Matrices of nonnegative rank at most three

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Nonnegative rank

- A fictional study on "Does watching football cause hair loss?"

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\begin{array}{r} \\ \leq 2 \text{ h} \\ 2-6 \text{ h} \\ \geq 6 \text{ h} \end{array}
\begin{array}{c} \quad\text{plenty} \quad \text{medium} \quad \text{less} \\ \begin{pmatrix} 51 & 45 & 33 \\ 28 & 30 & 29 \\ 15 & 27 & 38 \end{pmatrix} \end{array} =: U
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- We get a much better understanding if we consider the *hidden variable* gender.

• We get the sum of two **rank one** contingency tables:

$$U = \begin{pmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 7 & 21 & 35 \end{pmatrix} + \begin{pmatrix} 48 & 36 & 18 \\ 24 & 18 & 9 \\ 8 & 6 & 3 \end{pmatrix}$$

• So watching football and hair loss are **independent** given the gender.

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Main motivation:

Determine whether such an "expanatory random variable" exists and what the number of its states is?

Nonnegative rank

Given $M \in \mathbb{R}_{\geq 0}^{m \times n}$ its nonnegative rank is the smallest r such that there exist $A \in \mathbb{R}_{\geq 0}^{m \times r}$ and $B \in \mathbb{R}_{\geq 0}^{r \times n}$ with

$$M = A \cdot B.$$

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- Matrices of nonnegative rank at most r form a semialgebraic set $\mathcal{M}_{m \times n}^{r}$.

- If the nonnegative rank is 1 or 2, then it equals the rank.

- First interesting example is $\mathcal{M}_{m \times n}^{3}$.

• We want to solve optimization problems on $\mathcal{M}^3_{m\times n}$ (e.g. maximum likelihood estimation).

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• For this we need to understand the *Zariski closure of the boundary*.

$$\downarrow$$

topological boundary $\partial(\mathcal{M}^3_{m\times n})$

$$\downarrow$$

algebraic boundary $\overline{\partial(\mathcal{M}^3_{m\times n})}$

Algebraic boundary of $\mathcal{M}^3_{m\times n}$

Let us denote the multiplication map by

$$\mu : M_{m\times 3} \times M_{3\times n} \to M_{m\times n}$$

$$((a_{ik}), (b_{kj})) \mapsto (x_{ij}).$$

Theorem (Kubjas-Robeva-Sturmfels)

The irreducible components of $\overline{\partial(\mathcal{M}^3_{m\times n})}$ are

- components defined by $x_{ij} = 0$
- components parametrized by

$$(x_{ij}) = A \cdot B \text{ or } (x_{ij}) = B \cdot A,$$

where A has four zeros in three columns and distinct rows, and B has three zeros in distinct rows and columns.

Let $\mathcal{A} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & * \\ * & * & 0 \\ \vdots & \vdots & \vdots \end{pmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 0 & * & * & \cdots \\ * & 0 & * & \cdots \\ * & * & 0 & \cdots \end{pmatrix} \right\}$

Consider the following irreducible component

$$\begin{array}{ccc} \mu : M_{m\times 3} \times M_{3\times n} & \longrightarrow & M_{m\times n} \\ \cup\!| & & \cup\!| \\ \mathcal{A}\times\mathcal{B} & \longrightarrow & \overline{\mu(\mathcal{A}\times\mathcal{B})} \end{array}$$

$$\overline{\mu(\mathcal{A}\times\mathcal{B})} \dashleftarrow X$$

Generators of the ideal of X.

The following theorem was previously conjectured by Kubjas-Robeva-Sturmfels.

Main Theorem (Eggermont-H.-Kubjas)

The ideal of the variety X is generated by certain degree 6 polynomials and degree 4 determinants

$$\mathcal{I}(X) = (f_i, \det_{j,k})_{i,j,k}.$$

Moreover, these polynomials form a Gröbner basis with respect to the graded reverse lexicographic term order.

