Symbolic methods in statistics: elegance towards efficiency

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June, 10th 2015 – Genova

Algebraic Statistics 2015

How to convince people that in programming, simplicity and clarity - in short what mathematicians call elegance - are a crucial matter that decides between success and failure? (E. Dijkstra)

The symbolic approach to combinatorial enumerations

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Systematic relations between some of the major constructions of discrete mathematics (words, trees, graphs, and permutations) and operations on generating functions that exactly encode counting sequences.

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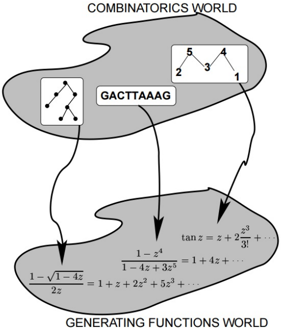

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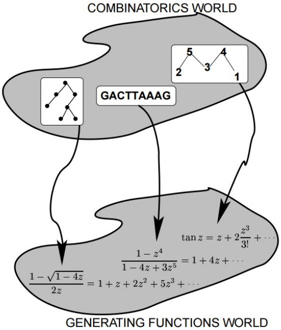

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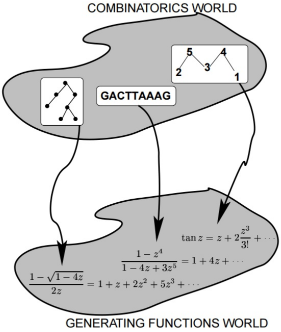

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- ▷ Flajolet, P. and Sedgewick, R. (2009) Analytic Combinatorics, Cambridge Univ. Press
- ▷ Roman, S.M. and Rota, G.-C. (1978) The umbral calculus, Adv.Math.

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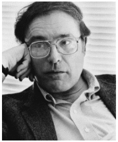
the "magic art" of lowering and raising exponents: $a_n \rightarrow a^n$



Gian-Carlo Rota

The classical umbral calculus

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The umbral calculus: an approach to combinatorial sequences using cumulants.

▷ Sturmfels, B. and Zwiernik, P. (2011) Binary cumulant varieties, Annals of Combinatorics

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Although the notation satisfied the most ardent advocate of spic-and-span rigor, the translation of "classical" umbral calculus into the newly found rigorous language made the method altogether unwieldy and unmanageable. Not only was the eerie feeling of witchcraft lost in the translation, but, after such a translation, the use of calculus to simplify computation and sharpen our intuition was lost by the wayside.

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What I mean by *symbolic methods*

A set of manipulation techniques aiming to perform algebraic calculations (possibly) through an algorithmic approach in order to find efficient mechanical processes to pass to a computer.

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What about computation cost?

- Efficiency is not so obvious.
- Sometimes a consequence of a different viewpoint.

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Why not a symbolic cumulant calculus?

⇝ Very close to the moment method for random matrices.

⇝ Commutative counterpart of free probability.

U-statistics

An appropriate choice of language and notation can simplify and clarify many statistical calculations.

▷ McCullagh, P. (1987) Tensor Methods in Statistics. Chapman & Hall, London

U-statistics

$$U = \frac{1}{(n)_m} \sum \Phi(X_{j_1}, X_{j_2}, \ldots, X_{j_m})$$

- symmetric polynomial in $(X_1, X_2, \ldots, X_n)$ i.i.d.r.v.'s;
- the sum ranges over the set of all permutations $(j_1, j_2, \ldots, j_m)$.

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Augmented symmetric polynomials vs moments

If $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m}) \vdash n$ is a partition of $r_1 + 2r_2 + \cdots + m\, r_m = n$ of length $r_1 + r_2 + \cdots + r_m = l(\lambda)$ and $E[X_i^j] = a_j$, $j = 1, 2, \ldots, m$ then

$$E\left[\sum \underbrace{X_s X_t \cdots}_{r_1} \underbrace{X_q^2 X_r^2 \cdots}_{r_2} \underbrace{X_u^m X_\nu^m \cdots}_{r_m}\right] = (n)_{l(\lambda)}\, a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}$$

▷ Stuart, A. and Ord, J.K. (1994) Kendall's Advanced Theory of Statistics. Vol. 1: Distribution Theory Edward Arnold, London (Section 12.5)

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assume that ⇑ could be "symbolically represented" by ⇓

$$\prod_{j=1}^{m} (\chi_1 X_1^j + \chi_2 X_2^j + \cdots + \chi_n X_n^j)^{r_j}$$

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$$\prod_{j=1}^{m} (\chi_1 X_1^j + \chi_2 X_2^j + \cdots + \chi_n X_n^j)^{r_j}$$

having a structure very similar to $a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}$

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How? $\triangleright\ \mathbb{E}[\chi_j^i] = \begin{cases} 1 & i = 0,1 \\ 0 & \text{otherwise} \end{cases}$

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$\overset{\text{No}}{\Rightarrow} \mathbb{E}[\chi_i^2] = 0$

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Computing

$$E\left[\left(\sum_{i\neq j}^{n} X_i^2 X_j\right)\left(\sum_{i=1}^{n} X_i^2 Y_i\right)^2\right]$$ with $(X_1, Y_1), \cdots, (X_n, Y_n)$ separately i.i.d.r.v.'s

