observed photon spectrum. In doing so we examine only one representative sample of the time series of spectra obtained, but we note that further physical information on the plasma heating and cooling processes is contained in the temporal evolution of the spectrum (Brown, 1974). The photon number flux energy spectrum from a non-isothermal plasma at photon energy  $\epsilon$  is

$$g(\epsilon) = \frac{A}{\epsilon} \int_0^\infty \frac{\xi(T)}{(kT)^{\frac{1}{2}}} \exp\left(-\frac{\epsilon}{kT}\right) dT, \quad (2)$$

where (Craig and Brown, 1986)

$$\xi(T) = \int_{S_T} \frac{n^2(\mathbf{r})}{|\nabla T|} dS_T \quad (3)$$

with  $S_T$  a constant temperature surface.

Equation (2) is an integral equation for  $\xi(T)$  given  $g(\epsilon)$ . Before undertaking its solution, it may be useful to clarify its relation to the determination of  $\overline{F}(E)$ , which has been a source of some confusion. Inversion of the Volterra integral equation relating  $g(\epsilon)$  to  $\overline{F}(E)$  will yield an  $\overline{F}(E)$  which applies to any model, including the thermal one (Brown and Emslie, 1988). However,

to determine the thermal model  $\xi(T)$  from  $\overline{F}(E)$  requires solution of a further integral equation to find what distribution of Maxwellians corresponds to this  $\overline{F}(E)$ . This second 'deconvolution' is of Fredholm type and its solution is much more unstable to noise in  $g(\epsilon)$  than deconvolution of  $\overline{F}(E)$ . Physically this is because the Maxwellian is very wide and smooth, with all temperatures in  $(0, \infty)$  making a contribution to every  $E$  in  $\overline{F}(E)$ , so that information on  $\xi(T)$  is heavily broad-band filtered in  $\overline{F}(E)$ . This property is reflected in the fact that Equation (2) can be transformed into a Laplace transform whose ill-posedness is well known (Bertero, Brianzi and Pike, 1985) in inverse problem circles.

Transposition to Laplace form is achieved by the change of variable  $y = \frac{1}{kT}$  and by the definition

$$f(y) = A \frac{\xi(\frac{1}{ky})}{ky^{\frac{3}{2}}} \quad (4)$$

to give

$$\frac{1}{\epsilon} \int_0^{\infty} f(y) \exp(-\epsilon y) dy = g(\epsilon) \quad (5)$$

which has to be considered in discretised form in  $\epsilon$  to allow for the  $N$  discrete finite bins in which the spectrum is recorded; in practice one should really integrate (5) over  $\epsilon$  in each bin but the bins are sufficiently narrow in the Lin and Schwartz data for a 'discrete point' representation to be adequate. This is:

$$g(\epsilon_n) = \frac{1}{\epsilon_n} \int_0^{\infty} f(y) \exp(-\epsilon_n y) dy \quad n = 1, \dots, N. \quad (6)$$

The severe ill-posedness of this Laplace inversion problem requires the use of regularisation methods to avoid the solution being swamped by amplified noise. In essence, we apply a smoothness assumption to suppress this noise,

thereby adding 'prior information' regarding the solution  $f(y)$  (Thompson et al., 1992). For such severely ill-posed problems, the choice of function space in which the solution is reconstructed and smoothed (i.e. the form of the a priori information) can be very important. To illustrate this point we consider the Tikhonov inversion technique first supposing that the solution is in a space of square integrable functions ( $L^2$  or weighted  $L^2$  space) before proceeding to describe the regularisation in a Sobolev space where a higher order smoothness assumption is used. Finally, we will make the inversion in these different function spaces first using simulated data and then the Lin and Schwartz data and show that Sobolev regularisation is much more satisfactory.

## 2. Tikhonov Regularisation in Spaces of Square Integrable Functions

The functional equation (6) can be written in operator form by introducing the map  $L, X \rightarrow Y$ , such that

$$(Lf)_n = \frac{1}{\epsilon_n} \int_0^\infty f(y) \exp(-\epsilon_n y) dy \quad n = 1, \dots, N. \quad (7)$$

In this way the linear inverse problem with discrete data becomes

$$\mathbf{g} = Lf \quad (8)$$

with  $(\mathbf{g})_n = g(\epsilon_n)$ . We assume that  $Y$  is a euclidean space (see Appendix A), while a first, extremely natural choice for the source space  $X$  is to suppose that it is the  $L^2(0, \infty)$  space of the functions such that  $\int_0^\infty |f(y)|^2 dy < \infty$ .