Steps of the proof

$$\mathrm{GL}_3 \curvearrowright \mathcal{A} \times \mathcal{B}$$

$$g \cdot (A, B) = (Ag^{-1}, gB)$$

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Pick $f \in \mathcal{I}(X)$ from K-R-S

$$\mu^* f = \det B_i \cdot f_{6,3}$$

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Stabilization of the boundary

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Given $M \in \mathbb{R}_{\geq 0}^{m \times n}$, let $W = \mathrm{Span}(M) \cap \Delta_{m-1}$ and $V = \mathrm{Cone}(M) \cap \Delta_{m-1}$.

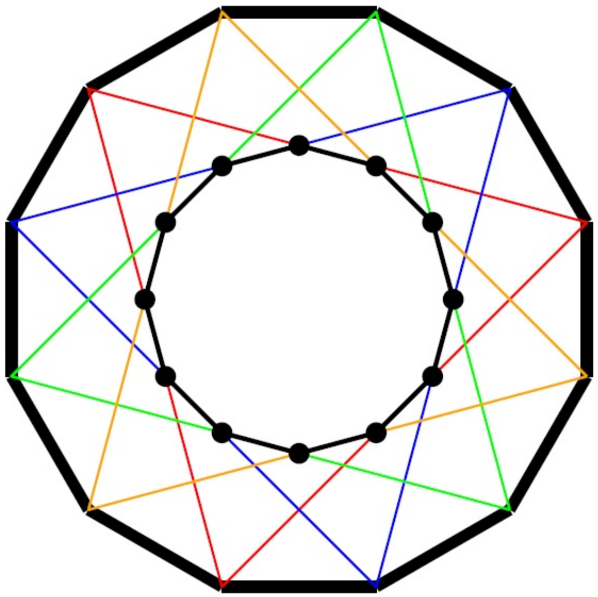
Lemma

Let $\mathrm{rank}(M) = r$, then M has nonneg. rank exactly r if and only if there exists a $(r-1)$-simplex Δ, such that $V \subseteq \Delta \subseteq W$.

Example

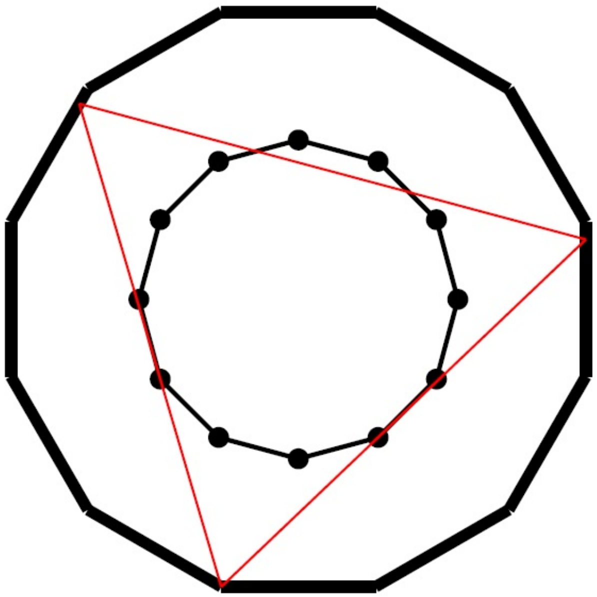
Here M^ϵ is a 12×12 matrix of rank 3 and nonnegative rank 4.

M



$3n = 12$ & bounding triangles

M^ϵ



After expanding by a factor ϵ

Any 12×5 (i.e $3n \times (\lceil \frac{3}{2} n \rceil - 1)$) submatrix has nonnegative rank 3.

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Claim

For $n > 4$, let $M \in \overline{\partial(\mathcal{M}^3_{m\times n})}$. Then we can find a column i_0 such that

$$M^{i_0} \in \overline{\partial(\mathcal{M}^3_{m\times n-1})},$$

where M^{i_0} is obtained by removing the i_0-th column of M.

Conjecture

For given $r \geq 3$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, and all matrices M on $\overline{\partial(\mathcal{M}^r_{m \times n})}$, there is a column i_0 such that

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Thank you!