▸ Push

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Push ... and then: (with $\mu_{i,j} = E[X^i Y^j]$)

$$2\,(n)_2\,[2\,\mu_{4,1}\,\mu_{3,1} + \mu_{5,2}\,\mu_{2,0} + \mu_{6,2}\,\mu_{1,0}] + 2\,(n)_3\,\mu_{3,1}\,\mu_{2,1}\,\mu_{2,0} + (n)_3\,[2\mu_{4,1}\,\mu_{2,1}\,\mu_{1,0} + \mu_{4,2}\,\mu_{2,0}\,\mu_{1,0}] + (n)_4\,\mu_{2,1}^2\,\mu_{2,0}\,\mu_{1,0}$$

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$$(n)_3\,[2\mu_{4,1}\,\mu_{2,1}\,\mu_{1,0} + \mu_{4,2}\,\mu_{2,0}\,\mu_{1,0}] + (n)_4\,\mu_{2,1}^2\,\mu_{2,0}\,\mu_{1,0}$$

- In a reasonable ammount of time
- In a form suitable for any symbolic language

...up to random matrices

Computing

▸ Algorithm

$$E\left\{\mathrm{Tr}\left[W_p(n)H_1\right]\mathrm{Tr}\left[W_p(n)H_2\right]^2\right\} \qquad \text{with} \quad H_1, H_2 \in \mathbb{C}^{p\times p}$$

$$W_p(n) = \sum_{i=1}^{n}(\boldsymbol{X}_i - \boldsymbol{m}_i)^{\dagger}(\boldsymbol{X}_i - \boldsymbol{m}_i) \text{ and } \boldsymbol{X}_i \sim N(\boldsymbol{m}_i, \Sigma)$$

<Wishart random matrix>

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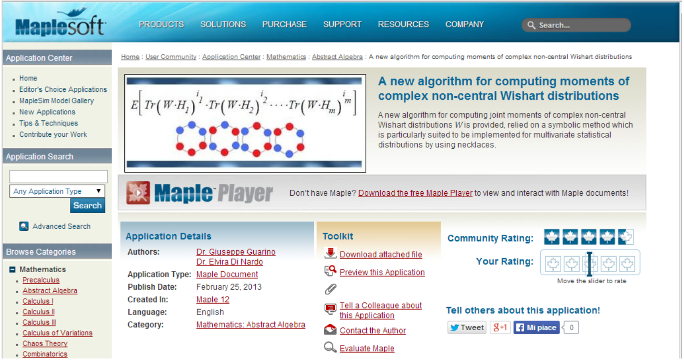
▸ Push ... and then: $\left(\text{with } \Omega = \Sigma^{-1}M \text{ and } M = \sum_{i=1}^{n} \boldsymbol{m}_i^{\dagger}\boldsymbol{m}_i \right)$

$$\begin{aligned}
& E\left\{\mathrm{Tr}\left[W_p(n)H_1\right]\mathrm{Tr}\left[W_p(n)H_2\right]^2\right\} = n\mathrm{Tr}\left(H_2\right)\mathrm{Tr}\left(\Omega H_1 H_2\right) - n\mathrm{Tr}\left(H_2\right)\mathrm{Tr}\left(\Omega H_2 H_1\right) \\
+ \quad & n\mathrm{Tr}\left(H_2\right)\mathrm{Tr}\left(\Omega H_1\right)\mathrm{Tr}\left(\Omega H_2\right) - n\mathrm{Tr}\left(\Omega H_2\right)\mathrm{Tr}\left(H_1 H_2\right) - n^2\mathrm{Tr}\left(\Omega H_2\right)\mathrm{Tr}\left(H_1\right)\mathrm{Tr}\left(H_2\right) \\
+ \quad & 2\,\mathrm{Tr}\left(\Omega H_2\right)\mathrm{Tr}\left(\Omega H_1 H_2\right) + 2\,\mathrm{Tr}\left(\Omega H_2\right)\mathrm{Tr}\left(\Omega H_2 H_1\right) - \mathrm{Tr}\left(\Omega H_1\right)\left(\mathrm{Tr}\left(\Omega H_2\right)\right)^2 \\
- \quad & \mathrm{Tr}\left(\Omega H_1 {H_2}^2\right) - \mathrm{Tr}\left(\Omega H_2 H_1 H_2\right) + \mathrm{Tr}\left(\Omega H_1\right)\mathrm{Tr}\left(\Omega {H_2}^2\right) + 2\,n^2\mathrm{Tr}\left(H_1 H_2\right)\mathrm{Tr}\left(H_2\right) \\
+ \quad & n^2/2\mathrm{Tr}\left(H_1\right)\mathrm{Tr}\left({H_2}^2\right) + n^3\mathrm{Tr}\left(H_1\right)\left(\mathrm{Tr}\left(H_2\right)\right)^2 + n\mathrm{Tr}\left(H_1 {H_2}^2\right) \\
+ \quad & n\mathrm{Tr}\left(H_1\right)\left(\mathrm{Tr}\left(\Omega H_2\right)\right)^2 + n^2\mathrm{Tr}\left(\Omega H_1\right)\left(\mathrm{Tr}\left(H_2\right)\right)^2 - n/2\,\mathrm{Tr}\left(\Omega H_1\right)\mathrm{Tr}\left({H_2}^2\right) \\
- \quad & n\mathrm{Tr}\left(H_1\right)\mathrm{Tr}\left(\Omega {H_2}^2\right)
\end{aligned}$$

Maple application center



$$E\left\{\text{Tr}\left[W_p(n)H_1\right]^{i_1}\cdots\text{Tr}\left[W_p(n)H_m\right]^{i_m}\right\} \text{ with } H_1,\ldots,H_m \in \mathbb{C}^{p\times p}$$

▷ Di Nardo, E. (2014) *On a symbolic representation of non-central Wishart random matrices with applications.* Jour. Mult. Anal.

In the literature

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A steep learning curve but...

