Algebraic Graph Limits

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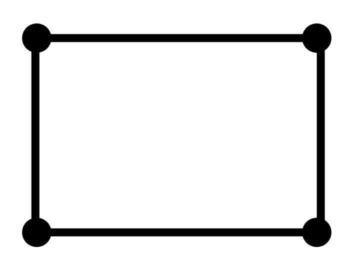
Motivation

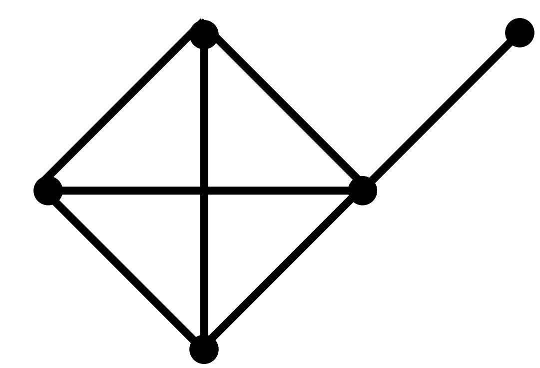
- Introduce new random graph models for large networks (finite simple graphs).
- Parameters of the model should be efficiently recoverable from observations.
- We want the model to have nice algebraic properties.

Subgraph densities

F



G



t(F,G)=
#subgraphs isomorphic to F in G /
#subgraphs isomorphic to F in the complete graphs with the same vertices as G

t(F,G)=3/15

Exchangeable random graphs

- Let W be a symmetric measurable function from $[0,1]^2$ to $[0,1]$.
- Let $X_1, X_2, \ldots, X_n$ be independent and uniform random variables on $[0,1]$.
- The **exchangeable random graph model** $G(n,W)$ gives graphs with vertex set $[n]$ and the edge ij exists with probability $W(X_i,X_j)$ independently of the other edges.

Expected subgraph densities

The expected value of t(F,G) when G comes from G(n,W) and n≥m=#V(F) is t(F,W)=

$$\int_{(x_1,x_2,\ldots,x_m)\in[0,1]^m} \prod_{ij\in E(F)} W(x_i,x_j)dx_1dx_2\cdots dx_m$$

Graph limits

- If W and W' have t(F,W)=t(F,W') for all graphs F then G(n,W)=G(n,W') for all n.
- Two functions W and W' are equivalent if G(n,W)=G(n,W') for all n. The set of equivalence classes is the space of **graph limits** or **graphons**.
- This space is infinite dimensional and a bit difficult to work with.

Algebraic graph limits

- We want to investigate exchangeable random graphs where W is algebraic.
- A polynomial P in **R**[x,y] is an **algebraic graph limit** if it takes values in [0,1] on the triangle $\{(x,y) \in [0,1]^2 : 1 \geq x+y\}$.
- An algebraic graph limit gives an exchangeable random graph by setting $W(x,y)=P(1-x,y)$ for $x \geq y$ and symmetrizing.

Theorem

Any graph limit can be approximated arbitrary well with an algebraic graph limit. The algebraic graph limits are dense it the space of graph limits.

Examples

- A constant α_{000} is an algebraic graph limit if and only if $1 \geq \alpha_{000} \geq 0$. This is the Erdös–Rényi model.
- The polynomial $\alpha_{100}x+\alpha_{010}y+\alpha_{001}(1-x-y)$ is an algebraic graph limit if and only if $1 \geq \alpha_{100}, \alpha_{010}, \alpha_{001} \geq 0$.

Bounded degree

- Let $\Delta=\{(x,y) \in[0,1]^2 : 1 \geq x+y\}$.
- A polynomial P give a graph limit if $P(\Delta)\subset[0,1]$. We want to understand the set of all such polynomials of a given degree.
- The algebraic graph limits of degree d form a convex set. What is the boundary?
- There is an easy explicit description of the polynomials with $P(\text{int}(\Delta))\subset(0,1)$.

Theorem

The polynomial

$$P(x,y) = \sum_{i+j+k=d} \binom{d}{i,j,k} \alpha_{ijk} x^i y^j (1-x-y)^k$$

satisfies $P(\mathrm{int}(\Delta)) \subset (0,1)$ if $1 \geq \alpha_{ijk} \geq 0$, unless the polynomial is identically 1 or 0. Furthermore all polynomials with $P(\mathrm{int}(\Delta)) \subset (0,1)$ are of the form above.