In this case it is possible to introduce the functions

$$\phi_n = \frac{1}{\epsilon_n} \exp(-\epsilon_n y) \quad n = 1, \dots, N \quad (9)$$

and so Equation (7) becomes

$$(Lf)_n = (f, \phi_n) \quad n = 1, \dots, N, \quad (10)$$

where with  $(f, \phi_n)$  one indicates the scalar product in  $L^2(0, \infty)$  (see Appendix B).  $L$  is a finite rank operator (that is, its range is contained in a finite dimensional vector space) and so its singular system can be introduced; it is the set of triples  $\{\sigma_n; v_n, \mathbf{u}_n\}_{n=1}^N$  such that

$$Lv_n = \sigma_n \mathbf{u}_n \quad ; \quad L^* \mathbf{u}_n = \sigma_n v_n \quad n = 1, \dots, N, \quad (11)$$

where  $L^*$  is the adjoint of  $L$  (Dunford and Schwartz, 1958). One can show (Bertero, De Mol and Pike, 1985) that the singular values  $\sigma_n$  and the singular vectors  $\mathbf{u}_n$  are, respectively, the square roots of the eigenvalues and the eigenvectors of the matrix representative of the operator  $LL^*$ . This matrix can be decomposed in the form

$$LL^* = G^T W, \quad (12)$$

where  $W$  is the weight matrix (see Appendix A) and  $G$  is the Gram matrix whose elements are:

$$G_{nm} = (\phi_n, \phi_m) \quad n, m = 1, \dots, N. \quad (13)$$

By using Equation (9) the elements of the Gram matrix can be analytically computed (see Appendix B). The knowledge of the singular system of the

operator  $L$  allows to obtain an estimate of the numerical instability of the problem, as the condition number  $C(L)$  is given by (Bertero, De Mol and Pike, 1985)

$$C(L) = \frac{\sigma_1}{\sigma_N} \quad (14)$$

with  $\sigma_1$  and  $\sigma_N$  respectively the largest and the smallest singular value. In general, the inversion of the Laplace transform is an extremely ill-conditioned problem: for example, under the conditions  $N = 20$ ,  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV, one obtains  $C \approx 10^9$ ; in the case of the Volterra problem studied in Piana (1994), for the same number of sampled points and in the same photon energy range, the order of magnitude of the condition number was  $C \approx 10^3$ . This confirms that one can obtain much less information from  $g(\epsilon)$  about  $\xi(T)$  than about  $\bar{F}(E)$ .

In order to find a stable estimate of the solution, Tikhonov regularisation technique can be applied (Tikhonov, 1963). It can be easily shown (Bertero, De Mol and Pike, 1988) that the regularised solution provided by this algorithm can be expressed in terms of the elements of the singular system through the formula

$$f_\lambda = \sum_{n=1}^N \frac{\sigma_n}{\sigma_n^2 + \lambda} (\mathbf{g}, \mathbf{u}_n) v_n, \quad (15)$$

where  $\lambda$  is the regularisation parameter.

If the function which must be recovered is not in  $L^2(0, \infty)$ , a first possibility is to truncate its support (that is, the interval in which the function is different from zero) at  $y = a$ , with  $a$  sufficiently large. Nevertheless, this approach may introduce some disadvantages: in particular, if one supposes that  $f$  is different from zero in a bounded interval, then the singular functions

are large at the edges of this interval. It is possible to avoid such limitation on the support of the solution, by making the hypotheses that it satisfies (Bertero, De Mol and Pike, 1986)

$$\int_0^\infty w^2(y)|f(y)|^2 dy < \infty, \quad (16)$$

where  $w(y)$  is an opportune weight function whose analytic shape depends on the characteristics of the function which has to be reconstructed. For example, in the case of the reconstruction of

$$f(y) = y, \quad (17)$$

a good choice for the weight function  $w(y)$  is

$$w(y) = \frac{1}{\sqrt{1+y^4}}. \quad (18)$$

If the weighted  $L^2(0, \infty)$  space, characterised by this weight function, is chosen as source space for the inverse problem (8), then the Gram matrix associated to the operator  $L$  defined in (7) can be analytically computed and the result is shown in Appendix B.

### 3. Tikhonov Regularisation in a Sobolev Space

From a general point of view, the inversion of the operator  $L$  can be made easier by inclusion of any a priori information on the solution in the regularising algorithm. An example of this fact is represented by reconstruction of functions which are assumed to be zero at the origin, by Tikhonov method. In this case one can choose, as the source space, the Sobolev space  $H^1(0, a)$



(Natterer, 1986), endowed with the scalar product defined in Appendix B and with the prescription that its functions satisfy the condition

$$f(0) = 0. \quad (19)$$

In this case, the action of the operator  $L$  described in terms of a scalar product as in Equation (10), and the use of integration by parts lead to the following boundary value problem for the functions  $\phi_n$ :

$$\begin{cases} \phi_n''(y) = -\frac{1}{\epsilon_n} \exp(-\epsilon_n y), \\ \phi_n'(a) = 0, \\ \phi_n(0) = 0. \end{cases} \quad (20)$$

Solution of this problem provides the analytic form of the functions  $\phi_n$ :

$$\phi_n(y) = \frac{1}{\epsilon_n^3} - \frac{1}{\epsilon_n^3} \exp(-\epsilon_n y) - \frac{y}{\epsilon_n^2} \exp(-\epsilon_n a) \quad n = 1, \dots, N. \quad (21)$$

The explicit form of the elements of the Gram matrix are described again in Appendix B.