When symbolic methods are used properly, they can give us more insights to problems and the efficiency could be reached as by-product.

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• mean $\rightarrow c_1$

Example:

n-th cumulant $c_n = n$-th coeff. of $\log$ MGF

In the literature

A steep learning curve but...

When symbolic methods are used properly, they can give us more insights to problems and the efficiency could be reached as by-product.

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- mean $\to c_1$
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k-statistics

Definition

The n-th k-statistic k_n is the unique symmetric unbiased estimator of the n-th cumulant c_n, i.e. $E[k_n] = c_n$.

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$$
\begin{aligned}
k_1 &= \frac{S_1}{n} \\
k_2 &= \frac{nS_2 - S_1^2}{(n)_2} \\
k_3 &= \frac{2S_1^3 - 3nS_1S_2 + n^2S_3}{(n)_3} \\
k_4 &= \frac{-6S_1^4 + 12nS_1^2S_2 - 3n(n-1)S_2^2 - 4n(n+1)S_1S_3 + n^2(n+1)S_4}{(n)_4}
\end{aligned}
$$

$$
\boxed{S_r = \sum_{i=1}^{n} X_i^r}
$$

k-statistics

A nice formula: cumulants in terms of moments

If $\lambda = (1^{r_1}, 2^{r_2}, \ldots, m^{r_m}) \vdash i \leq n$ then

$$c_i = i! \sum_{\lambda \vdash i} \frac{(-1)^{l(\lambda)-1} [l(\lambda) - 1)]!}{r_1! r_2! \cdots r_m!} \frac{a_1^{r_1} a_2^{r_2} \cdots a_m^{r_m}}{(1!)^{r_1} (2!)^{r_2} \cdots (m!)^{r_m}}$$

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$$\frac{\sum \underbrace{X_s X_t \cdots}_{r_1} \underbrace{X_q^2 X_r^2 \cdots}_{r_2} \underbrace{X_u^m X_\nu^m \cdots}_{r_m}}{n\,(n-1) \cdots (n - l(\lambda) + 1)}$$

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in terms of power sums

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▷ Too heavy from a computational point of view!

Computational results

(A&S) Andrews, D.F. and Stafford, J.E. (2000) *Symbolic computation for statistical inference.* Oxford University Press.

k-statistics	*A&S*
k_5	0.06
k_7	0.31
k_9	1.44
k_{11}	8.36
k_{14}	396.39
k_{16}	57982.4
k_{18}	–
k_{20}	–
k_{22}	–
k_{24}	–
k_{26}	–
k_{28}	–

PC Pentium(R)4, Intel(R)

CPU 2.08 Ghz

512MB Ram

Maple 10.0

Mathematica 4.2

Times in seconds

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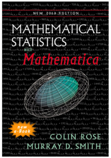
(Symbolic) Di Nardo, E., Guarino, G. and Senato, D. (2008) *A unifying framework for k-statistics, polykays and their multivariate generalizations.* Bernoulli.

(MathStat)



k-statistics	*A&S*	*MathStat*	*Symbolic*
k_5	0.06	0.01	0.01
k_7	0.31	0.02	0.01
k_9	1.44	0.04	0.01
k_{11}	8.36	0.14	0.01
k_{14}	396.39	0.64	0.02
k_{16}	57982.4	2,63	0.08
k_{18}	–	6.90	0.16
k_{20}	–	25.15	0.33
k_{22}	–	81.70	0.80
k_{24}	–	359.40	1.62
k_{26}	–	1581.05	2.51
k_{28}	–	6505.45	4.83

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A speeder way of computing

$$c_i = E[(C_{1,Z} + \cdots + C_{n,Z})^i]$$

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A speeder way of computing = a new formula and a new insight

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Statistics and Computing (2009)

"Every polynomial symmetric function can be expressed in terms of polykays."

▷ Tukey, J. (1956) *Keeping moment-like sample computations simple.* Ann.Math.Stat.

$k_{r,\ldots,s}$	AS Algorithms	MathStatica		Polyk-algorithm
$k_{3,2}$	0.06	0.02		0.02
$k_{4,4}$	0.67	0.06		0.06
$k_{5,3}$	0.69	0.08		0.07
$k_{7,5}$	34.23	0.79		0.70
$k_{7,7}$	435.67	2.52		2.43
$k_{9,9}$	-	27.41		23.32
$k_{10,8}$	-	30.24		25.06
$k_{4,4,4}$	34.17	0.64		0.77

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▷ Staude, B. and Rotter, S. (2010) *Cubic: cumulant based inference of higher-order correlations in massively parallel spike trains.* J. Comp. Neuroscience

Joint cumulants are zero for i.r.v.'s

Table: For `AS` Algorithms, missed computational times means "greater than 20 hours". For `MathStatica`, missed computational times means "procedures not available"

$k_{s_1 \ldots s_r; l_1 \ldots l_m}$	`AS` Algorithms	`MathStatica`	
$k_{3\,2}$	0.25	0.03	
$k_{4\,4}$	28.36	0.16	
$k_{5\,5}$	259.16	0.55	
$k_{6\,5}$	959.67	1.01	
$k_{7\,7}$	-	8.49	
$k_{8\,7}$	-	14.92	
$k_{3\,3\,3}$	1180.03	0.88	
$k_{4\,4\,3}$	-	4.80	
$k_{4\,4\,4}$	-	13.53	
$k_{2\,1;\,1\,1}$	0.20	-	
$k_{2\,2;\,2\,1}$	6.30	-	
$k_{2\,2;\,2\,2}$	33.75	-	
$k_{2\,2;\,2\,1;\,1\,1}$	126.19	-	
$k_{2\,2;\,2\,1;\,2\,1}$	398.42	-	
$k_{2\,2;\,2\,2;\,2\,1}$	1387.00	-	
$k_{2\,2;\,2\,2;\,2\,2}$	3787.41	-	

