The linear sampling method in a lossy background: an energy perspective

R. Aramini\textsuperscript{a}\textsuperscript{*}, M. Brignone\textsuperscript{b}, G. Caviglia\textsuperscript{b}, A. Massa\textsuperscript{a} and M. Piana\textsuperscript{b,c}

\textsuperscript{a}Dipartimento di Ingegneria e Scienza dell’Informazione (DISI), Universit\`a di Trento, Italy; \textsuperscript{b}Dipartimento di Matematica (DIMA), Universit\`a di Genova, Italy; \textsuperscript{c}CNR - SPIN, Genova, Italy

\textit{(Received 00 Month 200\textsuperscript{x}; in final form 00 Month 200\textsuperscript{x})}

In this paper we propose a physical approach to the linear sampling method for a possibly inhomogeneous and lossy background, whereby the modified far-field equation at the basis of the method can be regarded as a constraint on the power fluxes carried by the Poynting vector associated with the scattered field. Under appropriate assumptions on the flow lines of this Poynting vector, the general theorem inspiring the linear sampling method (and concerning the existence of approximate solutions to the modified far-field equation) can be reformulated in a different way, which is more appropriate to explain the performance of the method.

\textbf{Keywords:} qualitative inverse scattering, linear sampling method.

\textbf{AMS Subject Classification:} 45A05; 65R20; 65R30; 65R32; 78A46.

1. Introduction

The linear sampling method (LSM) \cite{1–3, 6} is a well-known algorithm for visualizing the support of an unknown object by processing the data of a time-harmonic scattering experiment, and it can be considered the first of a set of qualitative methods \cite{4} in the field of inverse scattering. The strong points of these methods (intrinsic linearity, short computational times, no need for a priori information on the scatterer, etc.) make them an effective approach for a large variety of inverse scattering problems.

However, qualitative methods are based on equations that are artificially formulated, i.e., do not derive from physical laws. This lack of a physical foundation seems to be responsible for at least one of the open problems \cite{4, 5} concerning, in particular, the LSM. Indeed, the far-field equation at the basis of the LSM is known to admit approximate solutions whose $L^2$-norms behave as good indicator functions for the support of the scatterer (i.e., functions bounded inside and arbitrarily large outside the unknown object), but there is a priori no reason why computing the $L^2$-norm of a (Tikhonov) regularized solution of the far-field equation, as requested by the implementation of the LSM, should provide one of these indicator functions.

In order to overcome this difficulty, the far-field equation at the basis of the LSM can be modified so to obtain a different qualitative method, i.e., the factorization
method (FM), first formulated in [7] and extensively described in [8]. The theoretical foundation of the FM is much sounder than that of the LSM; moreover, the FM can provide a deep insight into the LSM, as shown in [9–11]. In particular, in [10] the LSM is formulated and mathematically justified by means of results holding for the FM. However, this approach is restrictive from the viewpoint of scattering conditions, since the FM is, so far, significantly less general than the LSM [4, 5].

This gap motivated the physics-based approach to the LSM proposed in a recent paper [12], whereby the far-field equation of the LSM is regarded as a constraint on the power fluxes carried by the Poynting vector associated with the scattered electromagnetic field. It is shown there that, under some assumptions on the flow lines of this Poynting vector in the background, any approximate (and, in particular, the Tikhonov regularized) solution of the far-field equation behaves as a good indicator function for the support of the unknown target. This approach is explained in detail in the case of a homogeneous and lossless background, and for an experimental set-up consisting of emitting and receiving antennas placed in the far-field region. However, the LSM can also be formulated when a lossy and inhomogeneous background is taken into account, and all the antennas are placed in the near-field region [13–15]. Hence, in this paper we want to show how the investigation pursued in [12] can be adapted to this more complex scenario. The main advantage of this approach is that the material properties of the scatterer (and, to a large extent, of the background) are irrelevant: we focus on the two-dimensional, penetrable case only to fix ideas and notations, but the key-role in our framework is just played by the behaviour of the flow lines of the Poynting vector in the background medium (and not inside the scatterer). However, so far, this is also a limitation of our approach, since such behaviour is a posteriori observed by means of numerical simulations, and not theoretically predicted. Progress in this direction would probably require sophisticated tools of topological dynamics (see e.g. [16]), and is beyond the scope of the present investigation.

The paper is organized according to the following plan. In Section 2 we formulate the direct and the inverse scattering problems we are interested in, we summarize the LSM and highlight the theoretical open issue we want to address. Section 3 is concerned with the propagation of electromagnetic power flux throughout the background: in particular, the role of the flow lines of the Poynting vector field associated with the scattered field is emphasized. Section 4 shows how the analysis of power fluxes can give an insight into the LSM when emitters and receivers are placed in the near-field and the far-field region respectively: to this end, the investigation of [12] is adapted to the case of a lossy and inhomogeneous background. In Section 5 we perform some numerical simulations in order to catch the typical behaviour of the flow lines of the Poynting vector field, and we explain how this behaviour is related to the performance of the LSM. Section 6 addresses the situation where both emitters and receivers are in the near-field region, by pointing out its link with the case of far-field receivers, considered in the previous sections. Finally, our conclusions and hints for future work are presented in Section 7.

2. The direct and inverse scattering problems

We consider a time-harmonic electromagnetic wave of angular frequency $\omega > 0$. First, we describe its propagation inside the background medium. The latter is assumed to be a penetrable, isotropic, infinitely long cylinder, whose physical and geometrical properties are the same on each plane perpendicular to its axis $\hat{a}$. Let us choose any of these planes and identify it with $\mathbb{R}^2$. Following the conventions
of Sec. 9.1 in [17], we assign the refraction index of the background as

$$n(x) := \frac{1}{\varepsilon_0} \left[ \varepsilon(x) + i \frac{\sigma(x)}{\omega} \right] \quad \forall x \in \mathbb{R}^2,$$

where $i = \sqrt{-1}$. We assume that there exists $R_0 > 0$ such that $\varepsilon(x) = \varepsilon_0 > 0$ and $\sigma(x) = 0 \ \forall x \in \Omega_0 := \mathbb{R}^2 \setminus B_{R_0}$, with $B_{R_0} := \{ x \in \mathbb{R}^2 : |x| < R_0 \}$. In $\Omega_0$ the wavenumber is $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$, while the refraction index is $n_0 = 1$. Then, if we define $m(x) := 1 - n(x)$, the support of $m$ (denoted by supp$m$) is compact: we also require that supp$m$ is piecewise homogeneous, i.e., that there exists a finite number $J$ of open $C^2$-domains $\Omega_j \subset \mathbb{R}^2$ where $n(x) = n_j \in \mathbb{C}$ $\forall x \in \Omega_j$, for $j = 1, \ldots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and supp$m = \cup_{j=1}^J \Omega_j$. Accordingly, the wavenumber is given by $k^2(x) = k_0^2 n_j =: k_j^2 \in \mathbb{C}$ (with $\text{Im}[k(x)] \geq 0$) $\forall x \in \Omega_j$ and $\forall j = 1, \ldots, J$. The magnetic permeability is assumed to be constant everywhere, i.e., $\mu(x) = \mu_0 > 0 \ \forall x \in \mathbb{R}^2$.