## 4. Numerical Results: Inversion of Simulated Data

First of all, we have applied Tikhonov theory to the inversion of Equation (6) in different function spaces, in the case of simulated data. A first example is given by the function

$$f(y) = y \exp(-\beta_0 y) \quad (22)$$

with  $\beta_0$  a constant. This function corresponds to the differential emission measure (Brown and Emslie, 1988)

$$\xi(T) \sim \frac{k}{(kT)^{\frac{5}{2}}} \exp\left(-\frac{\beta_0}{kT}\right), \quad (23)$$

where  $k$  is the Boltzmann constant and all the other multiplicative constants have been put equal to unity. The data vector is obtained by introducing Equation (22) into Equation (7) and then, by affecting the components with gaussian noise of zero mean and standard deviation equal to  $2\eta(Lf)_n$ , where  $\eta$  is the relative error on the data; the simulated noisy data vector  $\mathbf{g}$  is then inserted into Equation (15) in order to obtain the regularised reconstruction of (22). It must be noted that it is not particularly informative to provide a reconstruction of the function  $f$  for only one noise realisation. Rather, the way in which the algorithm propagates the error from the data to the solution is better shown by applying Tikhonov regularisation to several random realisation of the simulated data. The result of this procedure is represented by a 'confidence strip' such as that plotted in Figure 1, where the solid line is the theoretical function (22). We used  $N = 20$  points in the energy interval  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV with  $\eta = 0.02$  and  $\beta_0 = 1$ . The optimal value of the regularisation parameter ( $\lambda = 4.7 \times 10^{-4}$ ) has been obtained through the comparison between  $f_\lambda$  and  $f$ . The value of  $\lambda$  determines which of the singular functions effectively contribute to the regularised reconstruction through Equation (15) since, from a certain value of the index  $n$ , the coefficients  $\frac{\sigma_n}{\sigma_n^2 + \lambda}(\mathbf{g}, \mathbf{u}_n)$  in it become extremely small. More precisely, the sum (15) can be truncated at the value  $n = M$  so that the relative variation between the corresponding  $f_\lambda^M$ ,

$$f_\lambda^M = \sum_{n=1}^M \frac{\sigma_n}{\sigma_n^2 + \lambda}(\mathbf{g}, \mathbf{u}_n)v_n, \quad (24)$$

and the complete sum in which  $n$  runs up to  $N$  (Equation (15)) is smaller than  $\eta$ , i.e. than the relative error on the data. In the case of the recon-

struction of function (22), this happens at  $M = 4$ . On the other hand, the singular function of order  $n$  is characterised by  $n - 1$  zeroes in  $(0, \infty)$  (Bertero and Grünbaum, 1985) and so the reconstructed solution does not contain details (i.e. is unresolved) in the intervals whose amplitudes coincide with the distances between two adjacent zeroes of the last significant singular function. The estimate of the 'temperature'  $y$ -resolution obtained in this way is represented by the horizontal resolution bars plotted in Figure 1, over the 'confidence strip', for the two values of  $y$  representing the geometrical means between the three zeroes of the fourth singular function; the vertical error bars are the upper and lower bounds of the 'confidence strip' at these points.

Another form of the differential emission measure used in Brown and Emslie (1988), is

$$\xi(T) \sim \frac{k}{(kT)^{\frac{5}{2}}} \quad (25)$$

so that  $f(y)$  is given by (17). Obviously, this function is not in  $L^2(0, \infty)$ , and so, as already explained in Section 2, a first approach is to replace it by

$$f(y) = \begin{cases} y & y < a, \\ 0 & y \geq a, \end{cases} \quad (26)$$

with  $a$  sufficiently large. Tikhonov regularisation has been applied to function (26) in the case  $N = 20$ ,  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV,  $\eta = 0.02$ ,  $a = 20$  keV<sup>-1</sup>. The result is represented in Figure 2a (here again  $M = 4$ ). As one can see, for these data (but also in the case of Figure 1, even if less seriously), the error and resolution bars (plotted again in correspondence with the geometrical means between the zeroes of the last significant singular function) do not include the true solution and, in general, the recovery is very bad. The reason

resides in the fact that the Laplace inversion is a very ill-posed problem and so the filtering provided by the zero-order regularisation (that is, the Tikhonov regularisation in  $L^2(0, \infty)$ ) is too rough for a good recover of the true solution.

The reconstruction of function (17) gets better if the inversion is made in a weighted  $L^2(0, \infty)$  with  $w(y)$  given by Equation (18). Figure 2b shows that the improvement obtained is evident particularly in the 'confidence strip', whose oscillations around the theoretical solution are narrower than in the reconstruction of Figure 2a. The improvement in the resolution is not equally significant, even if, in this case, five and not four singular functions contribute to the regularised solution.

