Hypergraph Decompositions and Toric Ideals

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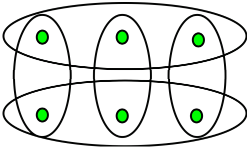
Toric ideal of a hypergraph



- $H = (E, V)$ hypergraph
- I_H is kernel of the monomial map

$$\mathbb{K}[p_e : e \in E] \to \mathbb{K}[q_v : v \in V]$$
$$p_e \mapsto \prod_{v \in e} q_v$$

- the toric ideal is parametrized by the vertex-edge incidence matrix

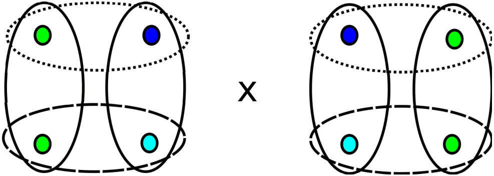
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Main problem

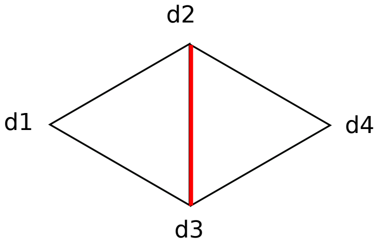
Sonja Petrović and Despina Stasi (2014), Toric algebra of hypergraphs, *J. Algebraic Combin.*, **39**, 187-208:

Question

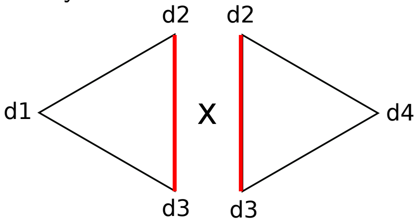
Given a hypergraph H that is obtained by identifying vertices from two smaller hypergraphs H_1 and H_2, is it possible to obtain generating set of I_H from the generating set of I_{H_1} and I_{H_2}?



Hierarchical models



grade by the values on the intersection



Toric fiber products

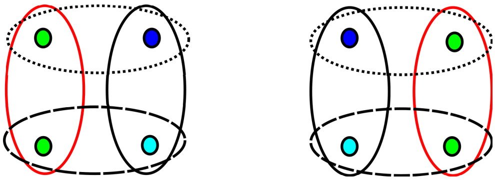
- $\mathbb{K}[x] = \mathbb{K}[x^i_j : i \in [r], j \in [s_i]], \mathbb{K}[y] = \mathbb{K}[y^i_k : i \in [r], k \in [t_i]]$
- multigraded by

$$\deg(x^i_j) = \deg(y^i_k) = \mathbf{a}^i \in \mathbb{Z}^d$$

- $I \subseteq \mathbb{K}[x], J \subseteq \mathbb{K}[y]$ homogeneous wrt the multigrading
- $\mathbb{K}[z] = \mathbb{K}[z^i_{jk} : i \in [r], j \in [s_i], k \in [t_i]]$
- $\phi_{I,J} : \mathbb{K}[z] \to \mathbb{K}[x]/I \otimes_{\mathbb{K}} \mathbb{K}[y]/J, z^i_{jk} \mapsto x^i_j \otimes y^i_k$
- toric fiber product: $I \times_{\mathcal{A}} J = \ker(\phi_{I,J})$
- Sullivant 2007; Engström, Kahle, Sullivant 2014; Kahle, Rauh 2014; Rauh, Sullivant 2014+

Possibility for toric fiber products?

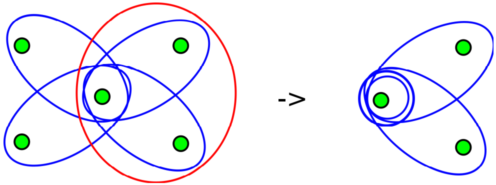
- One idea:
  - $V = V_1 \cup V_2$
  - H_i is the subhypergraph induced by V_i
  - grade edges in H_i by their incidence vectors for $V_1 \cap V_2$
- However, there are problems:
  - the ideals are not homogeneous
  - knowing the incidence vectors for the intersection $V_1 \cap V_2$ is not enough information