The incident electric field is generated by a line source parallel to $\hat{a}$, whose intersection $x_0$ with the plane of interest is placed in $\Omega_0$. Accordingly, we can choose a coordinate system such that the incident field is expressed as

$$E^i(x; x_0) = (0, 0, u^i(x; x_0)), \quad \text{with} \quad u^i(x; x_0) = \Phi_0(x; x_0) := \frac{i}{4} H_0^{(1)}(k_0|x - x_0|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind and of order zero (the dependence of the fields on the source point $x_0$ will be often omitted in the following). This means that we are considering TM-polarized waves: then, the scattered and total electric field can be expressed as $E^s(x) = (0, 0, u^s(x))$ and $E(x) = (0, 0, u(x))$ respectively. Starting from the time-harmonic and rescaled Maxwell equations (see Sec. 9.1 in [17])

$$\text{curl} E - ik_0 H = 0, \quad \text{curl} H + ik_0 n(x) E = 0,$$

and imposing the continuity of the tangential components of the total fields $E$ and $H$ across the surfaces where $n(x)$ is discontinuous, easy computations show that the scattering problem can be formulated as follows: given the incident field $u^i$ as in (2), find the total field $u \in C^1(\mathbb{R}^2 \setminus \{x_0\})$ such that [13]

$$\begin{align*}
\Delta u(x) + k^2(x) u(x) &= 0 \quad \text{for} \quad x \in \mathbb{R}^2 \setminus \{x_0\} \quad (a) \\
u(x) &= u^i(x) + u^s(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus \{x_0\} \quad (b) \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - ik_0 u^s \right) &= 0, \quad (c)
\end{align*}$$

where the last relation expresses the Sommerfeld radiation condition. More precisely, Eq. (4)(a) is written for each open domain $\Omega_j$ and complemented by the transmission conditions imposing the continuity of both $u$ and its normal derivative $\frac{\partial u}{\partial n}$ across the discontinuity surfaces of $n(x)$. Moreover, for each $j = 1, \ldots, J$, $k^2(x) = k_j^2 \in \mathbb{C}$ $\forall x \in \Omega_j$; hence, from (4)(a), $u$ satisfies the Helmholtz equation in each homogeneous portion $\Omega_j$ of the background, then $u$ is real-analytic in $\Omega_j$ itself [4] (of course, it is also real-analytic in $\Omega_0 \setminus \{x_0\}$). Since an analogous argument shows that $u^i$, given by (2), is real-analytic in $\mathbb{R}^2 \setminus \{x_0\}$, the regularity properties of $u$ also hold for the scattered field $u^s = u - u^i$, except that $u^s$ is real-analytic in $x_0$ too.
The direct scattering problem (4) is equivalent [13, 17] to the Lippmann-Schwinger equation:

\[ u(x) = \Phi_0(x, x_0) - k_0^2 \int_{B_m} \Phi_0(x, y) m(y) u(y) \, dy, \]  

(5)

where \( B_m := \text{supp} \, m \). In particular, the integral formulation (5) of problem (4) allows proving the well-posedness of the latter: if \( B \) is any open ball, centred at the origin, such that \( \overline{B} \supset B_m \) and \( x_0 \notin \overline{B} \), the solution \( u \) of (4) in \( \overline{B} \) depends continuously on \( u_i \) with respect to the maximum norm in \( C(\overline{B}) \).

Assume now that a smooth, penetrable and cylindrical scatterer (with the same axis \( \hat{a} \) and the same magnetic permeability \( \mu_0 \) as the background) is located in \( \mathbb{R}^2 \setminus \overline{\Omega}_0 \): more precisely, we assume that it takes up the closure \( \overline{D} \) of an open, bounded and \( C^2 \)-domain \( D \subset \mathbb{R}^2 \setminus \overline{\Omega}_0 \), such that \( \mathbb{R}^2 \setminus \overline{D} \) is connected and the lines of discontinuity of the refraction index are \( C^2 \)-curves. Accordingly, definition (1) is replaced by:

\[ \tilde{n}(x) := \begin{cases} n(x) & \forall x \in \mathbb{R}^2 \setminus \overline{D} \\ n_D(x) & \forall x \in \overline{D}, \end{cases} \]  

(6)

with \( n_D \in C^1(D) \), while the corresponding wavenumber is now given by \( \tilde{k}^2(x) = k_0^2 \tilde{n}(x) \). The previous discussion is easily extended to this new framework: it suffices to replace everywhere \( n(x) \) with \( \tilde{n}(x) \). We consider the same incident field (2) as before, and we denote the new total and scattered fields as \( \tilde{u} \) and \( \tilde{u}^s \) respectively. In particular, it is easy to realize [13, 14] that

\[ u^d := \tilde{u}^s - u^s = \tilde{u} - u^t - u^s = \tilde{u} - u \in C^1(\mathbb{R}^2) \]  

(7)

is the field scattered by \( \overline{D} \) when \( u \), which is the Green’s function for the background medium described by (1), is regarded as the incident field.

The qualitative inverse scattering problem considered here is to visualize the support \( \overline{D} \) of the scatterer by processing the data of an appropriate scattering experiment, starting from the knowledge of the background, i.e., of the refraction index \( n(x) \). We assume that the source point \( x_0 \) can be moved along a circle \( \Lambda := \{ x \in \mathbb{R}^2 : |x| = R_\Lambda \} \), while the receivers are located on a circle \( \Gamma := \{ x \in \mathbb{R}^2 : |x| = R_\Gamma \} \), with \( R_\Lambda, R_\Gamma > R_0 \).

The LSM (as outlined, in particular, in [13, 14]) is a procedure for solving this inverse problem. It is based on the \textit{modified far-field operator}

\[ F : L^2(\Lambda) \to L^2(\Gamma), \quad (Fg)(x) := \int_{\Lambda} u^d(x; y) g(y) \, ds(y) \]  

(8)

and the corresponding \textit{modified far-field equation}

\[ (Fg_z)(x) = u(x; z), \quad x \in \Gamma, \]  

(9)

written for each \( z \) in the investigation domain \( T \), contained in the disk of boundary \( \Gamma \). We point out that, in spite of the name of Eq. (9), emitters and receivers are usually placed in the near-field region, as is reasonable when an inhomogeneous background is taken into account. In general, Eq. (9) is unsolvable for almost all \( z \in T \) [14]. However, the following theorem [13, 14] ensures that (9) admits \( \epsilon \)-approximate solutions \( g^\epsilon_z \) whose \( L^2(\Lambda) \)-norm can be used as an indicator function.
Theorem 2.1: let $D, F, k_0$ be as above and assume that $k_0$ is not a transmission eigenvalue. Then:

(i) if $z \in D$, for every $\epsilon > 0$ there exists a solution $g_\epsilon^z \in L^2(\Lambda)$ of the inequality

$$\|(Fg_\epsilon^z)(\cdot) - u(\cdot, z)\|_{L^2(F)} \leq \epsilon$$  \hspace{2cm} (10)

such that, for each $z^* \in \partial D$,

$$\lim_{\epsilon \to 0} \|g_\epsilon^z\|_{L^2(\Lambda)} = \infty;$$  \hspace{2cm} (11)