Finally, Figure 2c shows the regularised reconstruction of the function (17) when the inversion is made in the Sobolev space previously described; here again  $N = 20$ ,  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV,  $\eta = 0.02$ ,  $a = 20$  keV<sup>-1</sup>. In this case,  $M = 8$  and the value of the regularisation parameter has again been chosen through the comparison with the theoretical function. The improvement compared with the reconstructions of Figures 2a and 2b is evident both in the better accuracy with which the regularised solution fits the theoretical function and in the improved resolution. We used the regularised inversion in the Sobolev space also in the case of the function

$$f(y) = y^2. \tag{27}$$

The result is represented in Figure 3, again with 8 significant singular functions.

## 5. Inversion of the Real Data

Our application to real data was for the photon spectrum presented in Figure 4. This is an X-ray spectrum emitted by the solar flare of 27 June 1980 (Lin and Schwartz, 1987), characterised by  $N = 37$  points logarithmically sampled between  $\epsilon_1 = 14.09$  keV and  $\epsilon_N = 286.99$  keV. These energies are in a keV range about ten times higher than used in the simulations but since the mathematical properties of the problem depend essentially only on  $\frac{\epsilon}{T}$ , this just means that the equivalent solution support range is at ten times higher temperatures. In recovering the differential emission measure, we have eliminated the first two anomalous points and the last eight ones, which are strongly affected by noise and background. So the vector  $\mathbf{g}$  we have used is composed of  $N = 27$  points in the photon energy range  $\epsilon_1 = 16.62$  keV,  $\epsilon_N = 146.99$  keV. If the function  $f$  which produces this data vector belongs to  $L^2(0, \infty)$ , according to the Tikhonov technique it can be approximated through Equation (15), in which the singular system is obtained by diagonalising the Gram matrix given in (13). For real data, the optimal smoothing parameter of course cannot be selected by comparison with the unknown true solution. In order to determine in a sufficiently objective way the value of the regularisation parameter  $\lambda$ , we applied Morozov's Discrepancy Principle (Davies, 1992), according to which the 'best  $\lambda$ ' is that one which solves the equation

$$\|Lf_\lambda - \mathbf{g}\|_Y = \delta, \quad (28)$$

where the norm is the one introduced in the euclidean data space and  $\delta$  is the RMS error on the data. Nevertheless, we found that the regularised solution

corresponding to this value is characterised by a strong instability and by several negative components and so it must be considered unphysical. Moreover it is important to note that the use of other criteria for the choice of  $\lambda$ , such as, for example, the Generalised Cross Validation (Craven and Wahba, 1979; Golub, Heath and Wahba, 1979), does not provide any improvement, for the Discrepancy Principle typically has oversmoothing properties. The presence of strong numerical instability remains even if the inversion of Equation (6) is made in the weighted  $L^2(0, \infty)$  space (where the weight function  $w(y)$  is given by Equation (18)) and  $\lambda$  is again chosen according to the Discrepancy Principle.

The results are completely different if the Sobolev space described in Section 3 is assumed as source space. Yet, in order to apply Tikhonov regularisation in this space, it is necessary, first of all, to verify that condition (19) is compatible with the real data. To this aim, there are some 'abelian theorems' (Widder, 1946) which, from the knowledge of the asymptotic behaviour of a function  $F(t)$  (for  $t \rightarrow 0$  or for  $t \rightarrow \infty$ ) permit us to deduce the asymptotic behaviour of its Laplace transform  $(\mathcal{L}F)(s)$  (for  $s \rightarrow s_0$  with an opportune  $s_0$  or for  $s \rightarrow \infty$ ). In particular, one of these theorems says that if  $F(t)$ , for  $t \rightarrow 0$ , is

$$F(t) \sim At^\alpha \tag{29}$$

with  $\alpha > -1$ , then  $(\mathcal{L}F)(s)$ , for  $t \rightarrow \infty$ , is

$$(\mathcal{L}F)(s) \sim A \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \tag{30}$$

where  $\Gamma$  is defined as

$$\Gamma(z) = \int_0^{\infty} \exp(-t)t^{z-1}dt. \quad (31)$$

By applying this theorem to the present case, one finds that condition (19) on the function  $f$  implies that the spectrum  $g(\epsilon)$  must have an asymptotic behaviour (for large values of the photon energy) like

$$g(\epsilon) \sim \frac{1}{\epsilon^{\beta}} \quad (32)$$

with  $\beta > 2$ ; by comparing the real spectrum with the function  $g(\epsilon) = \frac{1}{\epsilon^2}$  one shows that this behaviour is verified (in the region containing the data points) and so condition (19) is compatible with this spectrum.

The inversion of the real data by this method provides the result plotted in Figure 5, where we have directly represented the 'confidence strip' for the differential emission measure  $\xi(T)$  itself, linked to  $f$  through the definition (4); the error and resolution bars are plotted over the 'confidence strip' (with six significant singular functions). As one can see, the value of the regularisation parameter  $\lambda = 6.7 \times 10^{-10}$ , chosen with the Discrepancy Principle, gives a Sobolev regularised solution which is smooth, positive and so physically meaningful.