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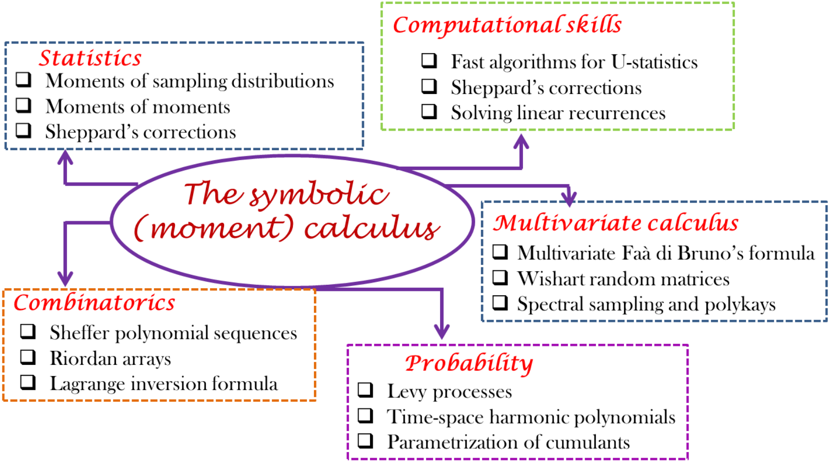
An overview on what we have done...



Symbolic moment calculus

a_i represented by a symbol $\alpha \in \mathcal{A}$

$\Rightarrow$

$$\mathbb{E}[\alpha^i] = a_i$$

Symbolic combinatorics

$a_i = |\{$ discrete structures $\}|$

gen.func. $1 + \sum_{i \geq 1} a_i \frac{t^i}{i!}$

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Probability in terms of r.v.'s

Take an ordered commutative algebra over $\mathbb{C}[\mathcal{A}]$ and endows it with a positive linear functional $\mathbb{E}$:

- an element of $\mathcal{A} \Rightarrow$ a r.v.
- the linear functional $\mathbb{E} \Rightarrow$ the expectation of a r.v.
- the sequence $\{a_i\} \Rightarrow$ the moments of a r.v.

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Special Umbrae	Moments
Augmentation umbra	$\mathbb{E}[\varepsilon^i] = 0$
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▷ Not all r.v.'s can be represented by umbrae.

▷ Not all umbrae are r.v.'s.

The algebra of non-commutative r.v.'s

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- $a \in \mathcal{A}$ (non-commutative r.v.'s)
- unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(a^i)$ i-th moment
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Free cumulants - Nc(i)

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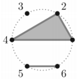

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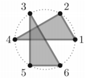

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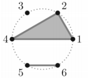

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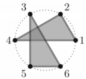

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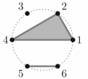

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Free cumulants - Nc(i)

Remark: a, b are free commutative r.v.'s iff at least one of them has vanishing variance.

The framework

Uncorrelation property

$$\mathbb{E}[\alpha^i \gamma^j \cdots \delta^k] = \mathbb{E}[\alpha^i]\,\mathbb{E}[\gamma^j]\,\cdots\,\mathbb{E}[\delta^k]$$

$\mathbb{E}$ factorizes on different symbols

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- α, α' similar $\Leftrightarrow$ $\mathbb{E}[\alpha^i] = \mathbb{E}[(\alpha')^i] = a_i$ for all i, in symbols $\alpha \equiv \alpha'$
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$$\mathcal{S}_N = X_1 + \cdots + X_N$$

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$\gamma.\alpha$ vs S_N

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Random matrices are non-commutative objects whose large-dimension asymptotic have provided the major applications of free probability:

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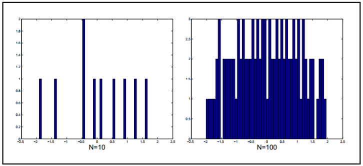

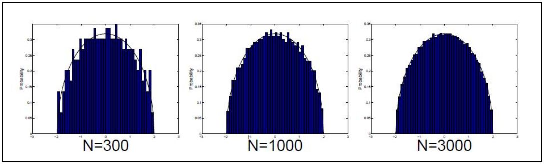

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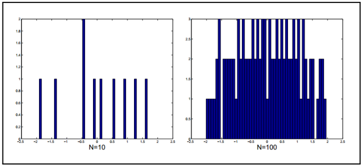

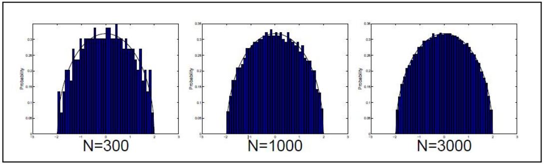

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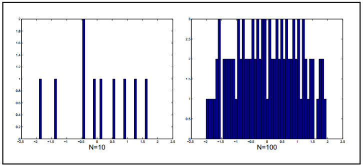

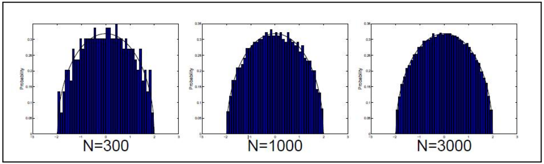

If A and B are indipendent diagonal matrices

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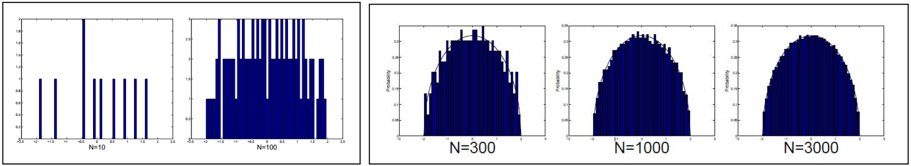

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- What about $c_i(A+B)$?
- How to define $c_i(A)$?