(ii) if $z \in \mathbb{R}^2 \setminus \bar{D}$, for every $\epsilon > 0$ there exists a solution $g_\epsilon^z \in L^2(\Lambda)$ of inequality (10); moreover, every $g_\epsilon^z \in L^2(\Lambda)$ satisfying (10) is such that

$$\lim_{\epsilon \to 0} \|g_\epsilon^z\|_{L^2(\Lambda)} = \infty.$$  \hspace{2cm} (12)

Inspired by this theorem, the LSM visualizes the scatterer by computing the Tikhonov regularized solution $g_\epsilon^z$ of equation (9) and by plotting its $L^2(\Lambda)$-norm for each sampling point $z \in T$. However, the main problem with this approach is that there is no guarantee that $g_\epsilon^z$ behaves as the $\epsilon$-approximate solution exhibited by the theorem (see [4, 5, 8–10, 15] for a detailed discussion of this point). Rather, remembering also that regularization prevents $\epsilon$ from vanishing (at least in presence of noise), an “ideal” version of Theorem 2.1 would be the following: (i) if $z \in D$ and $\epsilon > 0$ is small enough, then any $g_\epsilon^z \in L^2(\Lambda)$ satisfying (10) is such that limit (11) holds; (ii) if $z \in \mathbb{R}^2 \setminus \bar{D}$ and $\epsilon > 0$ is small enough, there exists no $g_\epsilon^z \in L^2(\Lambda)$ satisfying (10).

Nevertheless, without any further hypothesis, this “ideal” version is clearly false: e.g., the new point (ii) is made impossible by point (ii) of Theorem 2.1. Then, the key-problem is now: what further hypotheses can we reasonably add in order to obtain the desired statement? To face this problem, in the following sections we shall analyze how time-averaged electromagnetic power is transported in the background medium, from both the analytical and numerical viewpoints.

3. The Poynting vector and the transport of power fluxes

As suggested by (8) and (9), the field $u^d$ plays a crucial role in the implementation of the LSM: then, in our framework, it is important to analyze $u^d$ from the viewpoint of energy transport. From $E^d(x) = (0, 0, u^d(x))$ and the first of Eq.s (3), we find the corresponding magnetic field $H^d(x) = \frac{1}{i k_0} \left( \frac{\partial u^d}{\partial x_2}(x), -\frac{\partial u^d}{\partial x_1}(x), 0 \right)$. Then, remembering that $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$, we can compute the time-averaged Poynting vector associated to the scattered electromagnetic field $(E^d, H^d)$ as [18]:

$$S^d(x) = \frac{1}{4 \sqrt{\varepsilon_0 \mu_0}} \left[ E^d \times H^d + E^d \times H^d \right](x) = \frac{1}{4 \lambda \varepsilon_0 \mu_0} \left[ \bar{u}^d \nabla u^d - u^d \nabla \bar{u}^d \right](x).$$  \hspace{2cm} (13)

The previous regularity discussion implies that each component of $S^d$ is in $C^0(\mathbb{R}^2)$ and real-analytic in $\Omega_j \cap (\mathbb{R}^2 \setminus \bar{D})$ for each $j = 0, \ldots, J$. We shall come back to this point later.
It is well known [18] that the integral of the time-averaged Poynting vector field over a closed surface Σ with outward unit normal represents the real power outgoing from Σ. Of course, in our two-dimensional setting, we consider plane curves instead of surfaces: then, if γ is a non-auto-intersecting and piecewise smooth curve in \( \mathbb{R}^2 \) (as always assumed in the following), we define the power flux of the field \( u^d \) across γ as the power flux of the corresponding Poynting vector (13), i.e.,

\[
\mathcal{F}_\gamma(u^d) := \int_\gamma \mathcal{S}^d(x) \cdot \nu(x) \, dl(x) = \frac{1}{4i\omega \varepsilon_0 \mu_0} \int_{E_\gamma} \left[ \bar{u}^d \frac{\partial u^d}{\partial \nu} - u^d \frac{\partial \bar{u}^d}{\partial \nu} \right] (x) \, dl(x),
\]

(14)

where \( \nu(x) \) is the unit normal to γ at the point \( x \) and \( dl(x) \) denotes the standard linear measure on γ. In particular, if γ is the boundary (with outward unit normal) of a bounded, connected and regular domain \( E_\gamma \subset \mathbb{R}^2 \setminus D \), we can prove that the flux (14) is not greater than zero, since the background is, in general, lossy. To this end, it suffices to apply Gauss’s divergence theorem in (14) is not greater than zero, since the background is, in general, lossy. To this end, it suffices to apply Gauss’s divergence theorem in \( E_\gamma \): remembering (13), (14), and observing that \( u^d \) satisfies Eq. (4)(a) in \( \mathbb{R}^2 \setminus \bar{D} \), we get

\[
\mathcal{F}_\gamma(u^d) = \int_\gamma \mathcal{S}^d(x) \cdot \nu(x) \, dl(x) = \int_{E_\gamma} \text{div} \mathcal{S}^d(x) dx =
\]

\[
= \frac{1}{4i\omega \varepsilon_0 \mu_0} \int_{E_\gamma} \left[ |\nabla u^d|^2 + \bar{u}^d \Delta u^d - |\nabla u^d|^2 - u^d \Delta \bar{u}^d \right] (x) dx =
\]

\[
= -\frac{1}{2\omega \varepsilon_0 \mu_0} \int_{E_\gamma} \left\{ |u^d(x)|^2 \text{ Im } [k^2(x)] \right\} dx \leq 0,
\]

(15)

where we used \( k^2(x) = k_0^2 n(x) \), with \( k_0^2 > 0 \) and \( \text{Im } [n(x)] \geq 0 \) by (1). Of course, if the domain \( E_\gamma \) is a subset of the lossless region \( \Omega_0 \), equality holds true at the end of (15).

It is now interesting to discuss the particular case where \( E_\gamma \) is a finite flow strip of \( u^d \) in the background medium. To this end, we firstly recall that, given a point \( x_1 \in \mathbb{R}^2 \setminus D \), the flow line of the plane vector field \( \mathcal{S}^d \) starting from \( x_1 \) is defined as the solution \( x(\tau) \) (for \( \tau \geq 0 \)) of the Cauchy problem

\[
\frac{dx}{dt}(\tau) = \mathcal{S}^d(x(\tau)), \quad x(0) = x_1.
\]

(16)

For simplicity, we shall often refer to such \( x(\tau) \) as the flow line \( \zeta_{x_1}(\tau) \) of \( u^d \) starting from \( x_1 \), although it is actually a flow line of \( \mathcal{S}^d \). We observe that if \( x_1 \) is taken inside one of the domains \( \Omega_j \), the real-analyticity of \( \mathcal{S}^d \) in \( \Omega_j \) (remarked above) ensures the local existence and uniqueness of \( \zeta_{x_1}(\tau) \); instead, if \( x_1 \) is chosen just on the boundaries \( \partial D \) or \( \partial \Omega_j \), the continuity of \( \mathcal{S}^d \) (in particular, the fact that it is not Lipschitz in a neighbourhood of \( x_1 \)) does not suffice to ensure the uniqueness of \( \zeta_{x_1}(\tau) \). In other terms, when \( x_1 \) belongs to a discontinuity curve of \( \bar{n}(x) \), it may be a ramification point of \( u^d \), i.e., a point whence more than one flow line of \( u^d \) can start. We shall denote with \( R := \left( \bigcup_{j=0}^J \partial \Omega_j \right) \cap (\mathbb{R}^2 \setminus D) \cup \partial D \) the subset of \( \mathbb{R}^2 \setminus D \) where ramification points may occur.

