On patterns of conjunctive forks

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conjunctive forks play a central role in Reichenbach's causal theory of time

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\mathrm{P}(A \cap C|B) &= \mathrm{P}(A|B) \cdot \mathrm{P}(C|B)\,, \\
\mathrm{P}(A \cap C|\Omega \setminus B) &= \mathrm{P}(A|\,\Omega \setminus B) \cdot \mathrm{P}(C|\,\Omega \setminus B)\,, \\
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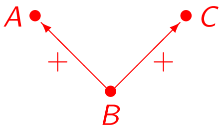

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PROBLEM: Given a ternary relation $\mathcal{R}$ on a ground set N decide whether it is *fork representable*, thus

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In algebraic language,

solve a system of quadratic equations and inequalities.

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A simpler problem is easily solvable:
given any pattern σ, there exist events A_i, $i \in N$, s.t.

$$\sigma_{ij} = \mathrm{sgn}\langle A_i, A_j\rangle\,, \quad ij \in \binom{N}{2}\,.$$

$(A, B, C)_{\mathrm{P}} \quad \text{iff} \quad (C, B, A)_{\mathrm{P}}$

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Corollary 1: $(A, B, C)_{\mathrm{P}}$ implies that A, B, C are nontrivial
(thus $(A, A, A)_{\mathrm{P}}$, $(B, B, B)_{\mathrm{P}}$ and $(C, C, C)_{\mathrm{P}}$)
and any two are positively correlated
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Proof: $\langle A, C\rangle\langle B, B\rangle = \langle A, B\rangle\langle B, C\rangle$ and
$\langle A, B\rangle\langle C, C\rangle = \langle A, C\rangle\langle B, C\rangle$ combine to
$\langle B, C\rangle^2 = \langle B, B\rangle\langle C, C\rangle$, then $B = C$ □

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Collecting the four implications ... weak betweenness

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$\mathcal{R}$ is called *solvable* if and only if the system

$$x_{ik} = x_{ij} + x_{jk} \quad \text{for } (i, j, k) \in \mathcal{R} \text{ pairwise distinct,}$$

has a solution with all involved x_{ij} positive.

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In general, $\mathcal{R}$ must be a 'regular' weak betweenness. Then a *quotient* $\mathcal{Q}$ of $\mathcal{R}$ is constructed. It is a weak betweenness that satisfies the nondegeneracy condition. $\mathcal{R}$ is fork representable iff $\mathcal{Q}$ is solvable.

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The conditions can be verified in time polynomial in $|N|$.

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P is constructed on $\mathbb{Z}_2^N$ explicitly, can be arbitrarily close to the uniform distribution; Fourier-Stieltjes transform of P is related to solvability + few other tricks

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Gaussian case is likely not difficult.