On uniqueness for anisotropic inhomogeneous inverse scattering problems

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Abstract. The problem of determining the shape of a two-dimensional inhomogeneous orthotropic scatterer from far-field data is considered. In particular, by using integral equation techniques we prove that the support of the scatterer is uniquely determined if the far-field pattern is known for all incident directions. We point out that this uniqueness result may be useful also for practical applications since in previously treated cases (isotropic inhomogeneous objects in the case of transverse magnetic incident waves and homogeneous orthotropic objects in the case of transverse electric incident waves) the theorems proving uniqueness have been quite straightforward extended to formulate a simple method for solving the inverse scattering problem.

1. Introduction

We think that one of the main present goals of scattering theory is represented by the solution of the following problem:

Problem 1.1. *Given an anisotropic inhomogeneous object and an electromagnetic wave with fixed frequency scattering with it, provide a reconstruction of the index of refraction from measurements of the scattered field at large distances.*

This inverse problem is very difficult to address and the reasons are mainly two. First of all the problem is non linear and while a theory for the solution of linear inverse problems is well-known and efficient algorithms have been tested in many significant applications, the formulation of analogous theory and methods in the case of non linear problems is still far from a satisfactory realization. Typically scientists follow two different approaches. One possibility is to exploit physical conditions in order to linearize the problem and then use the inverse methods formulated in linear theory. Nevertheless there are very interesting applications where linearization is not possible and, on the contrary, the frequency of the incident field is explicitly in the resonance region. An important example is represented by the detection of leukemia by using microwaves [5]. Another possible approach is to preserve the non linear nature of the problem and to apply non linear optimization iterative schemes for the inversion. These algorithms may be efficient but, in particular for three-dimensional problems, they are extremely heavy from a computational point of view and furthermore they typically suffer local minima.

The second difficulty related to the solution of Problem 1.1 is due to its ill-posedness. It is well-known [8] that a problem is said well-posed in the sense of Hadamard if

- the solution exists;
- the solution is unique;
- the solution depends continuously on the data.

The problem is said ill-posed if at least one of these three statements does not hold. In the case of Problem 1.1, none of these three statements is true. In particular, the fact that the solution of the problem is not unique is pointed out in a rather spectacular way in Figure 1, Joe Coyle of the Department of Mathematical Sciences of the University of Delaware obtained by using a finite element method code. In this figure the total field scattered by two different objects contained in the two circles is represented when the incident field is taken to be a point source. The first object is anisotropic, while in the bottom one the refraction index is a multiple of the identity (i.e., the object is isotropic). As you can see, the total fields are obviously different inside the circles, since the dielectrics are different, but they coincide outside the circles.

The lack of uniqueness has important consequences in the solution of the inverse scattering problem. In fact, let us describe Problem 1.1 in the synthetic form

$$u^{(s)} = \mathcal{F}(n) \tag{1.1}$$

where $u^{(s)}$ is the scattered field, n is the index of refraction and $\mathcal{F}$ is the non linear operator describing the scattering. For noisy data the error components typically put $u^{(s)}$ out of the range of the operator $\mathcal{F}$ so that in general the solution of the problem never exists for real applications. Furthermore the presence of noise on the data should imply numerical instability. Therefore, in order to obtain stable and reliable approximate solutions of equation (1.1) regularization techniques are necessary. However there exists several regularization methods providing different approximate solutions; it follows that since the exact solution of the problem is not unique we are not guaranteed that these different methods give approximate solutions converging to the same exact solution.

In order to avoid all these difficulties, a recently developed strategy is to deal with a modified version of Problem 1.1. This idea is based on the observation that in many applications it is important to determine only the support of the scatterer (that is the region where the index of refraction is different from the constant matrix describing the index of refraction of the surrounding medium) instead of the values of the index of refraction inside it. For example, in medical imaging at a first analysis it suffices to know whether the tumor exists or not and not directly the characteristics of its tissue. In other terms, we are now interested in the following inverse scattering problem:

Problem 1.2. *Given an anisotropic and inhomogeneous object and an electromagnetic wave with fixed frequency scattering with it, provide a reconstruction of the shape of the object from measurements of the scattered field at large distances.*

We notice that this problem can be considered the more general inverse scattering problem, since it comprehends the particular case when the scatterer is a not penetrable obstacle. Also Problem 1.2 is non linear and ill-posed. However recent results [1, 7, 6, 10] have proven for many cases that this problem is in some sense equivalent to a linear integral equation of the first kind. This allows to formulate an inversion method, called *simple method* or RSM (*regularized sampling method*) which so far has been applied to several simulated data with encouraging results. We point out that this method is exact in the sense that it does not require any high- or low- frequency linearization.

The present paper is devoted to prove uniqueness of solution of Problem 1.2 for two-dimensional inhomogeneous orthotropic scatterers. We notice that particular results have been already obtained for two-dimensional geometries. For example, in [14] uniqueness is proven for an isotropic inhomogeneous object in the case of TM mode (an easier proof of this same result is given in [13]) and in [4] for a homogeneous orthotropic object in the case of TE mode. In section 4 of the present paper (Theorem 4.3) we will prove that if the far-field scattering data are known in each point for all incident directions then the shape of a two-dimensional inhomogeneous orthotropic scatterer is uniquely determined. The procedure we will follow may be useful also for the inversion, since in the previously treated cases the theorems proving uniqueness have been extended in a quite straightforward way to formulate the RSM.