Then, let us consider finite flow strips of \( u^d \): they are open, connected and (almost everywhere) regular domains \( E_\gamma \subset \mathbb{R}^2 \setminus D \) obtained via the following construction. Take two distinct points \( x_1, x_2 \in \mathbb{R}^2 \setminus D \) and assume that they are not ramification points, nor do they belong to the same flow line. Consider the two corresponding flow lines \( \zeta_{x_1}(\tau) \), \( \zeta_{x_2}(\tau) \) and fix two other points, \( x_3 \) in the image of \( \zeta_{x_1}(\tau) \) and
in the image of $\zeta_2(\tau)$. Hence, the arcs $x_1x_3$ and $x_2x_4$ are finite portions of flow lines and form the lateral boundary of a flow strip. Next, we close the latter by tracing two transverse sections, i.e., we consider two non-intersecting curves $\gamma_1$ and $\gamma_2$ connecting $x_1$ with $x_2$ and $x_3$ with $x_4$ respectively. The closed curve $\gamma$ formed by $\gamma_1$, $\gamma_2$ and the arcs $x_1x_3$, $x_2x_4$ is the boundary of the open domain $E_\gamma$, which shall be called a finite flow strip of $u^d$. Now, by virtue of (15), we have that $F_\gamma(u^d) \leq 0$; on the other hand, the power flux across the two arcs of flow lines is clearly zero. This amounts to saying that $F_{\gamma_1}(u^d) + F_{\gamma_2}(u^d) \leq 0$, with outward unit normals $\nu_1$ to $\gamma_1$ and $\nu_2$ to $\gamma_2$; but we can change the orientation of one of them (say $\nu_1$, which then becomes inward), so that we find $F_{\gamma_2}(u^d) \leq F_{\gamma_1}(u^d)$. The last inequality expresses the property that the power outgoing from the finite flow strip through $\gamma_2$ is not greater than the power ingoing through $\gamma_1$, as expected in a possibly lossy background.

In general, the flow tubes of the Poynting vector field are responsible for the power transport throughout the propagation medium: in other terms, they transmit the information stored in the electromagnetic field from one point to another in space. In our physical interpretation of the LSM, we are interested in tracking this transport phenomenon in the case of the Poynting vector $S^d$ given by (13). To this end, we need to assume that the flow strips of $u^d$ starting from $\Gamma$ reach the observation circle $\Gamma$ with a regular behaviour. This means that a flow strip $E_\gamma$, with lateral boundary formed by two flow lines $\zeta_{x_1}(\tau)$ and $\zeta_{x_2}(\tau)$, should not intersect the scatterer $D$ (in particular, $\zeta_{x_1}(\tau)$ and $\zeta_{x_2}(\tau)$ should not re-fold on $D$); furthermore, both $\zeta_{x_1}(\tau)$ and $\zeta_{x_2}(\tau)$ must intersect $\Gamma$ in exactly one point. We shall fix these properties by means of appropriate definitions in the following.

We are now ready to use the concepts introduced above for investigating the modified far-field equation (9) from a power-flux perspective. Then, we conclude this section by pointing out that, although focused on the field $u^d$, the previous discussion can be easily extended to any scattered field.

4. Power fluxes and the modified far-field equation: far-field measurements

As a first step, let us assume, for the moment, that the observation circle $\Gamma$ is placed in the far-field region, i.e., $R_\Gamma \to \infty$: this assumption simplifies our framework and allows us to rely on the results of [12]. We shall discuss the case of near-field measurements in Section 6; in any case, the emitters are placed on the near-field circle $\Lambda$.

We begin by stating the concept of (far-field) regularity for the flow lines and flow strips of the field $u^d$.

**Definition 4.1:** given a point $x_1 \in \mathbb{R}^2 \setminus D$, a flow line $\zeta_{x_1}(\tau)$ of $u^d$ starting from $x_1$ is called regular if the following properties hold:

1. $\zeta_{x_1}(\tau) \cap D = \emptyset \ \forall \tau \in [0, +\infty)$;
2. $\forall R \geq \bar{R}$, with $\bar{R} > 0$ large enough, $\exists! \tau_1(R) > 0$ such that $P_{x_1}(R) := \zeta_{x_1}[\tau_1(R)] \cap \{ x \in \mathbb{R}^2 : |x| = R \} \neq \emptyset$;
3. if $(R, \varphi[P_{x_1}(R)])$ denote the polar coordinates of the point $P_{x_1}(R)$, then $\exists \lim_{R \to \infty} \varphi[P_{x_1}(R)] =: \varphi_\infty(x_1)$.

Moreover, an infinite flow strip $E_\gamma$ of $u^d$ is called regular if it is delimited by two distinct and regular flow lines $\zeta_{x_1}(\tau)$, $\zeta_{x_2}(\tau)$, and $E_\gamma \cap D = \emptyset$. If $\varphi_\infty(x_1)$ and $\varphi_\infty(x_2)$ are the asymptotic polar angles of $\zeta_{x_1}(\tau)$ and $\zeta_{x_2}(\tau)$ respectively, the quantity $w_\infty(x_1, x_2) := |\varphi_\infty(x_1) - \varphi_\infty(x_2)|$ is called the far-field width of the flow strip $E_\gamma$. 
Point (3) states that the flow line $\zeta_1(\tau)$ must approach a definite direction in the far-field region. This is in agreement with the Silver-Müller radiation condition [17], which entails the transversality of radiating electric and magnetic fields far enough from the source, and then the radiality of the corresponding Poynting vector field.

Then, by using for the field $u^d$, given by (7), the asymptotic behaviour of radiating solutions of the Helmholtz equation [17], i.e.,

$$u^d(x) = \frac{e^{ik_0r}}{\sqrt{r}} u^d_\infty(\varphi) + O\left(r^{-3/2}\right) \quad \text{as } r \to \infty$$  \tag{17}

(where $(r, \varphi)$ are the polar coordinates of $x$), and remembering Sommerfeld radiation condition (4)(c), it is possible to show that (cf. Theorem 8.17 in [17] for the three-dimensional case)

$$\bar{u}^d \frac{\partial u^d}{\partial r}(r, \varphi) = \frac{ik_0}{r} \left| u^d_\infty(\varphi) \right|^2 + o\left(r^{-1}\right) \quad \text{as } r \to \infty,$$  \tag{18}

which holds uniformly in $\varphi$.

