The geometry of the statistical model for range-based localization

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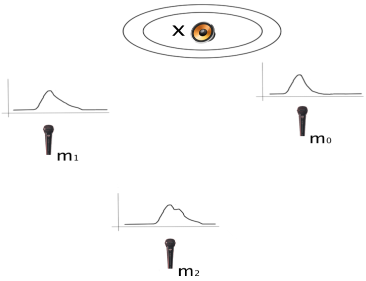
Range–based Localization

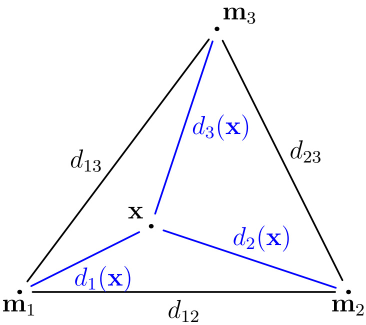
Problem: find the position of a point $\mathbf{x}$ from the range measurements between $\mathbf{x}$ and a set of given points $\mathbf{m_i}$, $i = 1, \ldots, n$.



Examples of applications:

- radar and active sonar
- molecular conformation
- wireless sensor networks

The Range Model



$$\mathbf{d_i}(\mathbf{x}) = \mathbf{x} - \mathbf{m_i} \qquad d_i(\mathbf{x}) = \|\mathbf{d_i}(\mathbf{x})\|$$

$$\mathbf{d_{ji}} = \mathbf{m_j} - \mathbf{m_i} \qquad d_{ji} = \|\mathbf{d_{ji}}\|$$

$$\mathcal{T}_{r,n} : \quad \mathbb{R}^r \longrightarrow \mathbb{R}^n$$
$$\mathbf{x} \longmapsto (d_1(\mathbf{x}), \ldots, d_n(\mathbf{x}))$$

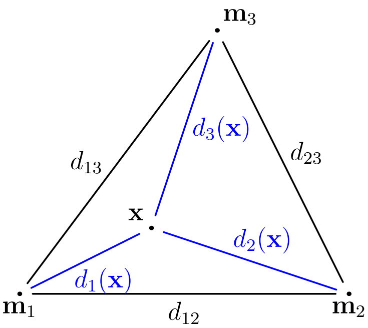
$\hat{d}_i(\mathbf{x}) =$ measured range
$\epsilon_i =$ measurement error $\Rightarrow$

$$\hat{d}_i(\mathbf{x}) = d_i(\mathbf{x}) + \epsilon_i$$

The model: $\widehat{\mathcal{T}}_{r,n}(\mathbf{x}) = (\hat{d}_1(\mathbf{x}), \ldots, \hat{d}_n(\mathbf{x})) \sim N(\mathcal{T}(\mathbf{x}), \mathbf{\Sigma})$

- **Deterministic problem:** if $\epsilon_i = 0$, find the conditions for existence and uniqueness of $\mathbf{x}$ (the identifiability problem).
- **Statistical problem:** if $\epsilon_i \neq 0$, efficiently estimate $\mathbf{x}$.

Euclidean Distance Geometry



The deterministic problem is a main topic of **Euclidean Distance Geometry** (DG) [Liberti and others, 2014].

Given a weighted graph $G = (V, E, W)$, with

- V the points $\mathbf{m_i}$ and $\mathbf{x}$
- E the available distances
- W the measured ranges

is G embeddable into some k-dimensional Euclidean space?

In DG the answer is usually given in terms of **Cayley–Menger determinant**.

The set of feasible ranges

Hypothesis:

- a point $\mathbf{x} \in \mathbb{R}^2$, thus $r = 2$;
- three known points $\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3} \in \mathbb{R}^2$, thus $n = 3$.

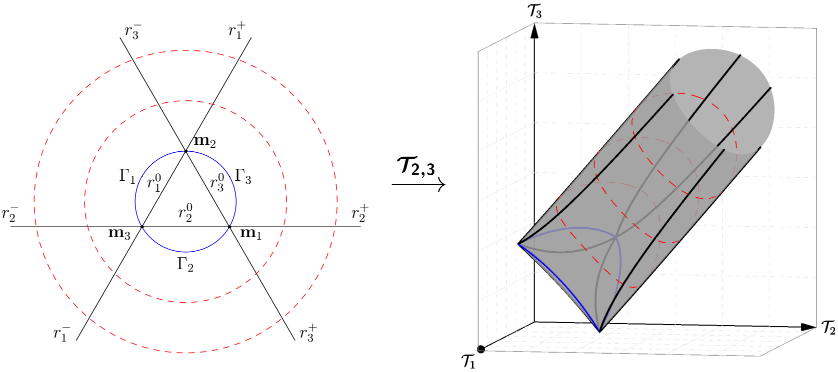
$\boldsymbol{\mathcal{T}} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathrm{Im}(\boldsymbol{\mathcal{T}_{2,3}})$ if and only if