Let us consider the case of a time-harmonic wave with wavenumber $k > 0$. The time-harmonic Maxwell equations outside the scatterer are

$$\nabla \times E_0 - ikH_0 = 0 \qquad \nabla \times H_0 + ikE_0 = 0 \tag{1.2}$$

while inside the scatterer we have

$$\nabla \times E - ikH = 0 \qquad \nabla \times H + iknE = 0 \quad . \tag{1.3}$$

n is the index of refraction. We assume that the dielectric is orthotropic, i.e.

$$n = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix} \tag{1.4}$$

where all the entries in general depend on x, a point in the scatterer D. On the boundary

∂D of D, the following continuity conditions for the tangential components of the fields hold:

$$\nu \times E_0 = \nu \times E \qquad\qquad \nu \times H_0 = \nu \times H \quad . \tag{1.5}$$

If the dielectric has a cylindrical simmetry the problem becomes two-dimensional. In the case of a transverse magnetic (TM) incident field (i.e., the electric field has only one component different from zero, in the direction of the axis of the cylinder) the boundary value problem describing the scattering involves only the entry n_{33} and it has been already discussed in [14, 13]. In the case of a transverse electric (TE) mode (i.e. when the magnetic field has only one component along the axis) it is not difficult to show that equations (1.2)-(1.5) lead to the transmission problem

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad\qquad x \in \mathbb{R}^2 \setminus \overline{D} \tag{1.6}$$

$$\nabla \cdot N \nabla u + k^2 u = 0 \qquad\qquad x \in D \tag{1.7}$$

$$u = u_0 \qquad\qquad x \in \partial D \tag{1.8}$$

$$\nu \cdot \nabla u_0 = \nu \cdot M \nabla u \qquad\qquad x \in \partial D \quad . \tag{1.9}$$

Here $H_0 = (0, 0, u_0)$ and $H = (0, 0, u)$. The 2x2 matrix N is defined by

$$N = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix} \tag{1.10}$$

and $N = M$ on the boundary of the scatterer, with M a constant matrix. For reasons which will be clear later, the following conditions on N and M are assumed:

- the entries of N are functions in $C^{1,\alpha}(\mathbb{R}^2)$;
- N is semi-coercive, i.e. $\mathrm{Im}(a \cdot \overline{Na}) \geq \gamma(x)|a|^2$ for every $a \in \mathbb{C}^2$ where $\gamma(x) \geq 0$;
- N is elliptic, i.e. $a \cdot Na \neq 0$ for all $a \neq 0$, $a \in \mathbb{R}^2$.
- M is a real constant symmetric positive definite matrix and $\det M^{1/2} \neq 1$.

In the second section of the present paper we will introduce the fundamental solution for the partial differential equation (1.7). This will allow to define the single- and double-layer potentials useful in the sequel. In the third section the well-posedness of the direct problem will be shown. In the fourth section we will prove the uniqueness of solution of the inverse problem.

2. Fundamental solutions and potential theory

An essential tool for the study of the boundary value problem (1.6)-(1.10) is represented by the fundamental solutions of the two partial differential equation (1.6) and (1.7). The fundamental solution of the Helmholtz equation (1.6) is well-known and is related to the Hankel function of order zero and of the first kind, i.e.

$$\Phi_0(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|) \qquad x \neq y \quad . \tag{2.1}$$

The existence of a fundamental solution for equation (1.7) has been proven in the case of $N = I$ on the boundary of the scatterer D. In particular it is shown in [12] that the partial differential equation (1.7) with $N = I$ holds in the weak sense and can be transformed into the Lippmann-Schwinger type equation

$$u(x) = u^{(i)}(x) + \int_D \nabla\Phi_0(x,z)[I - N(z)]\nabla u(z)dz \tag{2.2}$$

which has solution in $H^1(D)$. This result implies that the solution of equation (1.7) exists also when $N = M \neq I$ for $x \in \partial D$. In fact let us introduce

$$\Gamma := M^{-1/2}(\partial D) \tag{2.3}$$

and choose $\Gamma = \{z(t) \;:\; t \in [0, 2\pi]\}$ a regular 2π-periodic parametric representation with counterclockwise orientation. If

$$\beta(z(t)) = \frac{|z'(t)|}{|M^{1/2}z'(t)|} \;, \quad t \in [0, 2\pi] \tag{2.4}$$

it is not difficult to prove [4] that

$$ds_\Gamma(\eta) = \beta(\eta)ds(M^{1/2}\eta), \quad \eta \in \Gamma \tag{2.5}$$

and

$$\nu(M^{1/2}\eta) = \beta(\eta)\det M^{1/2}M^{-1/2}\nu_\Gamma(\eta), \quad \eta \in \Gamma \tag{2.6}$$

with $ds = ds_{\partial D}$ and $\nu = \nu_{\partial D}$. Then the following theorem holds:

Theorem 2.1. *Let* $N = M$ *for* $x \in \partial D$ *and* $\Omega := M^{-1/2}D$. *If* w *is a solution of the equation*

$$\nabla \cdot M^{-1/2}N(M^{1/2}\xi)M^{-1/2}\nabla w(\xi) + k^2 w(\xi) = 0 \quad \xi \in \Omega \tag{2.7}$$

then $u(x) = w(M^{-1/2}x)$ *is a solution of equation* (1.7).
Proof. First of all we observe that for $\xi \in \Gamma$, $M^{-1/2}N(M^{1/2}\xi)M^{-1/2} = I$ and therefore equation (2.7) is solvable. From

$$\nabla u(x) = M^{-1/2}\nabla w(M^{-1/2}x) \tag{2.8}$$

we have

$$\begin{aligned} &\nabla \cdot N(x)\nabla u(x) + k^2 u(x) = \\ &= \nabla \cdot M^{-1/2}N(M^{1/2}\xi)M^{-1/2}\nabla w(\xi) + k^2 w(\xi) = 0 \quad . \end{aligned} \tag{2.9}$$