## 6. Discussion and Physical Interpretation

We have explored a variety of methods of obtaining optimal regularised solutions to the Laplace inversion problem of deriving the plasma emission measure function  $\xi(T)$ , differential in  $T$ , required to produce a bremsstrahlung

spectrum and found that Tikhonov regularisation in Sobolev space yields the best results. Application of the method to real solar flare hard X-ray data yields a solution and 'confidence strip' which is positive everywhere. This confirms the conclusion of Emslie, Coffey and Schwartz (1989), based on the spectrum derivative criteria of Brown and Emslie (1988), that the event could be described purely in terms of thermal bremsstrahlung for a physically reasonable  $\xi(T)$  (i.e.  $\xi(T) \geq 0$ ), and does not demand a model with a non-thermal component (Johns and Lin, 1992) though certainly not excluding it. Because of the severe ill-posedness of the Laplace Fredholm inversion, the detailed form of  $\xi(T)$  is not so well resolved as the source mean electron flux spectrum (Thompson et al., 1992; Johns and Lin, 1992; Piana, 1994) but it nevertheless carries important physical information. Noting the large dynamic range in the (logarithmic) plot of  $\xi(T)$  in Figure 5, we see that, in terms of emission measure, the dominant feature of the hard X-ray source plasma in a purely thermal interpretation is a rather sharp peak just above  $T = 10^7$  K with  $\xi(T)$  declining rapidly at higher  $T$  (by a factor of 100 between  $10^8$  K and  $10^9$  K). That is, to fit the Lin and Schwartz (1987) high resolution spectrum purely thermally, we require addition of only a small emission measure of very hot plasma to the large emission measure of hot plasma around  $10^7$  K already known to exist from soft X-ray line measurements. This extra plasma with steeply decreasing  $\xi(T)$  in  $10^7 \text{ K} \leq T \leq 10^8 \text{ K}$  can be loosely identified with the 'superhot' component suggested by Lin and Schwartz (1987). The approximately (steep) power law form of  $\xi(T)$  over this temperature range, evident in Figure 5, confirms Brown's (1974) argument,



based on the Laplace Transform argument and low resolution data, that an allegedly 'nonthermal' power law hard X-ray spectrum could be mimicked by a power law  $\xi(T)$  distribution.

More interesting, however, is the inferred behaviour of  $\xi(T)$  at larger  $T$ . Though small in relative term, the upturn, or at least inflection, in  $\xi(T)$  at around  $10^8$  K and its form above that are very interesting physically. This region, which we might term the 'ultrahot' component, is much flatter and wider in  $T$  than the 'superhot' component, falling by a factor  $\leq 10$  over a decade in temperature. At such very high temperatures, of course, it is necessary to be rather careful over the notion of the source electrons being 'thermal' - i.e. locally Maxwellian - since, depending on the geometry and the density involved in  $\xi(T)$ , the collisional mean free path may very well be larger than the source size (temperature scale length). This, however, does not preclude a thermal interpretation, even at the highest temperatures, if wave particle interactions are present which reduce the mean free path and 'Maxwellianise' the particles (Brown, 1974). This point is discussed further below. We also emphasise, however, that the feature which appears in our  $\xi(T)$  above  $10^8$  K for a purely thermal model is not a unique interpretation of the spectral data. It could instead be, in whole or in part, nonthermal in origin, such as due to electron runaway as suggested by Lin and Schwartz (1987).

If we adhere to the purely thermal model, which is consistent with the data, it is interesting to consider what can be said quantitatively about the source plasma from the inferred  $\xi(T)$ . As emphasised by Craig and Brown

(1976),  $\xi(T)$  defined by Equation (3) is a density weighted spatial average of the inverse temperature gradient over the 2-D isothermal surface, which may be multiply connected. It is therefore only possible to draw conclusions about  $T(\mathbf{r})$ , which is the property of physical interest, from  $\xi(T)$  by making strong assumptions about the source geometry and density distribution. The most simplifying possible assumptions are that the geometry is 1-D and that the density or the pressure is spatially uniform, with the latter more likely for a very hot source in a geometrically small volume because of the small thermal particle crossing time. These assumptions are commonly made in modelling the thin quiet sun transition region from  $\xi(T)$  derived from collisionally excited XUV atomic line data (Gabriel, 1992). In the ultrahot highly magnetised transient plasma of flare energy release, possibly in complex tearing mode island geometry (Spicer, 1975), they are more suspect but about the best we can do at present, and what we use in the analysis below. We note, however, that further light should be shed on the plasma structure by utilising the time evolution of  $\xi(T)$  which we do not consider here (consider, for example, Lin and Schwartz (1987) and Johns and Lin (1992) removal of an alleged 'gradual' component from the spectrum).