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & \mathcal{T}_1^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & \mathcal{T}_2^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & \mathcal{T}_3^2 \\ 1 & \mathcal{T}_1^2 & \mathcal{T}_2^2 & \mathcal{T}_3^2 & 0 \end{vmatrix} = 0, \qquad \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \geq 0.$$

Proposition: the set of feasible ranges is the semialgebraic surface $X \subset \mathbb{R}^3$ defined by

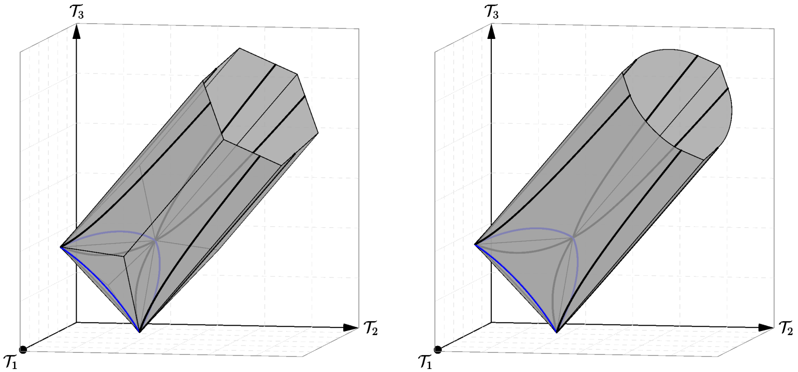
$$\begin{cases} d_{32}^2\mathcal{T}_1^4 + d_{31}^2\mathcal{T}_2^4 + d_{21}^2\mathcal{T}_3^4 - 2\mathbf{d_{32}}\cdot\mathbf{d_{31}}\mathcal{T}_1^2\mathcal{T}_2^2 + 2\mathbf{d_{32}}\cdot\mathbf{d_{21}}\mathcal{T}_1^2\mathcal{T}_3^2 - 2\mathbf{d_{31}}\cdot\mathbf{d_{21}}\mathcal{T}_2^2\mathcal{T}_3^2 - \\ -2\mathbf{d_{21}}\cdot\mathbf{d_{31}}d_{32}^2\mathcal{T}_1^2 + 2\mathbf{d_{32}}\cdot\mathbf{d_{21}}d_{31}^2\mathcal{T}_2^2 - 2\mathbf{d_{32}}\cdot\mathbf{d_{31}}d_{21}^2\mathcal{T}_3^2 + d_{21}^2d_{31}^2d_{32}^2 = 0, \\ \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \geq 0. \end{cases}$$

The Kummer's surface I



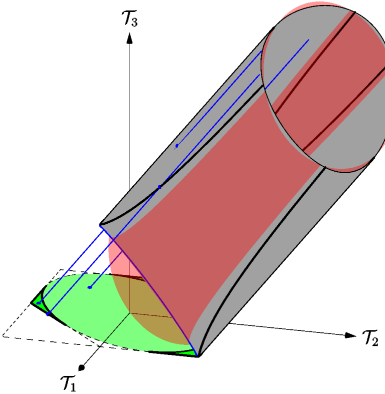
- $\bar{X}$ is a quartic surface with 16 nodes, thus $\bar{X}$ is a **Kummer's surface**. The nodes on X are the images of $\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}$.
- There exist 16 conics on $\bar{X}$. The conics on X are the images of $r_i^{\pm}$ and Γ_i, $i = 1, 2, 3$. They are asymptotic curves of X and divide the positive and negative curvature regions of X.

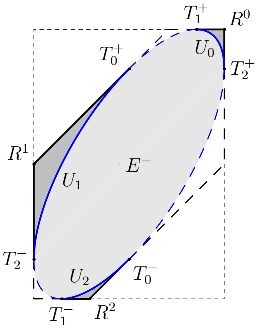
The Kummer's surface II



- There exist 16 planes (the tropes), each one tangent to $\bar{X}$ along one conic. The 12 tropes tangent to X come from the triangular inequalities plus some other geometrical arguments and they define a convex polyhedron Q_3 containing X.
- The boundary of the convex hull of X is the union of the positive curvature regions of X and slices of each facet of Q_3.