□

Equation (2.7) can be reduced to the Lippmann-Schwinger type equation

$$w(\xi) = w^{(i)}(\xi) + \int_\Omega \nabla\Phi_0(\xi,\eta)[I - M^{-1/2}N(M^{1/2}\xi)M^{-1/2}]\nabla w(\eta)d\eta \quad (2.10)$$

and if we choose the fundamental solution of the Helmholtz equation as the incident field $w^{(i)}$, equation (2.10) provides a fundamental solution of equation (2.7). Finally from Theorem 2.1 it follows that the fundamental solution of equation (1.7) is given by

$$\Psi(x,y) = \frac{1}{\det M^{1/2}}\Phi_0(M^{-1/2}x, M^{-1/2}y) + B(M^{-1/2}x, M^{-1/2}y) \quad (2.11)$$

where

$$B(\xi,\eta) = \int_\Omega \nabla\Phi_0(\xi,\eta')[I - M^{-1/2}N(M^{1/2}\eta')M^{-1/2}]\nabla\Psi(\eta',\eta)d\eta' \quad (2.12)$$

and the normalization of Φ_0 with $1/\det M^{1/2}$ will be useful in the future. We note that

$$\Phi(x,y) = \frac{1}{\det M^{1/2}}\Phi_0(M^{-1/2}x, M^{-1/2}y) \quad (2.13)$$

is the fundamental solution of equation (1.7) when $N = M$ in D [4] and thus, by defining $C(x,y) = B(M^{-1/2}x, M^{-1/2}y)$ we have

$$\Psi(x,y) = \Phi(x,y) + C(x,y) \quad . \quad (2.14)$$

The availability of a fundamental solution allows to define the orthotropic single-layer potential

$$u(x) = \int_{\partial D} \Psi(x,y)\varphi(y)ds(y) \qquad x \in \mathbb{R}^2 \setminus \partial D \quad (2.15)$$

and the orthotropic double-layer potential

$$v(x) = \int_{\partial D} \nu(y)\cdot N(y)\nabla_y\Psi(x,y)\psi(y)ds(y) \quad x \in \mathbb{R}^2 \setminus \partial D \quad . \quad (2.16)$$

These two potentials satisfy the jump relations described in the following two theorems.

Theorem 2.2. *The single-layer potential* (2.15) *satisfies the jump relations*

$$u_+ = u_- \quad (2.17)$$

$$[\nu(x)\cdot M\nabla u(x)]_\pm = \int_{\partial D} \nu(x)\cdot M\nabla_x\Psi(x,y)\varphi(y)ds(y) \mp \frac{1}{2}\varphi(x) \quad (2.18)$$

for $x \in \partial D$.

Proof. By introducing equation (2.14) into (2.15) we have

$$u(x) = \int_{\partial D} \Phi(x,y)\varphi(y)ds(y) + \int_{\partial D} C(x,y)\varphi(y)ds(y) \quad (2.19)$$

and

$$\nu(x) \cdot M\nabla_x u(x) = \int_{\partial D} \nu(x) \cdot M\nabla_x \Phi(x,y)\varphi(y)ds(y) + \\ + \int_{\partial D} \nu(x) \cdot M\nabla_x C(x,y)\varphi(y)ds(y) \quad . \tag{2.20}$$

Now, the terms involving $C(x,y)$ in equations (2.19)-(2.20) are continuous across ∂D while the jump relations for the potentials corresponding to Φ are obtained in [4], Lemma 2.2. For sake of completeness here we sketch the proof of this Lemma. We introduce the density

$$\psi(\eta) = \frac{\varphi(M^{1/2}\eta)}{\beta(\eta)} \tag{2.21}$$

and the corresponding single-layer potential

$$u_0(\xi) = \int_\Gamma \Phi_0(\xi,\eta)\psi(\eta)ds_\Gamma(\eta) \quad . \tag{2.22}$$

By using equations (2.5)-(2.6) we easily obtain

$$\int_{\partial D} \Phi(x,y)\varphi(y)ds(y) = \frac{1}{\det M^{1/2}} u_0(M^{-1/2}x) \quad . \tag{2.23}$$

Then the continuity of $u_0(\xi)$ across Γ [3] implies the continuity of the single-layer potential corresponding to Φ across ∂D. On the other hand the jump relation for the normal derivative of $u_0(\xi)$ across Γ is

$$[\nu_\Gamma \cdot \nabla u_0]_\pm = \int_\Gamma \nu_\Gamma(\xi) \cdot \nabla_\xi \Phi_0(\xi,\eta)\psi(\eta)ds_\Gamma(\eta) \mp \frac{1}{2}\psi \tag{2.24}$$

and then, again from (2.5)-(2.6)

$$[\nu(x) \cdot M\nabla_x \int_{\partial D} \Phi(x,y)\varphi(y)ds(y)]_\pm = \\ = \int_{\partial D} \nu(x) \cdot M\nabla_x \Phi(x,y)\varphi(y)ds(y) \mp \frac{1}{2}\varphi(x) \quad . \tag{2.25}$$

□

The other theorem is

Theorem 2.3. *The double-layer potential* (2.16) *satisfies the jump relations*

$$[v(x)]_\pm = \int_{\partial D} \nu(y) \cdot M\nabla_y \Psi(x,y)\phi(y)ds(y) \pm \frac{1}{2}\varphi(x) \tag{2.26}$$

$$[\nu(x) \cdot M\nabla v(x)]_+ = [\nu(x) \cdot M\nabla v(x)]_- \tag{2.27}$$

with $x \in \partial D$.