- Multigrading has to remember which edges can be glued

Modified construction

- $H = (V, E)$ hypergraphs
- $V = V_1 \cup V_2$
- $H_i = (V_i, E_i)$ is the subhypergraph induced by V_i
- $E_i = \{e \cap V_i : e \in E, e \cap V_i \neq \emptyset\}$
- we view E_1 and E_2 as multisets



Modified construction

- $\mathbb{K}[H_i] = \mathbb{K}[x_e : e \in E, e \cap V_i \neq \emptyset]$
- multigraded by

$$\deg(x_e) = \deg(y_e) = u_e \text{ if } e \cap V_1 \neq \emptyset \text{ and } e \cap V_2 \neq \emptyset,$$
$$\deg(x_e) = \deg(y_e) = 0 \text{ otherwise}$$

- $\mathbb{K}[H] = \mathbb{K}[z_e : e \in E]$,
- ring homomorphism $\phi_{I_{H_1}, I_{H_2}} : \mathbb{K}[H] \to \mathbb{K}[H_1]/I_{H_1} \otimes \mathbb{K}[H_2]/I_{H_2}$ defined by

$$z_e \mapsto x_e \otimes y_e \text{ if } e \cap V_1 \neq \emptyset \text{ and } e \cap V_2 \neq \emptyset,$$
$$z_e \mapsto x_e \otimes 1 \text{ if } e \subseteq V_1 \setminus V_2,$$
$$z_e \mapsto 1 \otimes y_e \text{ if } e \subseteq V_2 \setminus V_1$$

- $I_H = \ker(\phi_{I_{H_1}, I_{H_2}})$

Graver basis

Definition

A *Graver basis* of a toric ideal I consists of all the binomials $p^+ - p^- \in I$ such that there is no other binomial $q^+ - q^- \in I$ such that q^+ divides p^+ and q^- divides p^-.

$$Gr(I_{H_1}) = \{x^{u_i^+} - x^{u_i^-} : i \in I\}, Gr(I_{H_2}) = \{y^{v_j^+} - y^{v_j^-} : j \in J\}$$

- $(\alpha_i^+, \alpha_i^-) = (\deg(x^{u_i^+}), \deg(x^{u_i^-}))$ for all $i \in I$
- $(\beta_j^+, \beta_j^-) = (\deg(y^{v_j^+}), \deg(y^{v_j^-}))$ for all $j \in J$

Graver basis

$$L = \{(a_1, \ldots, a_I, b_1, \ldots, b_J) \in \mathbb{Z}^{I+J} : \sum a_i \alpha_i^{\text{sign}(a_i)} = \sum b_j \beta_j^{\text{sign}(b_j)}, \\ \sum a_i \alpha_i^{-\text{sign}(a_i)} = \sum b_j \beta_j^{-\text{sign}(b_j)}\}$$

Define a partial order on $\mathbb{R}^{I+J}$ by

$$x \preceq x' \Leftrightarrow \text{sign}(x_i) = \text{sign}(x_i') \text{ and } |x_i| \leq |x_i'| \text{ for } i = 1, \ldots, I + J$$

Let S be equal to the set of minimal elements in L wrt the partial order.

Gluing

- $E' = \{e \in E : e \cap V_1 \neq \emptyset, e \cap V_2 \neq \emptyset\}$
- $f = \prod_{e \subseteq V_1 \setminus V_2} x_e^{a_e^+} \prod_{e \in E'} x_e^{c_e^+} - \prod_{e \subseteq V_1 \setminus V_2} x_e^{a_e^-} \prod_{e \in E'} x_e^{c_e^-}$
- $g = \prod_{e \subseteq V_2 \setminus V_1} y_e^{b_e^+} \prod_{e \in E'} y_e^{c_e^+} - \prod_{e \subseteq V_1 \setminus V_2} y_e^{b_e^-} \prod_{e \in E'} y_e^{c_e^-}$
- $\text{glue}(f, g) = \prod_{e \subseteq V_1 \setminus V_2} z_e^{a_e^+} \prod_{e \subseteq V_2 \setminus V_1} z_e^{b_e^+} \prod_{e \in E'} z_e^{c_e^+} - \prod_{e \subseteq V_1 \setminus V_2} z_e^{a_e^-} \prod_{e \subseteq V_2 \setminus V_1} z_e^{b_e^-} \prod_{e \in E'} z_e^{c_e^-}$
- we say f and g are compatible
- $\mathcal{F}_1 \subseteq I_{H_1}$ and $\mathcal{F}_2 \subseteq I_{H_2}$ consist of binomials
- $\text{Glue}(\mathcal{F}_1, \mathcal{F}_2) = \{\text{glue}(f, g) : f \in \mathcal{F}_1, g \in \mathcal{F}_2 \text{ compatible}\}$