Identifiability

- Any graph limit can be recovered by knowing all expected graph densities.
- Algebraic graph limits can be recovered by knowing a finite number of graph densities. This can be done with algebraic methods as the densities $t(F,W)$ are polynomials in the parameters α_{ijk}.

Identifiability

- Degree 0 algebraic graph limits can be recovered from knowing the edge density.
- Degree 1 algebraic graph limits can be recovered from knowing the edge, 2-path, and triangle densities.

A conjecture

- Algebraic graph limits of degree d can be recovered from $(d+2)(d+1)/2$ subgraph densities.
- This is a lower bound.
- Not all $(d+2)(d+1)/2$ work. For example stars do not work.

A second conjecture

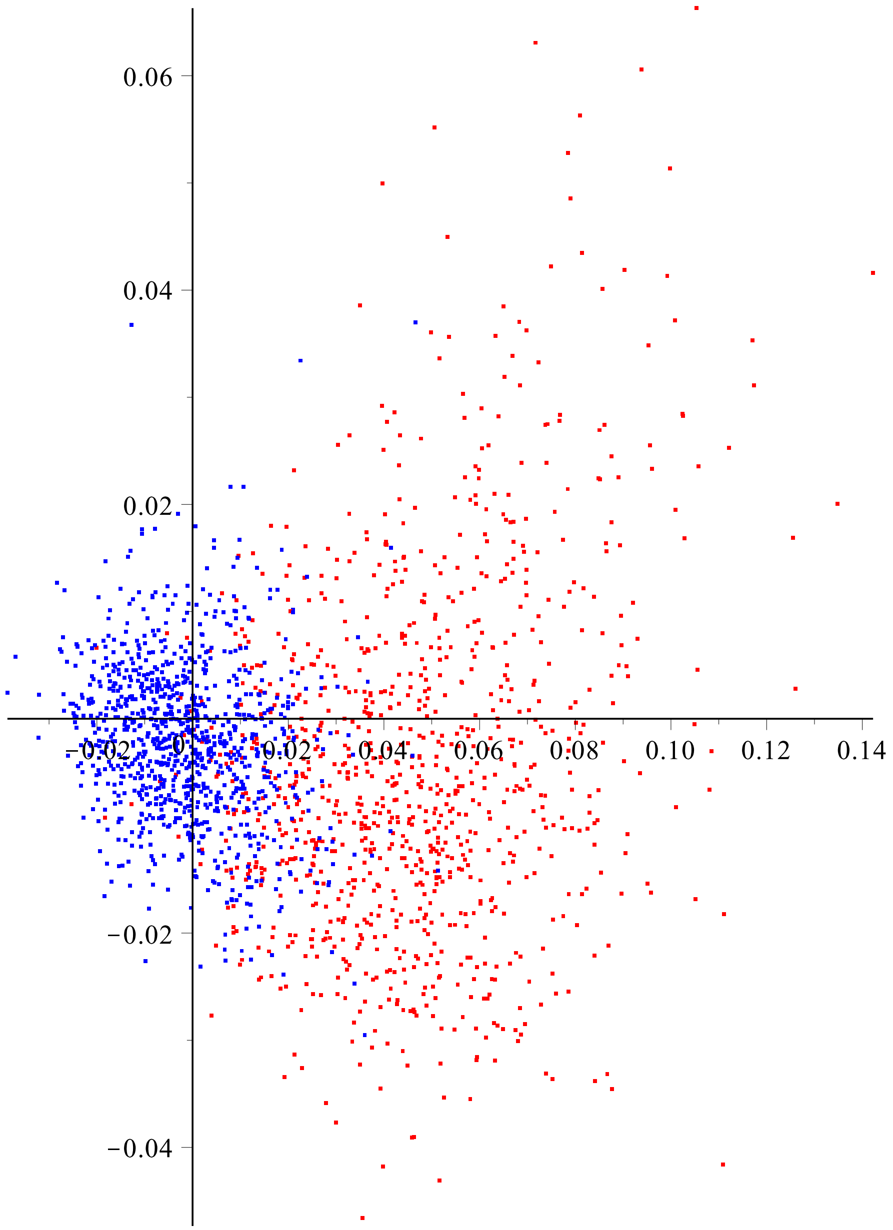
- If P(x,y) is an algebraic graph limit then P(y,x) is an algebraic graph limit and as graph limits they are equivalent.
- We conjecture that there is no other algebraic graph limit equivalent to P.