For a 1-D structure of cross sectional area  $A$  at constant pressure ( $n(z)T(z) = N_o T_o$ ) we have, from Equation (3) (Gabriel, 1992)

$$\left| \frac{dT}{dz} \right| = \frac{A(n_o T_o)^2}{T^2 \xi(T)}. \quad (33)$$

So, if we make the further assumption that  $T(z)$  is monotonic (say decreasing with increasing  $z$  from the point  $z = 0$  where  $n = n_o, T = T_o$ ) we can infer the shape of  $\frac{dT}{dz}$  as a function of  $T$ , within a scale factor  $A(n_o T_o)^2$  from Equation

(33) and integrate it numerically to get  $z(T)$ . One of the most important things about  $\frac{dT}{dz}$ , however, is that it is an indicator of the heat flux in the source and hence of plasma heating processes. At the high temperatures involved here, heat fluxes are large (even for anomalous conductivity) and we will assume for simplicity that conductive cooling completely dominates bremsstrahlung radiative cooling (which in any case decreases with increasing  $T$  in constant pressure conditions). Then the conductive heat flux term and the plasma heating function alone theoretically determine  $T(z)$  (as a function of time though here we will make the final simplifying assumption of a quasi-steady state  $T(z)$ ). So we may hope ultimately to infer properties of the plasma heating process from  $\xi(T)$ . Note especially from Equation (33) that  $\frac{dT}{dz}$  is inversely related to  $\xi(T)$  so that regions where  $T^2\xi(T)$  is smallest are those where  $\frac{dT}{dz}$  is largest. Thus small emission measure  $\frac{dT}{dz}$  does not necessarily imply lack of importance in terms of energy transport. At the 'superhot' end of the  $\xi(T)$  distribution, the temperature gradient is smallest and the density highest so that collisional diffusive conduction should well represent the heat flux  $F_c$  (Brown, 1974)

$$F_c(T) = -\kappa_o T^{5/2} \frac{dT}{dz} = A\kappa_o (n_o T_o)^2 \frac{T^{1/2}}{\xi(T)} \quad (34)$$

and the corresponding conductive volumetric cooling rate (that is the plasma heating rate  $H(T)$  for a quasisteady state) gives

$$H(T) = \frac{A^2 \kappa_o (n_o T_o)^4}{T^{5/2} \xi(T)} \left[ \frac{1}{2} - \frac{d \log \xi}{d \log T} \right]. \quad (35)$$

In Figure 6 and Figure 7 we show the scaled forms of  $F_c(T)$  and  $H(T)$  obtained from Equations (34) and (35) for the data based  $\xi(T)$  in Figure 5

for the entire range of  $T$ . Insofar as this is valid, the most interesting feature is the steep rise of  $F_c(T)$  and  $H(T)$  with  $T$  at higher temperatures, up to about  $2 \times 10^8$  K, where a maximum occurs.

As already noted, however, at the ultrahot end of the  $T$  range we would expect the diffusive conduction approximation to break down due to the low density and high  $\frac{dT}{dz}$  there. The heat flux then becomes saturated and possibly anomalous (Brown, Melrose and Spicer, 1979). It is beyond the scope of the present paper to discuss the possible anomalous forms of conductivity  $\kappa_{an}(T)$  but we illustrate how  $H(T)$  would change from Figure 7 in the case of saturated heat flux  $F_{sat}$

$$F_{sat}(T) \approx nv_e kT/6 \approx \left(\frac{k}{m_e}\right)^{\frac{1}{2}} nT^{\frac{1}{2}}, \quad (36)$$

again in the case of constant pressure. In contrast to Equation (34), the saturated heat flux (36) is not directly related to  $\xi(T)$  through  $\frac{dT}{dz}$  but the heating rate  $H = -\left(\frac{dF_c}{dT}\right)\left(\frac{dT}{dz}\right)$  is, namely:

$$H_{sat}(T) \approx \frac{A}{2} \left(\frac{k}{m_e}\right)^{\frac{1}{2}} \frac{(n_o T_o)^3}{T^{\frac{5}{2}} \xi(T)}. \quad (37)$$

In Figure 8 we show the results for  $H_{sat}(T)$  for the observed  $\xi(T)$  again over the entire range of  $T$ . A very important feature of both Figure 7 and Figure 8 is that, because  $\xi(T)$  is rather flat there, they exhibit a peak in  $H(T)$  at  $T \geq 2.5 \times 10^8$  K.

Neither Equation (35) nor Equation (37) (Figure 7 or Figure 8) will in fact be valid over the whole  $T$  range and we should really carry out a general consistent heat flux treatment over the whole range, allowing for decrease in

the density and increase in the ratio of mean free path to  $\frac{T}{(\frac{dT}{dz})}$  with increasing  $T$  (this will in general depend also on the assumed geometry). Here we simply emphasise the fact that in both limiting approximations the data on  $\xi(T)$  imply the existence of a maximum in plasma heating rate per unit volume (and therefore a fortiori per unit mass or per particle since the density is lowest there) at a temperature around  $2 \times 10^8$  K. This maximum should have important implications for primary flare energy release mechanisms.