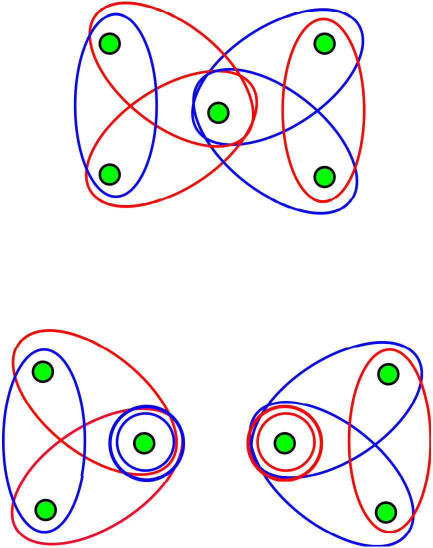
Graver basis

Theorem

The Graver basis of I_H is given by

$$\{glue(f,g) \in k[H] : f = \prod_{i \in I} (x^{u_i^{sign(a_i)}})^{a_i} - \prod_{i \in I} (x^{u_i^{-sign(a_i)}})^{a_i},$$
$$g = \prod_{j \in J} (y^{v_j^{sign(b_j)}})^{b_j} - \prod_{j \in J} (y^{v_j^{-sign(b_j)}})^{b_j}$$
$$\text{for } (a_1, \ldots, a_I, b_1, \ldots, b_J) \in S\}.$$

Graver basis



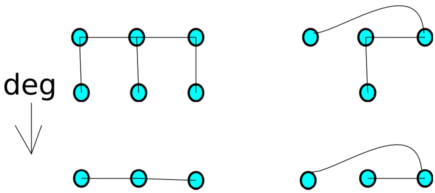
The compatible projection property

Definition

Let $\mathcal{F}_1 \subset I_1$ and $\mathcal{F}_2 \subset I_2$. The pair $\mathcal{F}_1$ and $\mathcal{F}_2$ has compatible projection property if for all compatible pairs $x^{u^+} - x^{u^-} \in I_{H_1}$ and $y^{v^+} - y^{v^-} \in I_{H_2}$ there exist $x^{u_i^+} - x^{u_i^-}$, monomial multiples of elements of $\mathcal{F}_1, i = 1, \ldots, m$, and $y^{v_j^+} - y^{v_j^-}$, monomial multiples of elements of $\mathcal{F}_2, j = 1, \ldots, n$, such that

1. $x^{u^+} - x^{u^-} = \sum x^{u_i^+} - x^{u_i^-}$ and $y^{v^+} - y^{v^-} = \sum y^{v_j^+} - y^{v_j^-}$,
2. if $i_1 < i_2 < \ldots < i_k$ are indices where $\deg(x^{u_i^+}) - \deg(x^{u_i^-}) \neq 0$ and $j_1 < j_2 < \ldots < j_l$ are indices where $\deg(y^{v_j^+}) - \deg(y^{v_j^-}) \neq 0$, then $k = l$ and $\deg(x^{u_{i_h}^+}) - \deg(x^{u_{i_h}^-}) = \deg(y^{v_{j_h}^+}) - \deg(y^{v_{j_h}^-})$ for all $h \in [k]$.

Markov basis



Theorem

Let $\mathcal{F}_1 \subset I_{H_1}$ and $\mathcal{F}_2 \subset I_{H_2}$ be Markov bases. Then $\mathbf{Glue}(\mathcal{F}_1, \mathcal{F}_2)$ *is a Markov basis of I_H if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the compatible projection property.*

Monomial sunflower

Consider the monomial sunflower. We can construct larger monomial sunflowers by taking an even number of copies of H and identifying all copies of the vertex v_1. We consider 128 copies of the sunflower. If we split it into two, then computing a Markov basis using `Macaulay2` interface for `4ti2` is 10 times faster compared to computing a Markov basis for the original hypergraph.