Pseudorange–based localization





- In some applications only the range differences or **pseudoranges** are available: $\tau_1(\mathbf{x}) = d_1(\mathbf{x}) - d_3(\mathbf{x}),\ \tau_2(\mathbf{x}) = d_2(\mathbf{x}) - d_3(\mathbf{x})$ [Compagnoni and others, 2013].
- The set of feasible pseudoranges is the projection $\pi(X)$ of X from its ideal singular point, where $\pi(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = (\mathcal{T}_1 - \mathcal{T}_3, \mathcal{T}_2 - \mathcal{T}_3)$.

Near and Far Field

In several applications one distinguishes between near and far field scenarios (e.g. distributed sensors versus compact arrays).

- **Near Field:** the point $\mathbf{x}$ is closed to (at least one) $\mathbf{m_i}$, $i = 1, 2, 3$. The **range model is singular**.
- **Far Field:** the point $\mathbf{x}$ is far away from $\mathbf{m_i}$, $i = 1, 2, 3$. A good approximation of the Kummer's surface is given by **the tangent cone to the ideal singular point of X**, i.e. the elliptic cylinder C having equation

$$d_{32}^2 \mathcal{T}_1^2 + d_{31}^2 \mathcal{T}_2^2 + d_{21}^2 \mathcal{T}_3^2 -$$

$$-2\mathbf{d_{31}} \cdot \mathbf{d_{32}} \mathcal{T}_1 \mathcal{T}_2 + 2\mathbf{d_{21}} \cdot \mathbf{d_{32}} \mathcal{T}_1 \mathcal{T}_3 - 2\mathbf{d_{21}} \cdot \mathbf{d_{31}} \mathcal{T}_2 \mathcal{T}_3 - \|\mathbf{d_{31}} \wedge \mathbf{d_{32}}\|^2 = 0.$$

Far Field estimation

Maximum Likelihood Estimation (MLE):

$$\overline{\mathcal{T}} = \underset{\mathcal{T}\in X}{\mathbf{argmin}} \|\widehat{\mathcal{T}} - \mathcal{T}\|^2$$

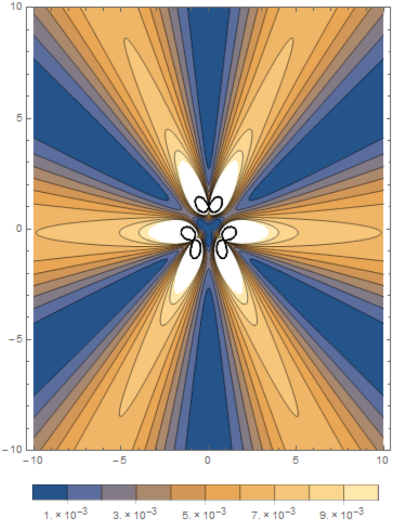
- asymptotically efficient estimator;
- nonconvex optimization;
- X has Euclidean Distance degree 20.

Squared–Range–based Least Square (SR–LS):
[Beck,Stoica,Li 2008]

$$\overline{\mathcal{T}} = \underset{\mathcal{T}\in X}{\mathbf{argmin}} \|\widehat{\mathcal{T}}^2 - \mathcal{T}^2\|^2$$

- it is not first order efficient;
- although nonconvex, there exist efficient solution methods;
- it is equivalent to MLE with respect to Cayley–Menger variety, an elliptic paraboloid with Euclidean Distance degree 5.

SR-LS performance



Scenario:

$\mathbf{m_1}=(-\frac{\sqrt{3}}{2},-\frac{1}{2})$, $\mathbf{m_2}=(\frac{\sqrt{3}}{2},-\frac{1}{2})$, $\mathbf{m_3}=(0,1)$

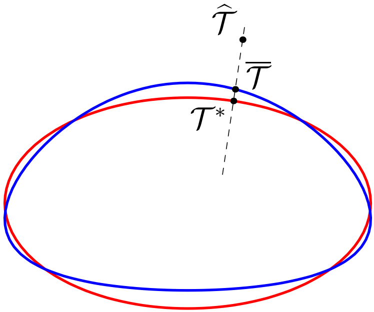
$\widehat{\mathcal{T}}(\mathbf{x})\sim N(\mathcal{T}(\mathbf{x}),\sigma^2\,\mathbf{I})$, $\sigma=0.1$

Asymptotic Inference:

- the inverse $G(\mathbf{x})$ of the Fisher matrix gives the asymptotic mean square error of the MLE;
- by Cramér-Rao inequality, the asymptotic mean square error $\bar{G}(\mathbf{x})$ of any consistent and unbiased estimator satisfies $\bar{G}(\mathbf{x})-G(\mathbf{x})\succeq 0$.

Proposition: $\bar{G}(\mathbf{x})-G(\mathbf{x})$ has only a non–zero eigenvalue $\lambda(\mathbf{x})$.

