From Factorial Designs to Hilbert Schemes

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Abstract

This talk is meant to explain the evolution of research which originated a few years ago from some problems in statistics.

In particular, the inverse problem for factorial designs gave birth to new ideas for the study of special schemes, called Border Basis Schemes.

They parametrize zero-dimensional ideals which share a quotient basis, and turn out to be open sets in the corresponding Hilbert Schemes.

General References and Advertising

G. Pistone – E. Riccomagno – H. Wynn: Algebraic Statistics: Computational Commutative Algebra in Statistics, Chapman&Hall (2000)

M. Kreuzer – L. Robbiano: Computational Commutative Algebra 1, Springer (2000)

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PART 1

The Inverse Problem in DoE (Design of Experiments)

Points and Statistics

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Definition

Let $\ell_i \geq 1$ for $i = 1, \ldots, n$ and $D_i = \{a_{i1}, a_{i2}, \ldots, a_{i\ell_i}\}$ with $a_{ij} \in K$.

- The affine point set $D = D_1 \times \cdots \times D_n \subseteq K^n$ is called the full design on $(D_1, \ldots, D_n)$ with levels $\ell_1, \ldots, \ell_n$.
- The polynomials $f_i = (x_i - a_{i1}) \cdots (x_i - a_{i\ell_i})$ with $i = 1, \ldots, n$ generate the vanishing ideal $\mathcal{I}(D)$ of D . They are called the canonical polynomials of D.

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Proposition

- *For every term ordering σ on $\mathbb{T}^n$, the canonical polynomials are the reduced σ -Gröbner basis of $\mathcal{I}(D)$.*
- *The order ideal (canonical set, factor closed set of power products,...) $\mathcal{O}_D = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i < \ell_i \text{ for } i = 1, \ldots, n\}$ is canonically associated to D and represents a K -basis of $P/\mathcal{I}(D)$.*

Points and Statistics II

- The main task is to identify an unknown function $f: D \to K$, called the model.
- In general it is not possible to perform all experiments corresponding to all points in D and measuring the value of f each time.
- [illegible] have some extra knowledge about the shape of f.
- In particular, we need to describe the sets of power products whose residue classes form a K-basis of $P/I(D)$. Statisticians express this property by saying that such sets of power products are identified by D.

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- In general it is not possible to perform all experiments corresponding to the points in D and measuring the value of $\bar{f}$ each time.
- A subset F of a full design D is called a fraction.
- We want to choose a fraction $F \subseteq D$ that allows us to identify the model if we have some extra knowledge about the shape of $\bar{f}$.
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The Inverse Problem

- Conversely, given $\mathcal{O}$, how can we choose the fractions F such that the matrix of coefficients is invertible?
- In other words, given a full design D and an order ideal $\mathcal{O} \subseteq \mathbb{T}^n$, which fractions $F \subseteq D$ have the property that the residue classes of the elements of $\mathcal{O}$ are a K-basis of $P/\mathcal{I}(F)$?
- This is called the inverse problem of DoE.

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The following conditions are equivalent.

- *The order ideal $\mathcal{O}$ is identified by the fraction F.*
- *The vanishing ideal $\mathcal{I}(F)$ has an $\mathcal{O}$-border basis.*
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This problem was partially solved in

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Proc. of ISSAC-1997 (Maui, Hawaii, July 1997) (New York, N.Y.),
W.W. Küchlin, Ed. (1997), 404–409.

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An Example

- Let D be the full design $D = \{-1, 0, 1\} \times \{-1, 0, 1\}$. The task it to solve the inverse problem for the order ideal $\mathcal{O} = \{1, x, y, x^2, y^2\}$.
- It turns out that [illegible] there are exactly 81 five-tuples which solve the inverse problem.