Proof. We introduce equation (2.14) into the explicit expressions for v and $\nu \cdot M\nabla v$ and observe that the terms involving $C(x,y)$ are continuous across ∂D. In order to find the jump relations for the potential involving Φ we follow the proof of Lemma 2.3 in [4]. That is, we define

$$v_0(\xi) = \int_\Gamma \nu(\eta) \cdot \nabla \Phi_0(\xi,\eta) ds_\Gamma(\eta) \quad . \tag{2.28}$$

By using equations (2.5)-(2.6) it is not difficult to see that

$$v_0(M^{-1/2}x) = \int_{\partial D} \nu(y) \cdot M \nabla_y \Phi(x,y) \varphi(y) ds(y) \tag{2.29}$$

and, from the jump relation for v_0 across Γ, that

$$[v_0(M^{-1/2}x)]_\pm = \int_{\partial D} \nu(y) \cdot M \nabla_y \Phi(x,y) \varphi(y) ds(y) \pm \frac{1}{2}\varphi \quad . \tag{2.30}$$

From the Maue identity for $v_0(\xi)$ [11]

$$\nu(\xi) \cdot \nabla v_0(\xi) = \frac{\partial}{\partial s(\xi)} \int_\Gamma \Phi_0(\xi,\eta) \frac{\partial \psi(\eta)}{\partial s(\eta)} ds_\Gamma(\eta) + \\ + k^2 \nu(\xi) \cdot \int_\Gamma \Phi_0(\xi,\eta) \psi(\eta) \nu_\Gamma(\eta) ds_\Gamma(\eta) \tag{2.31}$$

and again from (2.5) and (2.6) we have

$$\nu(x) \cdot M \nabla_x v_0(M^{-1/2}x) = \det M \frac{\partial}{\partial s(x)} \int_{\partial D} \Phi(x,y) \frac{\partial \varphi(y)}{\partial s(y)} ds(y) + \\ k^2 \nu(x) \int_{\partial D} \Phi(x,y) \varphi(y) M \nu(y) ds(y) \quad . \tag{2.32}$$

This implies that $\nu(x) \cdot M \nabla_x v_0(M^{-1/2}x)$ is continuous across ∂D.

□

Now for $x \in \partial D$ we define the set of operators

$$(S\varphi)(x) = 2 \int_{\partial D} \Psi(x,y) \varphi(y) ds(y) \tag{2.33}$$

$$(K\varphi)(x) = 2 \int_{\partial D} \nu(y) \cdot M \nabla_y \Psi(x,y) \varphi(y) ds(y) \tag{2.34}$$

$$(K'\varphi)(x) = 2\nu(x) \cdot M \nabla_x \int_{\partial D} \Psi(x,y) \varphi(y) ds(y) \tag{2.35}$$

$$(T\varphi)(x) = 2\nu(x) \cdot M \nabla_x \int_{\partial D} \nu(y) \cdot M \nabla_y \Psi(x,y) \varphi(y) ds(y) \tag{2.36}$$

$$(\tilde{T}\varphi)(x) = \frac{1}{\pi \det M^{1/2}} \nu(x) \cdot \\ M \nabla_x \int_{\partial D} \nu(y) \cdot M \nabla_y \log \frac{1}{|M^{-1/2}(x-y)|} \varphi(y) ds(y) \quad . \tag{2.37}$$

Correspondingly S_0, K_0, K_0', T_0 and $\tilde{T}_0$ denote the same operators for the isotropic case $M = I$. As a consequence of the logarithmic singularity of the fundamental solution Ψ we have the following regularity properties [4]: if ∂D is of class C^2 then

- S, K, K', S_0, K_0, K_0' are continuous from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$.
- S, K, S_0, K_0 are continuous from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$.
- T, T_0 are continuous from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$.
- $T - \tilde{T}$, $T_0 - \tilde{T}_0$ are continuous from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$ and from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$.
- If ∂D is of class $C^{3,\alpha}$ then $\frac{1}{\det M^{1/2}}\tilde{T} - \tilde{T}_0$ is continuous from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$.

3. Well-posedness of the direct problem

In this section we will consider the problem of determining the fields u_0, u satisfying the boundary value problem

$$\triangle_2 u_0 + k^2 u_0 = 0 \qquad x \in \mathbb{R}^2 \setminus \overline{D} \tag{3.1}$$

$$\nabla \cdot N \nabla u + k^2 u = 0 \qquad x \in D \tag{3.2}$$

$$u_0 - u = f \qquad x \in \partial D \tag{3.3}$$

$$\nu \cdot \nabla u_0 - \nu \cdot M \nabla u = g \qquad x \in \partial D \tag{3.4}$$

when u_0 is a radiating solution of the Helmholtz equation, i.e.

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_0}{\partial r} - iku_0\right) = 0 \tag{3.5}$$

and $f \in C^{1,\alpha}(\partial D)$, $g \in C^{0,\alpha}(\partial D)$. In particular we want to show that this problem is well-posed in the sense of Hadamard.

As far as uniqueness is concerned, the following theorem holds.

Theorem 3.1. *If $f = g = 0$ in the boundary conditions* (3.3)-(3.4) *then $u_0 = u = 0$, that is the solution of the direct problem is unique.*

Proof. By applying the divergence theorem to $F = u\overline{N}\nabla\overline{u}$ and by using equation (3.2) we have

$$\begin{aligned} &- \int_D k^2 |u(y)|^2 dy + \int_D \overline{N(y)} \nabla \overline{u(y)} \cdot \nabla u(y) dy = \\ &= \int_{\partial D} u(y) \nu(y) \cdot M \nabla \overline{u(y)} ds(y) \quad . \end{aligned} \tag{3.6}$$

For the coercivity property of N the imaginary part of the left hand side of equation (3.6) is non-negative and therefore also the imaginary part of the right hand side is non-negative. From the homogenenous boundary condition it follows

$$\mathrm{Im}\left(\int_{\partial D} u_0(y)\nu(y) \cdot \nabla \overline{u}_0(y)\right) \geq 0 \quad . \tag{3.7}$$

But u_0 is a radiating solution of the Helmholtz equation and then, from Theorem 2.12 in [3] (which ensures uniqueness for solutions to exterior boundary value problems when equation (3.7) holds) $u_0 = 0$.