Now, let us consider an infinite regular flow strip delimited by two regular flow lines $\zeta_{x_1}(\tau)$ and $\zeta_{x_2}(\tau)$ of $u^d$, with asymptotic polar angles $\varphi_\infty(x_1) \leq \varphi_\infty(x_2)$ respectively: then, by virtue of (14) and (18), we can compute the power outgoing from the far-field observation angle $[\varphi_\infty(x_1), \varphi_\infty(x_2)]$ as

$$F_{[\varphi_\infty(x_1), \varphi_\infty(x_2)]}(u^d) = \frac{1}{4i\omega \varepsilon_0 \mu_0} \lim_{R \to \infty} \int_{\varphi[\varphi_\infty(R)]} \left[ \bar{u}^d \frac{\partial u^d}{\partial \nu} - u^d \frac{\partial \bar{u}^d}{\partial \nu} \right] (R, \varphi) R \, d\varphi =$$

$$= \frac{2ik_0}{4i\omega \varepsilon_0 \mu_0} \int_{\varphi_\infty(x_1)}^{\varphi_\infty(x_2)} \left| u^d_\infty(\varphi) \right|^2 \, d\varphi = \frac{k_0}{2\omega \varepsilon_0 \mu_0} \left\| u^d_\infty \right\|^2_{L^2[\varphi_\infty(x_1), \varphi_\infty(x_2)]}.$$  \tag{19}

In the far-field approximation, the modified far-field operator (8) becomes

$$F_\infty : L^2(\Lambda) \to L^2[0, 2\pi], \quad (F_\infty g)(\varphi) := \int_{\Lambda} u^d_\infty(\varphi; y) g(y) \, ds(y)$$  \tag{20}

by virtue of (17), and the corresponding modified far-field equation reads

$$(F_\infty g_\epsilon)(\varphi) = u_\infty(\varphi; z), \quad \varphi \in [0, 2\pi],$$  \tag{21}

where $u_\infty$ is the far-field pattern of the field $u$ solving problem (4). With obvious changes, Theorem 2.1 remains valid in this far-field approximation [15]: in particular, an $\epsilon$-approximate solution $g_\epsilon^\epsilon$ of (21) satisfies

$$\|(F_\infty g_\epsilon^\epsilon)(\cdot) - u_\infty(\cdot; z)\|_{L^2[0, 2\pi]} \leq \epsilon.$$  \tag{22}

Hence, for any observation interval $[\varphi_1, \varphi_2] \subset [0, 2\pi]$ in the far-field region, we have

$$\left\| (F_\infty g_\epsilon^\epsilon)(\cdot) \right\|_{L^2[\varphi_1, \varphi_2]} - \left\| u_\infty(\cdot; z) \right\|_{L^2[\varphi_1, \varphi_2]} \left\| (F_\infty g_\epsilon^\epsilon)(\cdot) - u_\infty(\cdot; z) \right\|_{L^2[\varphi_1, \varphi_2]} \leq \epsilon.$$  \tag{23}
Since, in general, the following implications hold:

\[ a, b \geq 0, \ |a - b| \leq \epsilon \ \Rightarrow \ a + b \leq 2b + \epsilon \ \Rightarrow \ |a^2 - b^2| = |a - b| (a + b) \leq \epsilon^2 (2b + \epsilon), \]

(24)

setting \( a = \|(F_\infty g^\epsilon_z)(\cdot)\|_{L^2[\varphi_1, \varphi_2]} \) and \( b = \|u_\infty(\cdot; z)\|_{L^2[\varphi_1, \varphi_2]} \) yields

\[ \left| \|(F_\infty g^\epsilon_z)(\cdot)\|_{L^2[\varphi_1, \varphi_2]}^2 - \|u_\infty(\cdot; z)\|_{L^2[\varphi_1, \varphi_2]}^2 \right| \leq \epsilon \left( 2 \|u_\infty(\cdot; z)\|_{L^2[\varphi_1, \varphi_2]} + \epsilon \right). \]  (25)

By decreasing \( \epsilon \), one can make the right-hand side of (25) arbitrarily small. We also note that, by superposition [17], \((F_\infty g^\epsilon_z)(\varphi)\) is the far-field pattern of the field

\[ u^{d, \epsilon}_z(x) := \int_\Lambda u^d(x; y) g^\epsilon_z(y) ds(y), \quad x \in \mathbb{R}^2. \]  (26)

Then, remembering (19), inequality (25) allows regarding the modified far-field equation (21) as a constraint on energy fluxes: indeed, (25) means that the power \( F_{[\varphi_1, \varphi_2]}(u^{d, \epsilon}_z) \) outgoing from any observation interval \([\varphi_1, \varphi_2]\) and radiated by the field \( u^{d, \epsilon}_z \) (defined in (26)) can be made arbitrarily close to the power \( F_{[\varphi_1, \varphi_2]}(u(\cdot; z)) \) outgoing from the same interval and radiated by an elementary source located at the sampling point \( z \). In this perspective, requiring that the presence of such source is always detectable in \([\varphi_1, \varphi_2]\), in spite of the power loss along the path from the source itself to the receivers, is not too restrictive: it simply prevents the term \( u_\infty(\varphi; z) \) in (20) from becoming (in part or at all) irrelevant. This requirement, formalized in the following Assumption 1, is ultimately made on \( n(x) \): it means that the absorption due to its imaginary part must not be too large.

**Assumption 1:** for any observation interval \([\varphi_1, \varphi_2]\) in the far-field region, there exists a constant \( c_{1.2} > 0 \) such that \( \|u_\infty(\cdot; z)\|_{L^2[\varphi_1, \varphi_2]}^2 \geq c_{1.2} \) for all \( z \) in the investigation domain \( T \).

We are now in a position to conclude that all the theorems and results presented in [12] are still true (by slight changes in the proofs) in the current framework, provided that the flow lines of \( u^d \) behave appropriately. In order to investigate this behaviour, a numerical approach seems to be more viable and direct than an analytical one: then, in the next section, we shall present and discuss the results of such investigation.

5. Numerical investigation: far-field measurements

We performed several numerical experiments in different scattering conditions, and the purpose of this section is to summarize their results by focusing on a single, but representative enough, simulation. The frequency of our experiment is \( \nu = 1.0 \) GHz, corresponding to a wavelength \( \lambda_0 \equiv \lambda = 3.0 \cdot 10^{-1} \) m in vacuum. The background is formed by an ellipse and the vacuum outside it. The ellipse, characterized by relative permittivity \( \epsilon_r = 2.0 \) and conductivity \( \sigma = 1.0 \cdot 10^{-2} \) S·m\(^{-1}\), is centred at the origin of the investigation domain, and its semiaxes have lengths \( a = 4.0 \cdot 10^{-1} \lambda, \ b = 2.5 \cdot 10^{-1} \lambda \). The scatterer is a circle centred in \((2.0, 0.0) \cdot 10^{-1} \lambda \) and with radius \( r = 1.0 \cdot 10^{-1} \lambda \); its electrical parameters are \( \epsilon_r = 5.0 \) and \( \sigma = 3.0 \cdot 10^{-2} \) S·m\(^{-1}\).

We use \( N = 20 \) emitters, uniformly spaced on a circle \( \Lambda \) of radius \( R_\Lambda = 1.0 \lambda \), and \( N = 20 \) receivers, uniformly spaced on a far-field circle \( \Gamma \) of radius \( R_\Gamma = 1.2 \cdot 10^1 \lambda \).