Non-asymptotic case

If A and B are asymptotically free, then the asymptotic spectrum of the sum can be obtained from the individual asymptotic spectra.

⇓

$\rightsquigarrow$ As free probability only covers the asymptotic regime in which n is sent to infinity, there are some aspects of random matrix theory to which the tools of free probability are not sufficient by themselves to resolve.

How to preserve the framework of free probability?

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If in $\cancel{\lim_{n\to\infty}} \frac{1}{n} E[\mathrm{Tr}(A^i)] = \tau(A^i)$ the symbolic moment method can be resorted in order to compute $\{\tau(A^i)\}_{i\geq 1}$.

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Assume to symbolically represent the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of A with $\{\mu_1, \ldots, \mu_n\}$ umbral monomials so that

$$\tau(A^i) = \frac{1}{n}\mathbb{E}[\mu_1^i + \cdots + \mu_n^i] \text{ power sum symmetric polynomials in } \{\mu_i\}$$

▷ Capitaine M., Casalis M. (2006) *Cumulants for random matrices as convolutions on the symmetric group.* Probab. Theory Relat. Fields.

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Cumulants of random matrices

If $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ represents the eigenvalues of A then $\mathfrak{c}_{\boldsymbol{\mu}} = (\mathfrak{c}_{1,\boldsymbol{\mu}}, \ldots, \mathfrak{c}_{n,\boldsymbol{\mu}})$

$$\mathrm{Tr}(A) \Leftarrow \mu_1 + \cdots + \mu_n \equiv n.\beta.(\mathfrak{c}_{1,\boldsymbol{\mu}} + \cdots + \mathfrak{c}_{n,\boldsymbol{\mu}}) \Rightarrow \mathrm{Tr}\left(\mathfrak{C}(A)\right)$$

represents the n-tuple of cumulants of A.

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$\star$ *convolution on symmetric group*

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$$\mathfrak{m}[A(\sigma)] = \mathbb{E}\left\{\prod_{c \in C(\sigma)} \mathrm{Tr}\left[A^{1(c)}\right]\right\} \qquad \mathfrak{c}[A(\sigma)] = \prod_{c \in C(\sigma)} \frac{\mathbb{E}\left\{[\mathrm{Tr}\left(\mathfrak{C}(A)\right)]^{1(c)}\right\}}{(1(c)-1)!} \Rightarrow \text{polykays}$$

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The non central Wishart distribution

- Let $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n\}$ be random row vectors independently drawn from a p-variate complex normal distribution with zero mean and full rank covariance matrix Σ with eigenvalues $\{\theta_1, \ldots, \theta_p\}$
- Let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_n$ be complex row vectors of dimension p.

$$W_p(n, \Sigma, M) = \sum_{i=1}^{n} (\boldsymbol{X}_i - \boldsymbol{m}_i)^\dagger (\boldsymbol{X}_i - \boldsymbol{m}_i)$$

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$$\Omega = \Sigma^{-1} M \Leftarrow$$

non-centrality matrix

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Symbolic representation with $\widehat{W}(n) = W_p(n, \Sigma, 0)$

$$\text{Tr}[\widehat{W}(n)] \Leftarrow \text{central comp.}$$

$$+ \quad \overbrace{n.(\theta_1 \bar{u}_1 + \cdots + \theta_p \bar{u}_p)}$$

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$$\text{Cum}_i\left(\text{Tr}[W(n)]\right) = -i!\text{Tr}(M\Sigma^{i-1}) + n(i-1)!\text{Tr}(\Sigma^i)$$

$\mathbb{E}[\alpha^i]$ $\text{Tr}[\widehat{W}(n)]$

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$$\mathrm{Tr}[W(\gamma)] \equiv \overbrace{-1.\beta.\alpha}^{\alpha \Leftarrow \text{ formal comp.}} + \overbrace{\gamma.(\theta_1 \bar{u}_1 + \cdots + \theta_p \bar{u}_p)}^{\mathrm{Tr}[\widehat{W}(\gamma)] \Leftarrow \text{ central comp.}}$$

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A different way to represent the central component

$$\mathrm{Tr}[\widehat{W}(n)] = \mathrm{Tr}[\boldsymbol{X}_1^{\dagger}\boldsymbol{X}_1 + \cdots + \boldsymbol{X}_n^{\dagger}\boldsymbol{X}_n]$$

with $\{\boldsymbol{X}_1^{\dagger}\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n^{\dagger}\boldsymbol{X}_n\}$ i.i.d. random matrices of order p.

A different way to represent the central component

$$\mathrm{Tr}[\widehat{W}(n)] \equiv n.(\theta_1 \bar{u}_1 + \cdots + \theta_p \bar{u}_p) \equiv n.\beta.\delta$$

▷ $\{\bar{u}_1, \ldots, \bar{u}_p\}$ uncorrelated umbrae similar to the boolean unity umbra $\bar{u}$ whose moments are equal to the number of permutations of a set.