In order to prove that also $u = 0$ we introduce the connected open set $E = E_1 \cup E_2$, where E_2 is a subset of D while $D \cap E_1 = \phi$. We define on E the function

$$u' = \begin{cases} u_1 & x \in E_1 \\ u_2 & x \in E_2 \end{cases} \tag{3.8}$$

where $u_1 = 0$ and u_2 is the restriction of u over E_2. It is not difficult to see that u' is a weak solution of equation (3.2). In fact, if $\varphi \in C_0^\infty(E)$, by using the divergence theorem we have

$$\int_{E_2} u_2[\nabla \cdot N\nabla\varphi + k^2\varphi]dy = \int_{E_2} \varphi[\nabla \cdot N\nabla u_2 + k^2 u_2]dy + \\ + \int_{\partial E_2} (u_2\nu \cdot M\nabla\varphi - \varphi\nu \cdot M\nabla u_2)ds(y) \quad . \tag{3.9}$$

At the right hand side of equation (3.9) the volume integral is zero since by definition u_2 is a solution of (3.2). But also the surface integral is zero, for the homogeneous boundary condition and the compactness of the support of φ. It follows that u' is a weak solution but since the elliptic equation (3.2) has Hölder continuous coefficient, it is also a classical solution. Then by invoking twice the unique continuation principle [9] we have that $u = 0$ in D.

□

In order to show that the solution of problem (3.1)-(3.5) exists we use the Riesz theory of compact operators. We define the solutions of equations (3.1) and (3.2)

$$u_0(x) = \int_{\partial D} \det M^{1/2}\nu(y) \cdot \nabla_y \Phi_0(x,y)\psi(y)ds(y) + \\ + \int_{\partial D} \Phi_0(x,y)\varphi(y)ds(y) \tag{3.10}$$

and

$$u(x) = \int_{\partial D} \nu(y) \cdot M\nabla_y \Psi(x,y)\psi(y)ds(y) + \int_{\partial D} \Psi(x,y)\varphi(y)ds(y) \quad . \tag{3.11}$$

These functions satisfy the boundary conditions (3.3)-(3.4) if and only if the densities ψ and φ satisfy the system

$$[\det M^{1/2} K_0 - K]\psi + [\det M^{1/2} + 1]\psi + [S_0 - S]\varphi = 2f \tag{3.12}$$

$$-[\det M^{1/2} T_0 - T]\psi + 2\varphi - [K_0' - K']\varphi = -2g \quad . \tag{3.13}$$

By using the regularity properties of the integral operators it is not difficult to see that if ψ and φ are two continuous functions satisfying this system of equations with

$f \in C^{1,\alpha}(\partial D)$ and $g \in C^{0,\alpha}(\partial D)$, then $\psi \in C^{1,\alpha}(\partial D)$ and $\varphi \in C^{0,\alpha}(\partial D)$. The previous system (3.12)-(3.13) can be written in the matrix form

$$(E+A)\begin{pmatrix}\psi\\ \varphi\end{pmatrix} = \begin{pmatrix}2f\\ 2g\end{pmatrix} \tag{3.14}$$

with

$$E = \begin{pmatrix}[\det M^{1/2}+1]I & 0\\ 0 & 2I\end{pmatrix} \tag{3.15}$$

and

$$A = \begin{pmatrix}\det M^{1/2}K_0 - K & S_0 - S\\ -\det M^{1/2}T_0 + T & -K_0' + K'\end{pmatrix} \quad . \tag{3.16}$$

E is a matrix of operators with continuous inverse and A is a matrix of operators compact in $C(\partial D)\times C(\partial D)$. In order to prove that $E+A$ is injective let us consider the homogeneous conditions $f = g = 0$. From the uniqueness Theorem 3.1 u_0 is zero in $\mathbb{R}^2 \setminus \overline{D}$ and $u = 0$ in D. Then the jump relations for u_0

$$(u_0)_+ - (u_0)_- = \det M^{1/2}\psi \qquad (\nu\cdot\nabla u_0)_+ - (\nu\cdot\nabla u_0)_- = -\varphi \tag{3.17}$$

and for u

$$(u)_+ - (u)_- = \det M^{1/2}\psi \qquad (\nu\cdot M\nabla u)_+ - (\nu\cdot M\nabla u)_- = -\varphi \tag{3.18}$$

implies

$$(u_0)_+ + \det M^{1/2}u_+ = 0 \qquad (\nu\cdot\nabla u_0)_- + (\nu\cdot M\nabla u)_+ = 0 \quad . \tag{3.19}$$

By applying the divergence theorem to the function $F = \overline{u}_0\nabla u_0$ in D and by using equation (3.19) we obtain

$$\det M^{1/2}\int_{\partial D}(u)_+(\nu\cdot M\nabla\overline{u})_+ds(y) = \int_D[-k^2|u_0|^2 + |\nabla u_0|^2]dy \tag{3.20}$$

which implies that

$$\mathrm{Im}\left(\int_{\partial D}(u)_+(\nu\cdot M\nabla\overline{u})_+ds(y)\right) = 0 \quad . \tag{3.21}$$

If the change of variable $\xi = M^{-1/2}x$ is adopted equation (3.21) becomes

$$\mathrm{Im}\left(\int_\Gamma[u(M^{1/2}\xi)]_+[\nu_\Gamma(\xi)\cdot\nabla_\xi\overline{u}(M^{1/2}\xi)]_+ds_\Gamma(\xi)\right) = 0 \quad . \tag{3.22}$$

But $v(\xi) = u(M^{1/2}\xi)$ is a radiating solution of the Helmholtz equation outside Γ so that equation (3.22) implies that v is zero outside Γ. This implies that u is zero outside D and the conditions (3.19) implies $(u_0)_- = (\nu\cdot\nabla u_0)_- = 0$. Now we follow the same procedure adopted in the final part of the proof of Theorem 3.1 and we have the following existence theorem

Theorem 3.2. *Provided ∂D is of class $C^{3,\alpha}(\partial D)$ then there exists a unique solution of the transmission problem* (3.1)-(3.5).

In the proof of uniqueness for the inverse problem treated in the following section we will use different well-posedness results. We collect these results in the following theorem.