Orthogonal projection on C

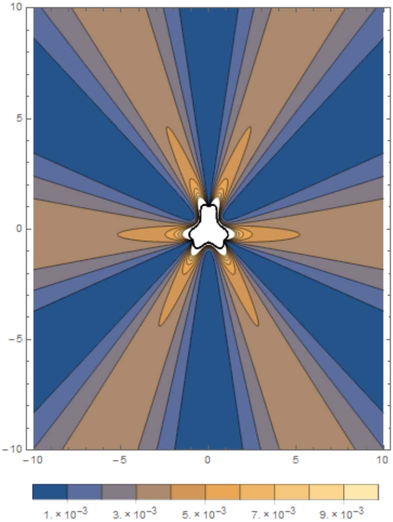


Algorithm (OPC):

- find the nearest point $\mathcal{T}^* \in C$ to $\widehat{\mathcal{T}}$;
- find the line $L_{\widehat{\mathcal{T}}}$ containing $\widehat{\mathcal{T}}, \mathcal{T}^*$;
- the estimate $\overline{\mathcal{T}}$ is the intersection of $L_{\widehat{\mathcal{T}}}$ and X closest to $\widehat{\mathcal{T}}$.

- OPC is a **consistent estimator**;
- the orthogonal projection on C is a two dimensional problem with Euclidean Distance degree 4, then to find $L_{\widehat{\mathcal{T}}} \cap X$ we have to solve a degree 4 polynomial equation;
- in far field regime we expect to have existence and uniqueness of the solution of OPC (at least in a local setting).

OPC performance



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Results:

- **OPC performs better than SR-LS in far field regime**, while it is not suitable for near field localization;
- OPC has a **lower algebraic computational complexity** with respect to MLE;
- similar results have been obtained for more general sensor configurations and in the analysis of the bias.

Conclusions and Perspectives

In our work:

- we studied the range-based localization problem with two and three sensors in terms of real algebraic geometry;
- we have characterized the measurements space using classical results on Kummer's surfaces;
- we began the study of the estimation problem.

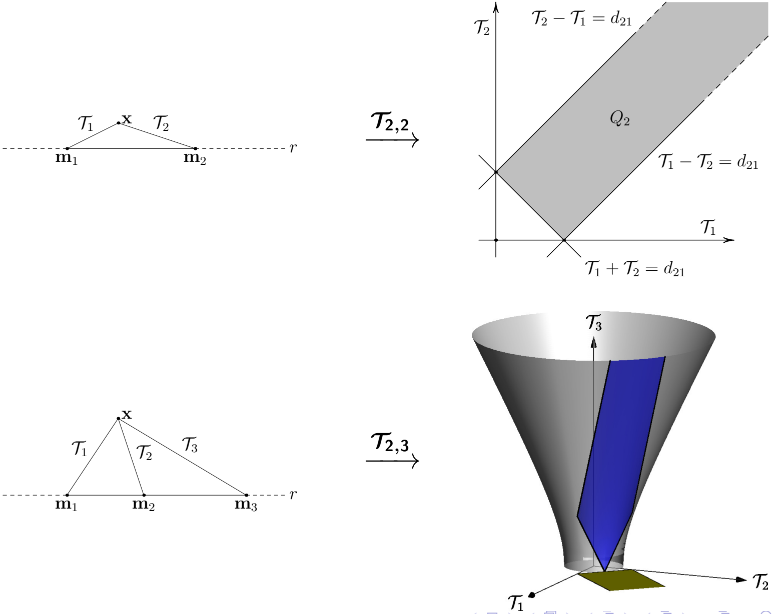
In future works we will:

- complete the analysis of near and far field estimation (singular model, second order efficient estimators [Kobayashi,Wynn 2013]);
- extend our analysis to the cases with $n > 3$ sensors.

Bibliography

- M.Compagnoni, R. Notari, A.A. Ruggiu, F.Antonacci, A.Sarti, *The Algebro–Geometric Study of Range Maps*, preprint, 2015.
- L. Liberti, C. Lavor, N. Maculan, and A. Mucherino. *Euclidean distance geometry and applications*, SIAM REVIEW, 56(1):3-69, 2014.
- R.W.H.T. Hudson. *Kummer's quartic surface*, Cambridge University Press, 1990.
- M.Compagnoni, R. Notari, F.Antonacci, A.Sarti, *A comprehensive analysis of the geometry of tdoa maps in localization problems*, Inverse Problems, 30(3):035004, 2014.
- A. Beck, P. Stoica, and Jian Li. *Exact and approximate solutions of source localization problems*, IEEE Transactions on Signal Processing (TSP), 56:1770-1778, 2008.

Aligned sensors



SR-LS versus OPC