The direct scattering data are computed by means of a code based on the method of moments [19], and are then collected into a \( N \times N \) matrix \( F \); to this end, the
investigation domain $T$ is discretized into square cells whose side is shorter than $\frac{\delta}{3}$ [20], where $\delta$ is the minimum skin depth of the propagation media. Each entry of $\mathbf{F}$ is then affected by 3% Gaussian noise. Next, these noisy data are inverted by means of a standard implementation of the LSM, as described e.g. in [2]: in particular, Tikhonov regularization, together with Morozov’s discrepancy principle, is applied to a discretized and noisy version of (21), in order to compute its regularized solution $g_{\alpha}(z) \in \mathbb{C}^N$ for each sampling point $z$ in a grid $Z$ covering $T$. In Figure 1 we show the visualization obtained by plotting $-\ln \|g_{\alpha}(z)\|_{\mathbb{C}^N}$ (rescaled to $[0, 1]$) as indicator function on a square domain $T$ of side $1.4 \lambda$. We superimpose a dashed and a solid line to represent the true profiles of the background ellipse and of the circular scatterer, respectively: as expected, only the latter is visualized.

The direct code and the LSM-algorithm provide a discretized version of the integrand in (26): then, for each $z$ of interest, we can easily compute the field $u_{d,\epsilon}^z$ and the corresponding Poynting vector field $S_{d,\epsilon}^z$ (cf. (13)) on an appropriate grid $\mathcal{X}$ of points. Hence, for a given $z \in Z$, the behaviour of the flow lines of $S_{d,\epsilon}^z$ can be visualized by plotting the unit vector $\hat{S}_{d,\epsilon}^z$ on the points of $\mathcal{X}$, where $\hat{S}_{d,\epsilon}^z$ is obviously obtained by normalizing $S_{d,\epsilon}^z$.

The typical behaviour of $\hat{S}_{d,\epsilon}^z$ can be described as follows. When the sampling point $z$ is taken inside the scatterer or even on its boundary $\partial D$, the flow lines are almost radially divergent from $z$ itself, thus resembling the behaviour of the Green’s function $u(\cdot, z)$ (cf. (4)) of the mere background: this situation is visualized in Figure 2 for $z \in D$ and in Figure 3 for $z \in \partial D$. More precisely, panel (a) of Figure 2 shows the behaviour of the vector field $\hat{S}_{d,\epsilon}^z$ when $z$ is chosen as the centre of the
circular scatterer, while panel (b) of Figure 2 shows the behaviour of the normalized Poynting vector field associated with the Green’s function $u(\cdot, z)$ for such $z$: the two panels are essentially identical. The same situation occurs in Figure 3, which is analogous to Figure 2, the only difference being that the sampling point is now on the boundary $\partial D$ of the scatterer. The circle $\Gamma$ where the receivers are placed cannot be contained in these pictures, but no anomaly of the flow lines is observed when the visualization domain is extended up to the far-field region. In particular, the flow lines starting from $\partial D$ are regular in the sense of Definition 4.1, and any $z \in \partial D$ is a ramification point of $u^d_{z^*}$, i.e., a beam of several flow lines originates from $z$.

Let us briefly recall how this behaviour of the flow lines allows concluding that the $L^2[0, 2\pi]$-norm of the approximate solution $g^\epsilon_z$ of Eq. (21) must blow up as the sampling point $z$ approaches a boundary point $z^* \in \partial D$ (for details, we refer to [12]). Consider a sequence of points $\{z_n\}_{n=0}^\infty \subset D$ such that $\lim_{n \to \infty} |z_n - z^*| = 0$ and a corresponding sequence of collapsing circles $C_{z_n}(z^*)$ centred at $z^*$ and with radius $r_n := |z_n - z^*| n \to \infty$. Let $\tilde{C}_{z_n}(z^*)$ be the portion of the circle not contained in $D$, i.e., $\tilde{C}_{z_n}(z^*) := C_{z_n}(z^*) \cap (\mathbb{R}^2 \setminus D)$, and let us assume that, for each $n \in \mathbb{N}$, the flow strip of $u^d_{z_n}$ starting from $\tilde{C}_{z_n}(z^*)$ is regular, with non-vanishing far-field width as $n \to \infty$, in the sense of Definition 4.1. If we assume, by contradiction, that the sequence $\{g^\epsilon_{z_n}\}_{n=0}^\infty$ is bounded in $L^2[0, 2\pi]$, two contradictory facts can be proved. First, we can show that the power flux $\mathcal{F}_\tilde{C}_{z_n}(u^d_{z_n})$ across $\tilde{C}_{z_n}$ vanishes as $n \to \infty$, since the length $|\tilde{C}_{z_n}|$ of the integration domain $\tilde{C}_{z_n}$ tends to zero, while the integrand function remains bounded. Second, the same flux $\mathcal{F}_\tilde{C}_{z_n}(u^d_{z_n})$ must be greater or equal to the far-field flux of the regular flow strip starting from $\tilde{C}_{z_n}$: but, according to inequality (25), by choosing $\epsilon$ small enough this far-field flux can be made arbitrarily close to that of the elementary source $u(\cdot; z_n)$ across the same (non-vanishing) far-field width of the flow strip. Then, by Assumption 1 and inequality (25) respectively, the far-field fluxes of both $u(\cdot; z_n)$ and $u^d_{z_n}$ are bounded from below by a positive constant, and then the same property holds for the near-field flux $\mathcal{F}_\tilde{C}_{z_n}(u^d_{z_n})$. This contradiction is removed by dropping the hypothesis of boundedness for the sequence $\{g^\epsilon_{z_n}\}_{n=0}^\infty$.

Let us now see what happens when the sampling point $z$ is taken outside the scatterer $D$. The situation is shown in Figure 4: its two panels (a) and (b) are now very different, i.e., the vector field $\mathcal{S}^\epsilon_{z^*}$ (panel (a)) does not resemble the Poynting vector field associated with the background Green’s function $u(\cdot, z)$ (panel (b)). Moreover, a ramification point $z_0$ of $u^\epsilon_{z^*}$ is detectable on the boundary $\partial D$, as highlighted by the small black box in panel (a).

The behaviour of the flow lines outgoing from $z_0$ suggests introducing the following

**Definition 5.1:** the field $u^d$ defined in (7) is called partially pseudo-radial with respect to a ramification point $z_0 \in R$ if there exist at least two regular flow lines of $u^d$, say $\zeta^1(t)$ and $\zeta^2(t)$, starting from $z_0$ and with asymptotic polar angles $\varphi^1_{z_0}(z_0)$, $\varphi^2_{z_0}(z_0)$ respectively, such that the flow strip delimited by them is regular with non-zero far-field width $w^2_{\infty}(z_0) := |\varphi^1_{\infty}(z_0) - \varphi^2_{\infty}(z_0)| > 0$. Moreover, if $\{\varphi^i_{\infty}(z_0)\}_{i \in I}$ is the set of the asymptotic polar angles of all such flow lines, we define the far-field width of this beam of flow lines as $w_{\infty}(z_0) := \sup_{i,j \in I} |\varphi^i_{\infty}(z_0) - \varphi^j_{\infty}(z_0)| > 0$.