Theorem 3.3. *If $f \in C^{1,\alpha}(\partial D)$* and *$g \in C^{0,\alpha}(\partial D)$ then the following inequalities hold*:

$$\|\psi\|_{C(\partial D)} + \|\varphi\|_{C(\partial D)} \leq c_1(\|f\|_{C(\partial D)} + \|g\|_{C(\partial D)}) \tag{3.23}$$

$$\|\psi\|_{C^{1,\alpha}(\partial D)} + \|\varphi\|_{C^{0,\alpha}(\partial D)} \leq c_2(\|f\|_{C^{1,\alpha}(\partial D)} + \|g\|_{C^{0,\alpha}(\partial D)}) \tag{3.24}$$

and

$$\|\psi\|_{L^2(\partial D)} + \|\varphi\|_{L^2(\partial D)} \leq c_3(\|f\|_{L^2(\partial D)} + \|g\|_{L^2(\partial D)}) \tag{3.25}$$

with c_1, c_2 and c_3 some constants depending on ∂D. Moreover the solutions of the transmission problem (3.1)-(3.5) *continuously depend on the data, in the sense that*

$$\|u\|_{C^{1,\alpha}(D)} + \|u_0\|_{C^{0,\alpha}(\mathbb{R}^2\setminus\overline{D})} \leq c(\|f\|_{C^{1,\alpha}(\partial D)} + \|g\|_{C^{0,\alpha}(\partial D)}) \quad . \tag{3.26}$$

where again c depends only on ∂D. Finally, if R_1, R_2 are two real numbers such that $0 < R_1 < R_2$ and B_{R_1}, B_{R_2} are the circles with radii respectively R_1, R_2, then the following modified well-posedness result holds:

$$\begin{aligned}&\|u\|_{C^{1,\alpha}(\partial D\setminus B_{R_2})} \leq \\ &\leq c'(\|f\|_{L^2(\partial D)} + \|g\|_{L^2(\partial D)} + \|f\|_{C^{1,\alpha}(\partial D\setminus B_{R_1})} + \|g\|_{C^{0,\alpha}(\partial D\setminus B_{R_1})})\end{aligned} \tag{3.27}$$

with $c' = c'(\partial D, R_1, R_2)$.

Proof. Equation (3.23) directly comes from from the Riesz theory since the fact that $E+A$ is injective in $C(\partial D)\times C(\partial D)$ implies that its inverse operator exists continuous. Inequality (3.24) is obtained by using the regularity properties of the operators (2.33)-(2.37) in the system of equations (3.12)-(3.13). In order to prove inequality (3.25) we consider the matrix equation (3.14) in $L^2(\partial D) \times L^2(\partial D)$. In this functional space E has again bounded inverse operator, A is again compact and furthermore it maps pairs in $L^2(\partial D) \times L^2(\partial D)$ into pairs in $C(\partial D) \times C(\partial D)$. This implies that if $(\psi \quad \varphi)$ in $L^2(\partial D) \times L^2(\partial D)$ satisfies the homogeneous matrix equation then it is in $C(\partial D) \times C(\partial D)$; but in this space $E + A$ is injective and therefore $(\psi \quad \varphi)$ is the zero vector.

As far as the two well-posedness results (3.26)-(3.27) are concerned, the first inequality easily comes from the Theorems 2.12, 2.16, 2.17 and 2.23 in [2] (these theorems prove the Hölder continuity of the single- and double-layer potentials and their derivatives together with the fact that the Hölder norms are bounded by the norms of the densities). In order to prove the last equation, we decompose the densities ψ and φ of the potential (3.11) in the form $\psi = \chi\psi + (1-\chi)\psi$ and $\varphi = \chi\varphi + (1-\chi)\varphi$ where χ is a continuous function equal to 1 in $B_{R_1} \cap \partial D$ and with support in $B_R \cap \partial D$, $R_1 < R < R_2$. We deal with the corresponding terms in the potential separately. The potentials corresponding to $\chi\psi$ and to $\chi\varphi$ are bounded in $C^{1,\alpha}(\overline{D} \setminus B_{R_2})$ by the L^2 norms of f and g. This follows from inequality (3.25), the Schwartz inequality and the fact that the fundamental solution Ψ is Hölder continuous in every region excluding the logarithmic singularity. Following the same procedure which led to the well-posedness (3.26) we have that the potential terms corresponding to $(1-\chi)\psi$ and $(1-\chi)\varphi$ can be bounded in $C^{1,\alpha}(\overline{D})$ by the sum of $\|f\|_{C^{1,\alpha}(\partial D \setminus B_{R_1})}$ and $\|g\|_{C^{0,\alpha}(\partial D \setminus B_{R_1})}$.

□

4. Uniqueness for the inverse problem

A radiating solution of the Helmholtz equation is characterized by the asymptotic behaviour

$$u_0(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty \quad . \tag{4.1}$$

The function u_∞ is defined on the unit circle $\Omega = \{x \in \mathbb{R}^2 \ : \ |x| = 1\}$ and is said the far-field pattern associated to the scattered wave u_0. The inverse problem we are interested in is the problem of determining the shape of the scatterer D from measures of the far-field pattern in correspondence with several directions $x/|x|$. In particular we want to prove that this problem has a unique solution. To this end we need some preparatory definitions and results. First of all, let us define the two sequences

$$x_n = x^* + \frac{R}{n}\nu(x^*) \tag{4.2}$$

and

$$z_n = x^* + \det M^{-1/2}\frac{R}{n}M\nu(x^*) \tag{4.3}$$

with $n = 1, 2, \ldots$, x^* a point on ∂D and $R > 0$. We note that for R small enough the z_n are in $\mathbb{R}^2 \setminus \overline{D}$ for all n and that they are image points $z_n = M^{1/2}\xi_n$ of points located on the exterior normal of Γ at the point $M^{-1/2}(x^*)$. Then the following lemma holds:

Lemma 4.1. *Let Φ_0 and Ψ the fundamental solutions of the equations* (3.1) *and* (3.2). *Then*

(i) *the sequence* $\|\Psi(\cdot,z_n)-\Phi_0(\cdot,x_n)\|_{C(\partial D)}$, $n=1,2,\ldots$ *is not bounded;*