  [illegible] of these 81 [illegible] are not of [illegible] type.
- This is a surprisingly high number which shows that border bases provide a much more flexible environment for working with zero-dimensional ideals than Gröbner bases do.
- The details are explained in
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- It turns out that we have to solve a system defined by 20 quadratic polynomials. Using CoCoA, we check that among the $126 = \binom{9}{5}$ five-tuples of points in D there are exactly 81 five-tuples which solve the inverse problem.
- It is natural to ask how many of these 81 fractions have the property that $\mathcal{O}$ is of the form $\mathbb{T}^n \setminus \mathsf{LT}_\sigma\{\mathcal{I}(F)\}$ with σ varying among the term orderings. One can prove that 36 of those 81 fractions are not of that type.
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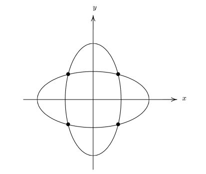
PART 2

Border Bases: The Continuous Case

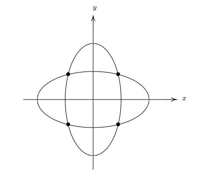
Two conics I



The set $\{x^2 - \frac{4}{5},\ y^2 - \frac{4}{5}\}$ is the universal reduced Gröbner basis of the ideal $I = (f_1, f_2) \subseteq \mathbb{C}[x, y]$, in particular with respect to $\sigma = \texttt{DegRevLex}$.

$\mathrm{LT}_\sigma(I) = (x^2, y^2)$, and the residue classes of the terms in $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{I\} = \{1, x, y, xy\}$ form a $\mathbb{C}$-vector space basis of $\mathbb{C}[x, y]/I$.

Two conics I



Example

Consider the polynomial system

$$\begin{array}{rcccl} f_1 & = & \frac{1}{4}x^2 + y^2 - 1 & = & 0 \\ f_2 & = & x^2 + \frac{1}{4}y^2 - 1 & = & 0 \end{array}$$

$\mathbb{X} = \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{4/5}, \pm\sqrt{4/5})\}$.

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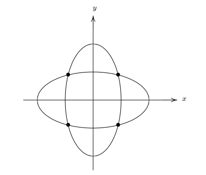

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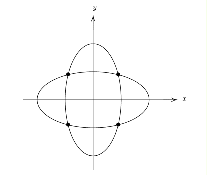

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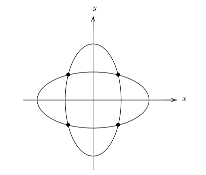

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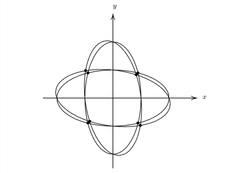

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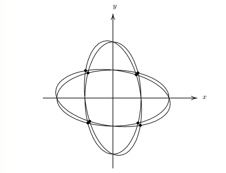

The intersection of $\mathcal{Z}(\tilde{f}_1)$ and $\mathcal{Z}(\tilde{f}_2)$ consists of four perturbed points $\tilde{\mathbb{X}}$ close to the points in $\mathbb{X}$.

- The ideal $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ has the reduced σ-Gröbner basis

$$\{x^2 - y^2,\ xy + \tfrac{5}{4\varepsilon}y^2 - \tfrac{1}{\varepsilon},\ y^3 - \tfrac{16\varepsilon}{16\varepsilon^2-25}x + \tfrac{20}{16\varepsilon^2-25}y\}$$

- Moreover, we have $\mathrm{LT}_\sigma(\tilde{I}) = (x^2, xy, y^3)$ and $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\tilde{I}\} = \{1, x, y, y^2\}$.

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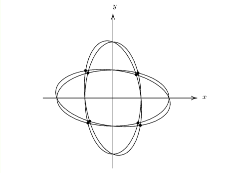

Now consider the slightly perturbed polynomial system

$$\begin{array}{rcl} \tilde{f}_1 & = & \frac{1}{4}x^2 + y^2 + \varepsilon\, xy - 1 & = & 0 \\ \tilde{f}_2 & = & x^2 + \frac{1}{4}y^2 + \varepsilon\, xy - 1 & = & 0 \end{array}$$

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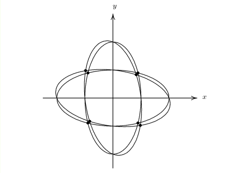

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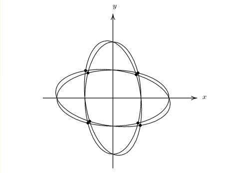

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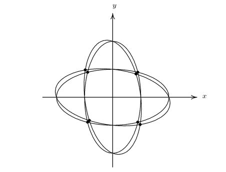

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Border Bases

The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of monomials $\mathcal{O}$ whose residue classes form a K-basis of P/I and by the multiplication matrices of this basis.