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As a summation of compound Poisson r.v.'s

$$\mathrm{Tr}[\widehat{W}(1)] = \mathrm{Tr}\left[\sum_{i=1}^{n} \boldsymbol{X}_i^{\dagger} \boldsymbol{X}_i\right] = Z_1 + \cdots + Z_{\mathrm{Po}(1)} \text{ with}$$

- $\{Z_i\}_{i=1}^{n}$ i.i.d. r.v.'s;
- $E[Z_i^k] = (k-1)!\mathrm{Tr}(\Sigma^k) = \mathrm{Cum}_k\left(\boldsymbol{X}_i^{\dagger} \boldsymbol{X}_i\right)$ for $k \in \mathbb{N}$

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⇝ The sequence of moments of $\text{Tr}[\widehat{W}(n)]$ is of binomial type.

⇝ The sequence of moments of $\text{Tr}[W(n)]$ is of Sheffer type.

Generalizing the computation of $\mathfrak{m}[W(n)]$ with multivariate notations

$$E\left\{\mathsf{Tr}\,[W(n)H_1]^{i_1}\cdots\mathsf{Tr}\,[W(n)H_m]^{i_m}\right\}$$

Solution

Generalizing the computation of $\mathfrak{m}[W(n)]$ with multivariate notations

$$E\left\{\mathrm{Tr}\,[W(n)H_1]^{i_1}\cdots\mathrm{Tr}\,[W(n)H_m]^{i_m}\right\} = \mathbb{E}[(-1.\beta.\tilde{\boldsymbol{\eta}} + n.\beta.\tilde{\boldsymbol{\rho}})^{\boldsymbol{i}}]$$

Solution

Univariate case: $E\left\{\mathrm{Tr}\,[W(n)]^k\right\} = \mathbb{E}[(-1.\beta.\alpha + n.\beta.\delta)^k]$

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Multivariate moments: *receipe ingredients*

For $\{g_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{N}_0^m} \in \mathbb{C}$ with $g_{\boldsymbol{i}} = g_{i_1,i_2,\ldots,i_m}$ and $g_{\boldsymbol{0}} = 1$, such that $\mathbb{E}[\boldsymbol{\nu}^{\boldsymbol{i}}] = g_{\boldsymbol{i}}$

- $\boldsymbol{\nu} = (\nu_1,\ldots,\nu_m)$ m-tuple of umbral monomials (not necessarely uncorrelated)
- $\boldsymbol{i} \in \mathbb{N}_0^m$ multi-index.

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▷ **Multinomial expansion:**

$$\sum_{\substack{\boldsymbol{t}_1,\boldsymbol{t}_2\in\mathbb{N}_0^m \\ \boldsymbol{t}_1+\boldsymbol{t}_2=\boldsymbol{i}}} \binom{\boldsymbol{i}}{\boldsymbol{t}_1,\boldsymbol{t}_2} \mathbb{E}[(-1.\beta.\tilde{\boldsymbol{\eta}})^{\boldsymbol{t}_1}]\mathbb{E}[(n.\beta.\tilde{\boldsymbol{\rho}})^{\boldsymbol{t}_2}]$$

Generalizing the computation of $\mathfrak{m}[W(n)]$ with multivariate notations

$$E\left\{\mathrm{Tr}\,[W(n)H_1]^{i_1}\cdots\mathrm{Tr}\,[W(n)H_m]^{i_m}\right\} = \mathbb{E}[(-1.\beta.\tilde{\boldsymbol{\eta}} + n.\beta.\tilde{\boldsymbol{\rho}})^{\boldsymbol{i}}]$$ Solution

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$$\boldsymbol{N}_m[\boldsymbol{i}] = \{\text{necklaces of type } \boldsymbol{i} \text{ on } [m]\} \quad \begin{cases} \boldsymbol{N}_3[(3,0,0)] = \{111\} \\ \boldsymbol{N}_3[(1,2,0)] = \{122\} \\ \boldsymbol{N}_3[(1,1,1)] = \{123, 132\} \end{cases}$$

Generalizing the computation of $\mathfrak{m}[W(n)]$ with multivariate notations

$$E\left\{\text{Tr}\,[W(\gamma)H_1]^{i_1}\cdots\text{Tr}\,[W(\gamma)H_m]^{i_m}\right\} = \mathbb{E}[(-1.\beta.\tilde{\boldsymbol{\eta}} + \gamma.\beta.\tilde{\boldsymbol{\rho}})^{\boldsymbol{i}}]$$ ▸ Solution

Multivariate moments: *receipe ingredients*

For $\{g_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{N}_0^m} \in \mathbb{C}$ with $g_{\boldsymbol{i}} = g_{i_1,i_2,\ldots,i_m}$ and $g_{\mathbf{0}} = 1$, such that $\mathbb{E}[\boldsymbol{\nu}^{\boldsymbol{i}}] = g_{\boldsymbol{i}}$

- $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$ m-tuple of umbral monomials (not necessarely uncorrelated)
- $\boldsymbol{i} \in \mathbb{N}_0^m$ multi-index.