We can now apply a contradiction argument very similar to the previous one, to conclude that, if $\epsilon$ is small enough, there cannot exist $g^\epsilon_z \in L^2[0, 2\pi]$ such that the field $u^d_{z^*}$ is partially pseudo-radial with respect to a ramification point $z_0 \in R$ and
Figure 2. (a) Behaviour of the vector field $\hat{S}_{d,\epsilon}^{z}$ (for far-field measurements) when the sampling point $z$ (red bullet) is taken inside the scatterer; (b) behaviour of the Poynting vector field associated with the background Green’s function $u(\cdot, z)$. 
Figure 3. (a) Behaviour of the vector field $\hat{S}_{d,\epsilon}^z$ (for far-field measurements) when the sampling point $z$ (red bullet) is taken on the boundary of the scatterer; (b) behaviour of the Poynting vector field associated with the background Green’s function $u(\cdot, z)$.
Figure 4. (a) Behaviour of the vector field $\mathbf{S}^d_{\mathbf{r},z}$ (for far-field measurements) when the sampling point $z$ (red bullet) is taken outside the scatterer: a ramification point (black box) appears on its boundary; (b) behaviour of the Poynting vector field associated with the background Green’s function $u'(\cdot,z)$. 
inequality (22) is satisfied.

We point out that in both cases, i.e., for \( z \to z^* \in \partial D \) and for \( z \notin \bar{D} \), the core of the argument is the fact that the far-field power flux outgoing from the regular flow strips of the field \( u_z^{d,e} \) is bounded from below by a positive constant, and Assumption 1 (which is trivially verified in the framework of [12]) is ultimately a way for ensuring this lower boundedness condition.

We can now summarize the above investigation by stating our version of the approximation Theorem 2.1: it is based on tracking the power loss in the flow strips of \( u_z^{d,e} \) starting from the boundary of the scatterer and reaching the observation curve \( \Gamma \), placed in the far-field region. This tracking is made possible by the behaviour of the flow lines, as observed in this section: in other terms, our theorem relies on some assumptions on the flow lines, but these assumptions do not follow from a theoretical investigation, but are rather the formalization of a numerically observed behaviour. In this sense, ours is an a posteriori approach. In spite of this limitation, the strong point of our new version of Theorem 2.1 is that any \( \epsilon \)-approximate solution of the far-field equation (9) behaves as a good indicator function (i.e., its \( L^2[0,2\pi] \)-norm blows up as \( z \) approaches \( \partial D \) and when \( z \notin \bar{D} \) for a small enough (but a priori non-vanishing) bound \( \epsilon \) on the discrepancy \( \| F_\infty g_z^\epsilon - u_\infty(\cdot, z) \|_{L^2[0,2\pi]} \).

**Theorem 5.2:** under Assumption 1 and the same hypotheses of Theorem 2.1:

(i) let \( z^* \in \partial D \) and let \( U_{z^*} \) be a neighbourhood of \( z^* \). For any \( \epsilon > 0 \) and for each \( z \in U_{z^*} \cap D \), let \( g_z^\epsilon \in L^2[0,2\pi] \) satisfy the inequality:

\[
\| Fg_z^\epsilon - u_\infty(\cdot, z) \|_{L^2[0,2\pi]} \leq \epsilon. \tag{27}
\]

Moreover, let \( C_z(z^*) \) be the circle with centre in \( z^* \) and radius \( r := |z - z^*| \), and define \( \bar{C}_z(z^*) := C_z(z^*) \cap (\mathbb{R}^2 \setminus D) \): for \( U_{z^*} \) small enough, \( \bar{C}_z(z^*) \) is an arc of extreme points \( y_1^z, y_2^z \in \partial D \) for each \( z \in U_{z^*} \cap D \). Finally, we assume that, for each \( z \in U_{z^*} \cap D \), the two flow lines \( \zeta_{y_1^z}(t), \zeta_{y_2^z}(t) \) of \( u_z^{d,e} \) (starting from \( y_1^z, y_2^z \) respectively) delimit a regular flow strip with far-field width \( w_\infty(z) := |\varphi_\infty(y_1^z) - \varphi_\infty(y_2^z)| \), such that \( w_\infty(z^*) := \lim_{z \to z^*} w_\infty(z) \neq 0 \), i.e., \( \varphi_\infty^1 := \lim_{z \to z^*} \varphi_\infty(y_1^z) \neq \varphi_\infty^2 := \lim_{z \to z^*} \varphi_\infty(y_2^z) \). Then, if \( \epsilon \) is small enough, it holds:

\[
\lim_{z \to z^*} \| g_z^\epsilon \|_{L^2(\Lambda)} = \infty; \tag{28}
\]

(ii) let \( z \in \mathbb{R}^2 \setminus D \); then, if \( \epsilon \) is small enough, there does not exist \( g_z^\epsilon \in L^2[0,2\pi] \) such that the field \( u_z^{d,e} \) is partially pseudo-radial with respect to a point \( z_0 \in \mathbb{R}^2 \) and inequality (27) is satisfied.

**Proof:** the proof is obtained by slight changes (obvious from the context) of the arguments proving Theorems 5.1 and 6.1 in [12]. \( \square \)

**Remark 1:** Theorem 5.2 deals with \( \epsilon \)-approximate solutions \( g_z^\epsilon \) of (21), then Tikhonov regularized solutions \( g_z^\alpha \) are a particular case. Since \( \| g_z^\alpha \|_{L^2[0,2\pi]} \) blows up only if \( \alpha \to 0 \), Theorem 5.2 foresees a vanishing \( \alpha \) for \( z \to z^* \in \partial D \) and for \( z \in \mathbb{R}^2 \setminus D \). This is in qualitative agreement with the well-known behaviour of \( \alpha \) in the LSM-algorithm: its values inside the scatterer are notably larger than those on its boundary or outside [2].
6. Power fluxes and the modified far-field equation: near-field measurements

It is now interesting to see how the above discussion on power fluxes can give an insight into the modified far-field equation (9) or, better, into its approximate version (10), for an observation circle \( \Gamma \) placed in the near-field region. Differently from the genuine far-field equation considered in [12], or from the case of far-field measurements as addressed in the previous section, at first sight inequality (10) seems to have nothing to do with power fluxes: indeed, as highlighted by definition (14), the power flux of any field involves both the field itself and its derivatives; on the other hand, having a control over the \( L^2(\Gamma) \)-norm of a field, as is the case for (10), does not entail, in general, any control over its derivatives and, consequently, over its flux.

We can give an alternative description of the above difficulty. The most direct way for adapting the results of the previous section to the case of near-field measurements consists in the following two steps: first, one should assume the (near-field) regularity of the flow strips, in the sense of

**Definition 6.1:** given a point \( x_1 \in \{ x \in \mathbb{R}^2 : |x| < R_\Gamma \} \setminus D \), a flow line \( \xi_x(\tau) \) of \( u^d \) starting from \( x_1 \) is called regular if:

1. \( \xi_x(\tau) \cap D = \emptyset \quad \forall \tau \in [0, +\infty) \);
2. \( \exists \tau_1 > 0 \) such that \( P_{x_1}(R_\Gamma) := \xi_x(\tau_1) \cap \Gamma \neq \emptyset \).

Moreover, a flow strip \( E_\gamma \) of \( u^d \) is called regular if it is delimited by two distinct and regular flow lines \( \xi_x(\tau) \), \( \xi_x(\tau) \), and \( E_\gamma \cap D = \emptyset \).

Second, it should be required that the power flux outgoing from the flow strips across their intersection with the observation circle \( \Gamma \) is bounded from below by a positive number. Nevertheless, the latter requirement is now very unnatural, since inequality (10) has a priori nothing to do with power fluxes.