**Acknowledgements.** It is a pleasure to thank R.P. Lin and C. Johns for providing the experimental data and an anonymous referee for very helpful comments.

This work has been partly supported by NATO Grant No. 930384, by an EEC Contract, by UK SERC/PPARC Grants and by Consorzio INFN.

## APPENDIX A

From a general point of view, a euclidean space  $Y$  is equipped with the scalar product

$$(\mathbf{g}, \mathbf{h}) = \sum_{n=1}^N g_n h_n w_n, \quad (\text{A.38})$$

where the form of the weights  $w_n$  depends on the kind of sampling. In the present case we have assumed the logarithmic sampling defined by

$$\epsilon_n = \epsilon_1 \Delta^{n-1} \quad n = 1, \dots, N \quad (\text{A.39})$$

and so the weights are (Bertero, Boccacci and Pike, 1984)

$$w_n = \log(\Delta) \epsilon_n \quad n = 1, \dots, N. \quad (\text{A.40})$$

Finally, it is possible to introduce the weight matrix  $W$ , whose entry  $(n,m)$  is

$$W_{nm} = w_n \delta_{nm} \quad n, m = 1, \dots, N. \quad (\text{A.41})$$

## APPENDIX B

The elements of the Gram matrix are given by Equation (13) where the scalar product is performed in the functional space which has been chosen as source space. In the case of the  $L^2(0, \infty)$  space, it is

$$(\phi_n, \phi_m) = \int_0^\infty \phi_n(y) \phi_m(y) dy. \quad (\text{B.1})$$

The Gram matrix associated with the operator  $L$  defined by Equation (7) is

$$G_{nm} = \frac{1}{\epsilon_n \epsilon_m} \frac{1}{\epsilon_n + \epsilon_m} \quad n, m = 1, \dots, N. \quad (\text{B.2})$$

In the case of a weighted  $L^2(0, \infty)$  space, the scalar product is given by

$$(\phi_n, \phi_m) = \int_0^\infty w^2(y) \phi_n(y) \phi_m(y) dy \quad (\text{B.3})$$

and the Gram matrix, for  $w(y)$  given by Equation (18), is

$$G_{nm} = \frac{1}{\epsilon_n \epsilon_m} \left[ \frac{1}{\epsilon_n + \epsilon_m} + \frac{24}{(\epsilon_n + \epsilon_m)^5} \right] \quad n, m = 1, \dots, N. \quad (\text{B.4})$$

Finally, in the case of the Sobolev space, the scalar product is defined as

$$(\phi_n, \phi_m) = \int_0^a \phi'_n(y) \phi'_m(y) dy \quad n, m = 1, \dots, N \quad (\text{B.5})$$

and the Gram matrix is given by

$$\begin{aligned} G_{nm} = & \frac{1}{(\epsilon_n \epsilon_m)^2} \left\{ -\frac{1}{\epsilon_n + \epsilon_m} \exp[-(\epsilon_n + \epsilon_m)a] + \right. \\ & + \frac{1}{\epsilon_n + \epsilon_m} + a \exp[-(\epsilon_n + \epsilon_m)a] + \\ & + \frac{1}{\epsilon_m} \exp[-(\epsilon_n + \epsilon_m)a] - \frac{1}{\epsilon_m} \exp(-\epsilon_n a) + \\ & \left. + \frac{1}{\epsilon_n} \exp[-(\epsilon_n + \epsilon_m)a] - \frac{1}{\epsilon_n} \exp(-\epsilon_m a) \right\}. \quad (\text{B.6}) \end{aligned}$$

## References

- Bertero, M. and Grünbaum, F.A.: 1985, *Inverse Problems* **1**, 181.
- Bertero, M., Boccacci, P. and Pike, E.R.: 1984, *Proc. R. Soc. Lond. A* **393**, 51.
- Bertero, M., Brianzi, P. and Pike, E.R.: 1985, *Proc. R. Soc. Lond. A* **398**, 23.
- Bertero, M., De Mol, C. and Pike, E.R.: 1985, *Inverse Problems* **1**, 301.
- Bertero, M., De Mol, C. and Pike, E.R.: 1986, *Inverse Problems* **2**, 247.
- Bertero, M., De Mol, C. and Pike, E.R.: 1988, *Inverse Problems* **4**, 573.
- Brown, J.C.: 1971, *Solar Phys.* **18**, 489.
- Brown, J.C.: 1974, in G.A. Newkirk (ed.), *Coronal Disturbances*, IAU Symp **57**, 395.
- Brown, J.C. and Emslie, A.G.: 1988, *Astrophys. J.* **331**, 554.
- Brown, J.C., Melrose, D.B. and Spicer, D.S.: 1979, *Astrophys. J.* **228**, 592.
- Chubb, T.A.: 1970, in E.R. Dyer (ed.), *Solar Terrestrial Physics*, Part 1, D. Reidel Publ. Co., Dordrecht.
- Craig, I.J.D. and Brown, J.C.: 1976, *Astron. Astrophys.* **49**, 239.
- Craig, I.J.D. and Brown, J.C.: 1986, *Inverse Problems in Astronomy*, Adam Hilger, Bristol.
- Craven, P. and Wahba, G.: 1979, *Num. Math* **31**, 377.
- Davies, A.M.: 1992, in M. Bertero and E.R. Pike (eds.), *Inverse Problems in Scattering and Imaging*, Adam Hilger, Bristol.
- Dunford, N. and Schwartz, J.T.: 1958, *Linear Operators*, Part 1, Interscience Publishers, New York.