(ii) *there exists* $c_1=c_1(\partial D)$ *such that*

$$\|\nu\cdot M\nabla\Psi(\cdot,z_n)-\nu\cdot\Phi_0(\cdot,x_n)\|_{C(\partial D)}\le c_1 \qquad n=1,2,\ldots \quad ; \tag{4.4}$$

(iii) *there exist* $c_2=c_2(\partial D)$ and $c_3=c_3(\partial D)$ *such that*

$$\|\Psi(\cdot,z_n)-\Phi_0(\cdot,x_n)\|_{L^2(\partial D)}\le c_2 \quad n=1,2,\ldots \tag{4.5}$$

and

$$\|\nu\cdot M\nabla\Psi(\cdot,z_n)-\nu\cdot\nabla\Phi_0(\cdot,x_n)\|_{L^2(\partial D)}\le c_3 \quad n=1,2,\ldots \quad . \tag{4.6}$$

Proof. From the definition of Ψ we have that

$$|\Psi(x^*,z_n)-\Phi_0(x^*,x_n)|\ge ||\Phi(x^*,z_n)-\Phi_0(x^*,x_n)|-|C(x^*,z_n)|| \quad . \tag{4.7}$$

Then (i) comes from the fact that for the asymptotical behaviour of Φ, $|\Phi(x^*,z_n)-\Phi_0(x^*,x_n)|\ge K\log n|1-\det M^{1/2}|$ with K a constant (note that here we use the condition $\det M^{1/2}\ne 1$).

To prove (ii) we observe that the leading singular terms of $\Phi_0(x,x^*+h\nu(x^*))$ and $\Psi(x,x^*+h\det M^{1/2}M\nu(x^*)))$ are

$$w_0(x,h)=\log\frac{1}{|x-x^*-h\nu(x^*)|} \tag{4.8}$$

and

$$w(x,h)=\frac{1}{\det M^{1/2}}\log\frac{1}{|M^{-1/2}[x-x^*-h\det M^{-1/2}M\nu(x^*)]|} \quad . \tag{4.9}$$

In [4] it is proven that the functions

$$x\to\nu(x)\cdot M\nabla w(x,h)-\nu(x)\cdot\nabla w_0(x,h)$$

are uniformly bounded for all small h. This implies the inequality (4.4).

Finally the inequalities (4.5) and (4.6) are consequences of the fact that the singularities of Φ_0 and Ψ are of the logarithmic type.

□

An useful result is represented by the following theorem.

Theorem 4.2. *Let us consider a bounded open domain* D *with* $\partial D\in C^2$ *and a point* x^* *on* ∂D. *If* $B_R=\{x\in\mathbb{R}^2 \quad , \quad |x-x^*|\le R \quad R>0\}$ *and* $u\in C^2(D)$ solves the equation

$$\nabla\cdot N\nabla u+k^2u=0 \tag{4.10}$$

then

$$\|u\|_{C(\partial D)} \leq C\left(\|\nu \cdot M\nabla u\|_{C(\partial D)} + \|u\|_{C(\partial D\setminus B_R)}\right) \tag{4.11}$$

with C a constant depending on D and R.
Proof. We consider equation (4.11) with the impedence boundary condition

$$\nu \cdot M\nabla u - i\eta u = g \quad x \in \partial D \tag{4.12}$$

where η is a continuous non-negative function whose support is in $\partial D\setminus B_R$ and such that $\eta \neq 0$. Following [6] we prove that the impedance problem (4.10), (4.12) is well-posed. In fact, let us take $g = 0$ and apply the divergence theorem to $F = u\overline{N}\nabla\overline{u}$ in D. We obtain

$$\int_D [-k^2|u|^2 + \overline{N}\nabla\overline{u} \cdot \nabla u]dy = -\mathrm{i}\int_{\partial D} \eta|u|^2 ds(y) \quad . \tag{4.13}$$

For the coercivity hypothesis on N, the imaginary part of the left hand side is non-negative while the imaginary part of the right hand side is non-positive. It follows that

$$\mathrm{Im}\int_{\partial D} \eta|u|^2 ds(y) = 0 \tag{4.14}$$

and therefore $u = 0$.

In order to prove the existence of the impedence problem, we use again the Riesz theory for compact operator. The potential

$$u(x) = \int_{\partial D} \Psi(x,y)\varphi(y)ds(y) \tag{4.15}$$

solves equation (4.10) and satisfies condition (4.12) if and only if the density $\varphi \in C(\partial D)$ satisfies

$$K'\varphi + \varphi - \mathrm{i}\eta S\varphi = 2g \quad . \tag{4.16}$$

The operator $K' - \mathrm{i}\eta S$ is compact in $C(\partial D)$. Moreover if $g = 0$

$$u_{=}[\nu \cdot M\nabla u]_{=}0 \tag{4.17}$$

for the uniqueness and from the continuity of the single-layer potential also $u_+ = 0$. But in $\mathbb{R}^2 \setminus \overline{D}$, u is a radiating solution of the Helmholtz equation and from the uniqueness of the solution of the exterior Dirichlet problem $u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. This implies that $[\nu \cdot M\nabla u]_+ = 0$ and then $\varphi = 0$. Therefore $I + K' - \mathrm{i}\eta S$ is injective and $(I + K' - \mathrm{i}\eta S)^{-1}$ exists continuous. Theorem 2.12 in [2] proves (in $\mathbb{R}^3$ but the result holds also in $\mathbb{R}^2$) that the single-layer potential u is uniformly Hölder continuous and that its Hölder norm is bounded by the uniform norm of the density φ, i.e.

$$\|u\|_{\alpha,\mathbb{R}^2} \leq C_\alpha \|\varphi\|_{\infty,\partial D} \tag{4.18}$$

with $0 < \alpha < 1$. This result, together with the impedence condition (4.12) and the compactness of the support of η straightforward lead to equation (4.11).