In order to overcome this difficulty, we observe that the investigation domain \( T \subset \mathbb{R}^2 \) where the sampling point can vary is a closed region surrounded by the circle \( \Gamma \) where the receivers are placed, i.e., \( \hat{T} \subset B_{R_\Gamma} := \{ x \in \mathbb{R}^2 : |x| < R_\Gamma \} \). Accordingly, for each \( z \in \hat{T} \) and for every \( \epsilon > 0 \), the radiating field (cf. definition (26)) \( u^d(z) - u(\cdot ; z) \) satisfies the Helmholtz equation (with wavenumber \( k_0 \)) in \( \mathbb{R}^2 \setminus \hat{B}_{R_\Gamma} \); moreover, it certainly holds \( |u^d(z) - u(\cdot ; z)| \in C(\mathbb{R}^2 \setminus \hat{B}_{R_\Gamma}) \cap C(\mathbb{R}^2 \setminus B_{R_\Gamma}) \). Hence, by virtue of Theorem 3.21 in [17], we can conclude that there exists a constant \( C \) such that

\[
\| (F_{\infty}g^x_\gamma)(\cdot) - u_\infty(\cdot ; z) \|_{L^2(0,2\pi)} \leq C \bigg\| u^d(\cdot) - u(\cdot ; z) \bigg\|_{L^2(\Gamma)} \leq C \epsilon,
\]

where \( (F_{\infty}g^x_\gamma)(\cdot) - u_\infty(\cdot ; z) \) is the far-field pattern of \( u^d(\cdot) - u(\cdot ; z) \). Hence, having a near-field control over the \( L^2(\Gamma) \)-norm of the latter field enables a control over the \( L^2(0,2\pi) \)-norm of its far-field pattern (while the converse is clearly false, cf. [17] and Remark 3.1 in [12]). In particular, as seen in the previous section (cf. relations (19), (23) and (25)), we can make the total fluxes \( F_{[\varphi_1,\varphi_2]}(u^d) \) and \( F_{[\varphi,\varphi_2]}(u(\cdot ; z)) \) as close to each other as we want, by choosing \( \epsilon \) small enough. This observation has a twofold consequence.

First, if we choose \( [\varphi_1,\varphi_2] = [0, 2\pi] \), we have that the difference between the total fluxes \( F_{[0,2\pi]}(u^d) \) and \( F_{[0,2\pi]}(u(\cdot ; z)) \) in the far-field region can be made arbitrarily small; but the observation circle \( \Gamma \) is placed in the outmost lossless region, then it holds \( F_{[0,2\pi]}(u^d) = F_{\Gamma}(u^d) \) and \( F_{[0,2\pi]}(u(\cdot ; z)) = F_{\Gamma}(u(\cdot ; z)) \). Hence, inequality (10) entails a constraint on the two total power fluxes across \( \Gamma \), whereby they must
be close enough to each other. Note that equation (9), whence inequality (10) follows, is linear in the unknown \( g_\epsilon \), while imposing a constraint directly on power fluxes, e.g. via the equation \( \mathcal{F}_\Gamma (u^{d,e}_z) = \mathcal{F}_\Gamma (u(\cdot; z)) \), would give rise to a non-linear problem.

Second, if we consider the flow lines of the field \( u^{d,e}_z \) beyond the observation circle \( \Gamma \), i.e., up to the far-field region, the current framework becomes completely analogous to that of the previous section. Of course, Assumption 1 is now inelegant in that the detectability of the point source is required in the far-field region, while the measurements are made in the near-field. However, we can first replace Assumption 1 by its near-field analogous, i.e., by requiring that, for any circular portion \( \Gamma_{1,2} \subset \Gamma \), there exists a constant \( c_{1,2} \) such that \( \mathcal{F}_{\Gamma_{1,2}} (u(\cdot; z)) \geq c_{1,2} \) for all \( z \) in the investigation domain \( T \). Then, we can take into account the typical behaviour of the flow lines of the field \( u(\cdot; z) \) radiated by a point source placed at \( z \): as shown e.g. in panels (b) of Figures 2, 3 and 4, although they are not exactly radial with respect to \( z \), these flow lines typically spread out from \( z \) and are regular up to the far-field region. Accordingly, we can reasonably conclude that Assumption 1 is implied by its near-field analogous, owing to the conservation of the power flux carried by each flow strip through the lossless medium, i.e., from \( \Gamma \) to the far-field region.

Summarizing, the case of near-field observations can be made analogous to that of far-field measurements, as long as the flow lines of the scattered field \( u^{d,e}_z \) are considered not only inside \( B_{R_\epsilon} \), but in all \( \mathbb{R}^2 \), up to the far-field region. In particular, the near-field version of Theorem 5.2 is simply obtained by replacing inequality (27) with the more general (10).

Then, as a last check, let us repeat the numerical experiment considered in Section 5, by only changing the radius \( R_\Gamma \) of the observation circle \( \Gamma \), which has now to be in the near-field: we choose \( R_\Gamma = 1.0 \lambda \). The visualization provided by the LSM is given in Figure 5 and is almost identical to the previous one, plotted in Figure 1.

Then, Figures 6, 7 and 8 show the behaviour of the vector field \( \hat{S}^{d,e}_z \) for \( z \in D \), \( z \in \partial D \) and \( z \not\in D \) respectively, for the same sampling points considered in panels (a) of Figures 2, 3 and 4: for each of the three corresponding pairs of pictures, an immediate comparison between the two images highlights that they are very similar. Hence, also from a numerical viewpoint, the case of near-field measurements can be discussed within the framework outlined for far-field measurements.

### 7. Conclusions

Our energy-based approach inspires a new form of the approximation theorem at the basis of the LSM. This reformulation provides a better explanation of the behaviour of the indicator function exploited by the LSM. Further developments concern both theoretical and numerical issues. From the theoretical viewpoint, a major achievement would consist in predicting the key-features (regularity, ramification points, etc.) of the flow lines, starting from the knowledge of the scattering conditions: this would make our physical interpretation a mathematical justification of the LSM. From a numerical viewpoint, it is interesting to investigate how much the LSM-algorithm (which accounts for discretization, noise, regularization, etc.) deviates from the continuous and noise-free framework. For example, our Theorem 5.2 requires that \( \epsilon \) is small enough: its detailed proof, easily achievable from [12], would show that, depending on \( z \), \( \epsilon \) must be less than some upper bounds related to the far-field widths of appropriate flow strips. On the other hand, such
ACKNOWLEDGMENTS

Massimo Brignone is supported by a grant from the Istituto Nazionale d’Alta Matematica (INdAM), Roma, Italy.

REFERENCES

Figure 6. Behaviour of the vector field $\hat{S}_{d,\epsilon}^z$ (for near-field measurements) when the sampling point $z$ (red bullet) is taken inside the scatterer.


Figure 7. Behaviour of the vector field $\mathbf{S}^{d,\epsilon}_{z}$ (for near-field measurements) when the sampling point $z$ (red bullet) is taken on the boundary of the scatterer.
Figure 8. Behaviour of the vector field $\hat{S}_{d,\epsilon}^{d,\epsilon}$ (for near-field measurements) when the sampling point $z$ (red bullet) is taken outside the scatterer: a ramification point (black box) appears on its boundary.