Some algebra

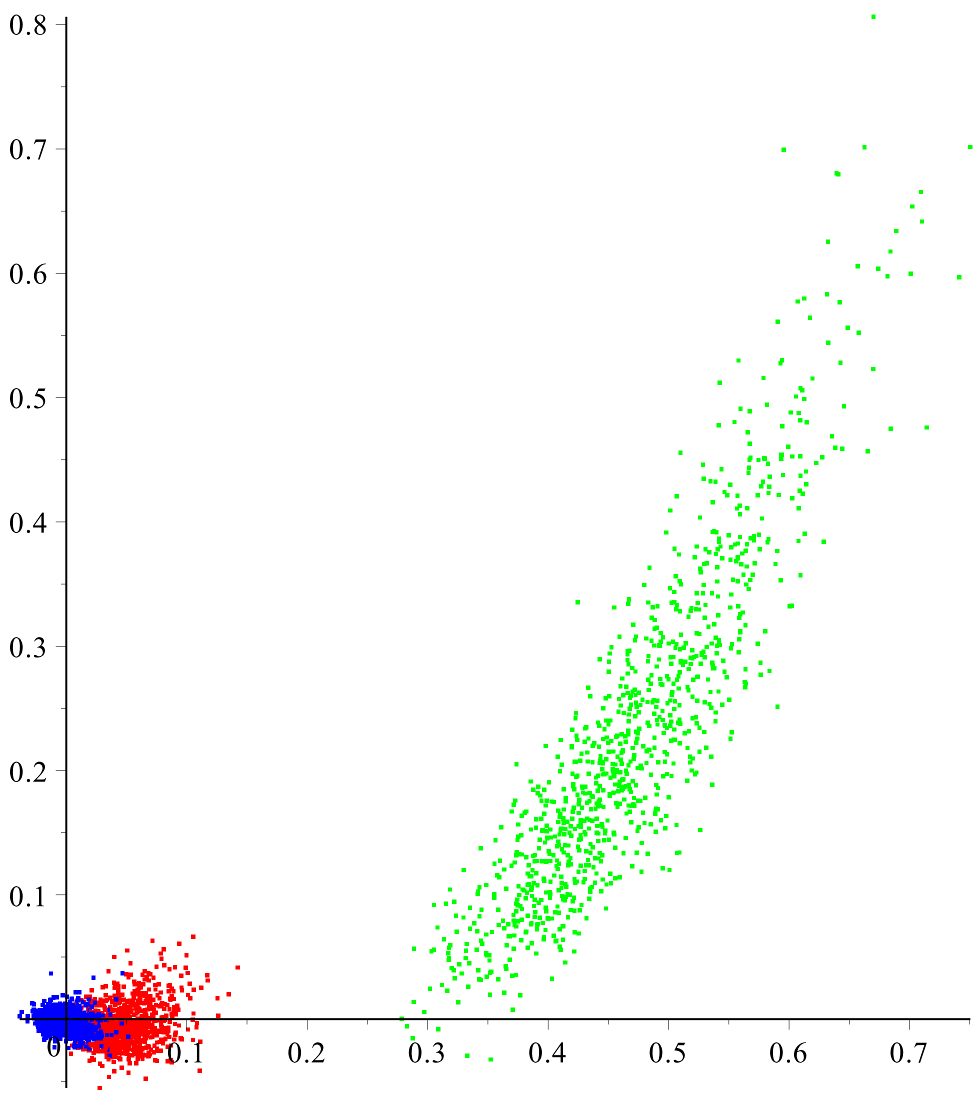
- There are algebraic relations among the expected subgraph densities from algebraic graph limits.
- For example $t(\text{edge},W)^2 - t(\text{2-path},W) = 0$ for constant W.
- For W from algebraic graph limits of degree 1 the following holds:

$$t(\text{3-star},W)+t(\text{3-path},W)+3t(\text{3-star},W)^3 \\ -5t(\text{edge},W)t(\text{2-path},W)=0$$

Some pictures



Some pictures



Thank You