Univariate case: $E\left\{\text{Tr}\,[W(\gamma)]^k\right\} = \mathbb{E}[(-1.\beta.\alpha + \gamma.\beta.\delta)^k]$

▷ **Multinomial expansion:** $$\sum_{\substack{\boldsymbol{t}_1,\boldsymbol{t}_2\in\mathbb{N}_0^m\\ \boldsymbol{t}_1+\boldsymbol{t}_2=\boldsymbol{i}}} \binom{\boldsymbol{i}}{\boldsymbol{t}_1,\boldsymbol{t}_2} \mathbb{E}[(-1.\beta.\tilde{\boldsymbol{\eta}})^{\boldsymbol{t}_1}]\mathbb{E}[(\gamma.\beta.\tilde{\boldsymbol{\rho}})^{\boldsymbol{t}_2}]$$

$$\boldsymbol{N}_m[\boldsymbol{i}] = \{\text{necklaces of type } \boldsymbol{i} \text{ on } [m]\} \quad \begin{cases} \boldsymbol{N}_3[(3,0,0)] = \{111\} \\ \boldsymbol{N}_3[(1,2,0)] = \{122\} \\ \boldsymbol{N}_3[(1,1,1)] = \{123, 132\} \end{cases}$$

Tricking: an example

$$\mathrm{Cum}_{\boldsymbol{i}}(\mathsf{Tr}[W(n)\,H_1], \ldots, \mathsf{Tr}[W(n)\,H_m]) = \boldsymbol{i}!(n\mathbb{E}[\boldsymbol{\rho}^{\boldsymbol{i}}] - \mathbb{E}[\boldsymbol{\eta}^{\boldsymbol{i}}])$$

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$$\begin{aligned}\mathrm{Cum}_{(1,2)}(\mathsf{Tr}[W(n)\,H_1],\mathsf{Tr}[W(n)\,H_2]) &= 2!\left\{n\mathsf{Tr}\left[(\Sigma H_1)(\Sigma H_2)^2\right] - \mathsf{Tr}\left[\Omega(\Sigma H_1)(\Sigma H_2)^2\right]\right.\\ &- \left.\mathsf{Tr}\left[\Omega(\Sigma H_2)^2(\Sigma H_1)\right] - \mathsf{Tr}\left[\Omega(\Sigma H_1)(\Sigma H_2)(\Sigma H_1)\right]\right\}\end{aligned}$$

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(Randomized Wishart distribution)

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$$\Uparrow$$

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(the central part)

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(the central part)

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$$2\,n^2\mathrm{Tr}\,(H_1H_2)\,\mathrm{Tr}\,(H_2) + n^3\mathrm{Tr}\,(H_1)\,(\mathrm{Tr}\,(H_2))^2 + n\mathrm{Tr}\left(H_1{H_2}^2\right) + n^2/2\mathrm{Tr}\,(H_1)\,\mathrm{Tr}\left({H_2}^2\right)$$

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Replace n^i *with* c_i

Simple random sampling

Simple random sample

A sub-vector $\boldsymbol{y}$ consisting of m components of $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, selected with equal probability $1/(n)_m$.

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$\sigma \in \mathfrak{S}_n$ a permutation
S the corresponding matrix

$$S_{ij} = \begin{cases} 1, & \text{if } \sigma(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

$$S = \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ \vdots & \vdots & \ldots & \vdots \\ s_{m,1} & s_{m,2} & \cdots & s_{m,n} \\ s_{m+1,1} & s_{m+1,2} & \cdots & s_{m+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n} \end{pmatrix}$$

A formal method
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Properties:

- ▷ $S_{[m\times n]}\, S^T_{[m\times n]} = I_m$,
- ▷ $S^T_{[m\times n]}\, S_{[m\times n]} \neq I_m$.

Example

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{corresponding to} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in \mathfrak{S}_4$$

Example

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{corresponding to} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in \mathfrak{S}_4$$

A simple random sampling is:

$$\begin{pmatrix} x_4 & 0 \\ 0 & x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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The full matrix is:

$$\begin{pmatrix} x_4 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 \\ 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Spectral sample

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Let H be a random unitary matrix uniformly distributed with respect to the Haar measure on the group $\mathcal{U}_n$ of $n \times n$ unitary matrices.

$$H = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ \vdots & \vdots & \ldots & \vdots \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \\ h_{m+1,1} & h_{m+1,2} & \ldots & h_{m+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{n,1} & h_{n,2} & \ldots & h_{n,n} \end{pmatrix}$$

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Spectral sample of size m

The eigenvalues (real r.v.'s)

$$\boldsymbol{y} = (y_1, \ldots, y_m) \text{ of } Y$$

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Open problem: Distribution of $\boldsymbol{y}$?

2014 Ipsen, J.R., Kieburg, M. ... *eigenvalue statistics for products of rectangular random matrices* Phys. Rev. E.

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- Generalization: replace $\mathrm{diag}(\boldsymbol{x})$ with a Hermitian random matrix X
- Meaning: a restriction operation $X \mapsto Y$ extracting a partial information from X

A second meaning

A random Hermitian matrix A of order n is said to be *freely randomized* if its distribution is invariant under unitary conjugation, i.e. $A \sim GAG^{\dagger}$ for each unitary G.

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Examples

a) if H is uniformly distributed with respect to Haar measure then, $HAH^{\dagger}$ is freely randomized.
b) if A is freely randomized, each leading sub-matrix is also freely randomized.

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A spectral sample comes from a freely randomized matrix $\Rightarrow H\text{diag}(\boldsymbol{x})H^{\dagger}$

A second meaning

A random Hermitian matrix A of order n is said to be *freely randomized* if its distribution is invariant under unitary conjugation, i.e. $A \sim GAG^{\dagger}$ for each unitary G.

Examples

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b) if A is freely randomized, each leading sub-matrix is also freely randomized.

A spectral sample comes from a freely randomized matrix $\Rightarrow H\text{diag}(\boldsymbol{x})H^{\dagger}$

- if $m = n$, the subsample $\boldsymbol{y}$ is a random permutation of $\boldsymbol{x}$.
- if $m < n$, the elements of $\boldsymbol{y}$ do not occur among the components of $\boldsymbol{x}$.

Statistics for spectral sampling?

$\rightsquigarrow$ Which spectral properties are preserved on the average by freely randomized matrix restriction? Example: the eigenvalue average is preserved.