□

Now we have all the tools to prove the uniqueness of the solution of the inverse problem.

Theorem 4.3. *If D_1 and D_2 are two orthotropic media such that, for fixed k, the far-field patterns $u_{\infty,1}$ and $u_{\infty,2}$ coincide for all incident directions d then $D_1 = D_2$.*
Proof. First of all let us consider the two boundary value problems

$$\triangle_2 u_{0,j} + k^2 u_{0,j} = 0 \qquad x \in \mathbb{R}^2 \setminus \overline{D}_j \tag{4.19}$$

$$\nabla \cdot N\nabla u_j + k^2 u_j = 0 \qquad x \in D_j \tag{4.20}$$

$$u_{0,j} - u_j = -\Phi_0(\cdot, x_0) \qquad x \in \partial D_j \tag{4.21}$$

$$\nu \cdot \nabla u_{0,j} - \nu \cdot M\nabla u_j = -\nu \cdot \nabla\Phi_0(\cdot, x_0) \qquad x \in \partial D_j \tag{4.22}$$

with $j = 1, 2$ and $x_0 \in G = \mathbb{R}^2 \setminus (\overline{D}_1 \cup \overline{D}_2)$. Since the far-field patterns are equal, when the boundary data are $f = -\exp(ikx \cdot d)$ and $g = -\nu \cdot \nabla\exp(ikx \cdot d)$ the scattered fields are equal. But the sources used in equations (4.21)-(4.22) can be approximated in Hölder norms by linear combinations of the corresponding boundary data for plane waves and therefore $u_{0,1} = u_{0,2}$.

Now by contradiction let us assume that $D_1 \neq D_2$. It follows that R and x^* exist such that $x^* \in \partial D_1$, $x^* \notin \overline{D}_2$, $x_n \in G$ and $B_R \cap \overline{D}_2 = \phi$. Then

$$\|\Phi_0(\cdot, x_n)\|_{C^{1,\alpha}(\partial D_2)} + \|\nu \cdot \nabla\Phi_0(\cdot, x_n)\|_{C^{0,\alpha}(\partial D_2)} \leq c \tag{4.23}$$

for all n, with c a constant. This is due to the fact that the singularity of Φ_0 is excluded.

Now we consider the boundary value problem for D_1

$$\triangle_2 u_{0,n} + k^2 u_{0,n} = 0 \qquad x \in \mathbb{R}^2 \setminus \overline{D}_1 \tag{4.24}$$

$$\nabla \cdot N\nabla u_n + k^2 u_n = 0 \qquad x \in D_1 \tag{4.25}$$

$$u_{0,n} - u_n = -\Phi_0(\cdot, x_n) \qquad x \in \partial D_1 \tag{4.26}$$

$$\nu \cdot \nabla u_{0,n} - \nu \cdot M\nabla u_n = -\nu \cdot \nabla\Phi_0(\cdot, x_n) \qquad x \in \partial D_1 \quad . \tag{4.27}$$

As just proven, $u_{0,n}$ coincides with the scattered field corresponding to the problem for D_2. Therefore, from the well-posedness of the direct problem and from inequality (4.23), there exists the constant c_1, such that

$$\|u_{0,n}\|_{C^{1,\alpha}(G)} \leq c_1 \quad . \tag{4.28}$$

Finally we define $\tilde{u}_n := u_n - \Psi(\cdot, z_n)$. It is obvious that $u_{0,n}$ and $\tilde{u}_n$ satisfies the equations (4.24)-(4.25) with the boundary conditions

$$u_{0,n} - \tilde{u}_n = \Psi(\cdot, z_n) - \Phi_0(\cdot, x_n) \qquad x \in \partial D_1 \tag{4.29}$$

$$\nu \cdot \nabla u_{0,n} - \nu \cdot M \nabla \tilde{u}_n = \nu \cdot M \nabla \Psi(\cdot, z_n) - \nu \cdot \nabla \Phi_0(\cdot, x_n) \qquad x \in \partial D_1 \quad . \tag{4.30}$$

By using the inequalities (3.27), (4.5), (4.6) we have that

$$\|\tilde{u}_n\|_{C^{1,\alpha}(\partial D_1 \setminus B_R)} \leq c \tag{4.31}$$

with c a constant and from Theorem 4.2

$$\begin{aligned} \|\tilde{u}_n\|_{C(\partial D_1)} &\leq a(\|\nu \cdot M \nabla \tilde{u}_n\|_{C(\partial D_1)} + \|\tilde{u}_n\|_{C(\partial D_1 \setminus B_R)}) \leq \\ &\leq b + d\|\nu \cdot M \nabla \tilde{u}_n\|_{C(\partial D_1)} \end{aligned} \tag{4.32}$$

with a, b and d constants. Then we use condition (4.30) and inequality (4.4) to obtain

$$\|\tilde{u}_n\|_{C(\partial D)} \leq a' + b'\|\nu \cdot \nabla u_{0,n}\|_{C(\partial D)} \tag{4.33}$$

with a', b' constants. Therefore from condition (4.29)

$$\|\Psi(\cdot, z_n) - \Phi_0(\cdot, x_n)\|_{C(\partial D)} \leq c' + d'\|u_{0,n}\|_{C^{1,\alpha}(\partial D)} \tag{4.34}$$

with c' and d' constants. From (4.34), by recalling that the sequence $\|u_{0,n}\|_{C^{1,\alpha}(\partial D)}$ is bounded we have that the sequence $\|\Psi(\cdot, z_n) - \Phi_0(\cdot, x_n)\|_{C(\partial D)}$ is bounded, which is in contrast with the result (i) of Lemma 4.1

□

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Figure captions

Figure 1 - Total field scattered by two different dielectrics. The first scatterer is anisotropic while in the bottom scatterer the refraction index is a multiple of the identity (i.e., the dielectric is isotropic). The incident field is a point source. The field is computed by using a finite element code. These pictures have been kindly provided by Joe Coyle.