- Emslie, A.G., Coffey, V. N. and Schwartz, R.A.: 1989, *Solar Phys.* **122**, 313.
- Gabriel, A.H.: 1992, in J.Schmelz and J.C. Brown (eds.), *The Sun: a Laboratory for Astrophysics*, NATO ASI, Kluwer, Dordrecht.
- Golub, G.H., Heath, M. and Wahba, G.: 1979, *Technometrics* **21**, 215.
- Johns, C.M. and Lin, R.P.: 1992, *Solar Phys.* **137**, 121.
- Lin, R.P. and Schwartz, R.A.: 1987, *Astrophys. J.* **312**, 462.
- Natterer, F.: 1986, *The mathematics of computerized tomography*, Wiley and Sons, New York.
- Piana, M: 1994, *Astron. Astrophys.* **288**, 949.
- Spicer, D.S.: 1975, *An Unstable Arch Model of a Solar Flare*, NRL Report, 8036.
- Thompson, A.M., Brown, J.C., Craig, I.J.D. and Fulber, C.: 1992, *Astron. Astrophys.* **265**, 278.
- Tikhonov, A.N.: 1963, *Sov. Math. Dokl.* **4**, 1035.
- Widder, D.V.: 1946, *The Laplace transform*, Princeton Univ. Press.

## Figure captions

Fig. 1. Confidence strip for the reconstruction of  $f(y) = y \exp(-\beta_0 y)$  (solid line) with  $\beta_0 = 1$ , in the  $L^2(0, \infty)$  space; the sampled points are  $N = 20$  in the range of photon energy  $\epsilon_1 = 1$  KeV,  $\epsilon_N = 10$  keV; the relative error on the simulated data is  $\eta = 0.02$ ; the value of the regularisation parameter is  $\lambda = 4.7 \times 10^{-4}$ . The resolution and error bars are plotted too; the number of significant singular functions is  $M = 4$ .

Fig. 2. Regularised reconstructions of  $f(y) = y$  (solid line) in different function spaces, with  $N = 20$ ,  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV,  $\eta = 0.02$ . (a) Regularisation in  $L^2(0, \infty)$ :  $\lambda = 1.9 \times 10^{-3}$ ,  $M = 4$ ; (b) regularisation in  $w-L^2(0, \infty)$  with the weight function  $w(y)$  given by Equation (18):  $\lambda = 3.9 \times 10^{-3}$ ,  $M = 5$ ; (c) regularisation in the Sobolev space  $H^1(0, a)$  with  $a = 20$  keV $^{-1}$ :  $\lambda = 2 \times 10^{-7}$ ,  $M = 8$ .

Fig. 3. Confidence strip for the reconstruction of  $f(y) = y^2$  (solid line) in the Sobolev space  $H^1(0, a)$ , with  $a = 20$  keV $^{-1}$ ,  $N = 20$ ,  $\epsilon_1 = 1$  keV,  $\epsilon_N = 10$  keV,  $\eta = 0.02$ ; here  $\lambda = 8 \times 10^{-8}$  and  $M = 8$ .

Fig. 4. Photon flux emitted by the solar flare of 27 June 1980; the data vector is composed by  $N = 37$  points in the photon energy range  $\epsilon_1 = 14.09$  keV,  $\epsilon_N = 286.99$  keV.

Fig. 5. Confidence strip representing the differential emission measure obtained through the Sobolev regularised inversion of the real data plotted in Figure 4; here only  $N = 27$  points of the data have been considered, between  $\epsilon_1 = 16.62$  keV and  $\epsilon_N = 146.99$  keV;  $\lambda = 6.7 \times 10^{-10}$  and  $M = 6$ .

Fig. 6. Heat flux for a 1-D geometric structure, in the case of the differential

emission measure obtained by the Sobolev regularised inversion of the real data of Figure 4.

Fig. 7. Distribution of the plasma heating rate per unit volume, for 1-D geometry, in the limits of classical diffusive conduction; the form of the differential emission measure here used is the same already used in Figure 6.

Fig. 8. Distribution of the plasma heating rate per unit volume, for 1-D geometry, in the limits of saturated heat flux.