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Natural statistics

A statistic T (a collection of functions $T_n : \mathbb{R}^n \to \mathbb{R}$) is said to be *natural*

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Spectral natural statistics

If $\boldsymbol{y}$ spectral sample and $\lambda \vdash i$, then $\mathbb{E}[\kappa_\lambda(\boldsymbol{y})] = \prod_{j=1}^{l(\lambda)} \mathbb{E}[(\mathfrak{c}_{1,\boldsymbol{y}} + \cdots + \mathfrak{c}_{m,\boldsymbol{y}})^{\lambda_j}]$

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Matricial polykays

Main theorem

Matricial polykays are the symmetric functions $\mathfrak{K}_\lambda(\boldsymbol{y})$ such that

$$\mathbb{E}[\mathfrak{K}_\lambda(\boldsymbol{y})](\sigma) = \texttt{const} \times \mathbb{E}\left\{\left[\mu(I_m)^{(-1)} \star \mu(Y)\right](\sigma)\right\}, \quad i \leq m$$

- $\sigma \in \mathfrak{S}_m$, a permutation with $|C(\sigma)|$ disjoint cycles;
- $(f \star g)(\sigma) = \sum_{\rho\,\omega = \sigma} f(\rho)\, g(\omega)$ convolution on $\mathfrak{S}_m$;
- $\mu(Y)(\sigma) = \prod_{c \in C(\sigma)} \text{Tr}\left(Y^{\mathfrak{l}(c)}\right)$ and $\mu(I_m)(\sigma) = m^{|C(\sigma)|}$;
- $f^{(-1)} \star f = f \star f^{(-1)} = \delta$ (indicator function)

The computation of $\mu(I_m)^{(-1)}$ requires to solve a system of m equations in m indeterminates. A different way: the so-called Weingarten function on $\mathfrak{S}_m$ (Open problem).

$$\mathfrak{K}_{(1)} = \frac{S_1}{n} \qquad \mathfrak{K}_{(2)} = \frac{nS_2 - S_1^2}{n\,(n^2-1)}$$

$$\mathfrak{K}_{(1^2)} = \frac{nS_1^2 - S_2}{n\,(n^2-1)}$$

$$\mathfrak{K}_{(3)} = 2\,\frac{2\,S_1^3 - 3\,n\,S_1\,S_2 + n^2 S_3}{n\,(n^2-1)\,(n^2-4)}$$

$$\mathfrak{K}_{(1,2)} = \frac{-2\,n\,S_3 + (n^2+2)S_1\,S_2 - nS_1^3}{n\,(n^2-1)\,(n^2-4)}$$

$$\mathfrak{K}_{(1^3)} = \frac{S_1^3(n^2-2) - 3\,n\,S_1\,S_2 + 4\,S_3}{n\,(n^2-1)\,(n^2-4)}.$$

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Connection with k-statistics

$$\mathfrak{K}_{(1)} = \frac{S_1}{n} = k_{(1)} \qquad \mathfrak{K}_{(2)} = \frac{nS_2 - S_1^2}{n\,(n^2-1)} = \frac{k_{(2)}}{(n+1)}$$

$$\mathfrak{K}_{(1^2)} = \frac{nS_1^2 - S_2}{n\,(n^2-1)} = \frac{k_{(1^2)}}{(n+1)}$$

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$$\mathfrak{K}_{(1^3)} = \frac{S_1^3(n^2-2) - 3\,n\,S_1\,S_2 + 4\,S_3}{n\,(n^2-1)\,(n^2-4)} = \frac{2k_{(1^3)} - 3k_{(1)}k_{(2)} + n(n+3)(k_{(1)})^3}{(n+1)(n+2)}.$$

The computation of $\mu(I_m)^{(-1)}$ requires to solve a system of m equations in m indeterminates. A different way: the so-called Weingarten function on $\mathfrak{S}_m$ (Open problem).

Properties

- $\mathfrak{K}_\lambda(\boldsymbol{y})$ are called `matricial polykays`, unbiased estimators of products of cumulants
- $\mathfrak{K}_\lambda(\boldsymbol{y})$ are natural statistics (the proof is strictly connected with the spectral sampling);
- The condition $i \leq m$ parallels the analogous condition for Fisher's k-statistics.
- $\mathbb{E}[\mathfrak{K}_\lambda(\boldsymbol{y})]$ tends towards the product of free cumulants when $m \to \infty$ as Fisher's polykays tends towards the product of classical cumulants.
- $\mathfrak{K}_\lambda(\boldsymbol{y})$ can be expressed as linear combination of generalized k-statistics with coefficients independent of n.

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Generalized spectral polykays

Generalized k-statistics are the sample version of the generalized cumulants.

Generalized spectral polykays

Generalized cumulants

$$c_{r,s\,t} = \mathrm{cov}(X^r, X^s X^t) \quad c_{r\,s,t\,u} = \mathrm{cov}(X^r\, X^s, X^t\, X^u)$$

application: in asymptotic approximations of distributions

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Generalized k-statistics are the sample version of the generalized cumulants.

- the generalized k-statistics are linearly independent;
- every polynomial symmetric function can be expressed uniquely as a linear combination of generalized k-statistics;
- any polynomial symmetric function whose expectation is independent of n can be expressed as linear combination of generalized k-statistics with coefficients independent of n.

A different choice of foundations can lead to a different way of thinking about the subject, and thus to ask a different set of questions and to discover a different set of proofs and solutions. Thus it is often of value to understand multiple foundational perspectives at once, to get a truly stereoscopic view of the subject.

From Terence Tao's blog

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Thanks for your attention!

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