Let K be a field, let $P = K[x_1, \ldots, x_n]$, let $\mathbb{T}^n$ be the monoid of terms, and let $\mathcal{O} \subseteq \mathbb{T}^n$ be an order ideal.

Let $\mathcal{O}$ have μ elements and $\partial\mathcal{O}$ have ν elements.
The border of $\mathcal{O}$ is the set $\partial\mathcal{O} = \mathbb{T}^n_1 \cdot \mathcal{O} \setminus \mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$.
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What are the border bases in the two cases of the conics and the perturbed conics?

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Existence and Uniqueness of Border Bases

Proposition

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ *be an order ideal, let* $I \subseteq P$ *be a zero-dimensional ideal, and assume that the residue classes of the elements of* $\mathcal{O}$ *form a* K *-vector space basis of* P/I *. Then there exists a unique* $\mathcal{O}$ *-border basis of* I *.*

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Let σ *be a term ordering on* $\mathbb{T}^n$ *, and let* $\mathcal{O}_\sigma(I)$ *be the order ideal* $\mathbb{T}^n \setminus \mathsf{LT}_\sigma\{I\}$ *. Then there exists a unique* $\mathcal{O}_\sigma(I)$ *-border basis* G *of* I *, and the reduced* σ *-Gröbner basis of* I *is the subset of* G *corresponding to the corners of* $\mathcal{O}_\sigma(I)$ *.*

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Commuting matrices

The following is a fundamental fact.

B. Mourrain: A new criterion for normal form algorithms, AAECC, Lecture Notes in Computer Science 1719 (1999), 430–443

Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal, let $G = \{g_1, \dots, g_\nu\}$ be an $\mathcal{O}$-border prebasis, and let $I = (g_1, \dots, g_\nu)$. Then the following conditions are equivalent.

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PART 3

Border Bases and the Hilbert Scheme

A glimpse at punctual Hilbert schemes

- Punctual Hilbert schemes are schemes which parametrize all the zero-dimensional projective subschemes of $\mathbb{P}^n$ which have the same multiplicity.

- There is a one-to-one correspondence between zero-dimensional ideals in $P = K[x_1, \dots, x_n]$ and zero-dimensional saturated homogeneous ideals in $\overline{P} = K[x_0, x_1, \dots, x_n]$. The correspondence is given by homogenization and dehomogenization.

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A glimpse at punctual Hilbert schemes

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An Example: Hilbert Polynomial = 4

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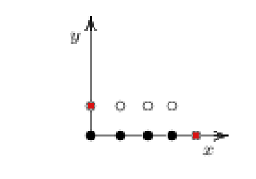

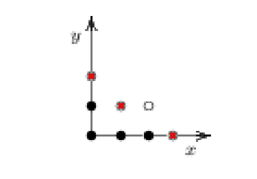

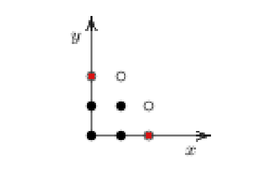

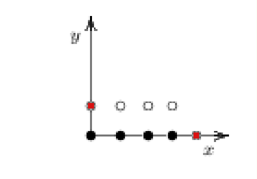
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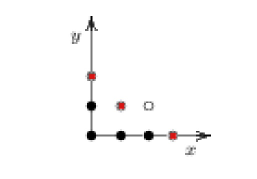
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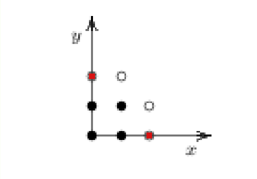
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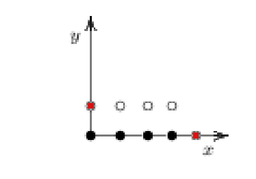

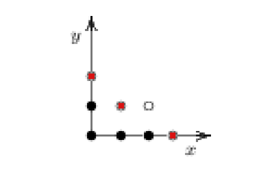

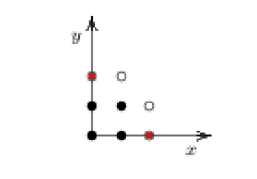

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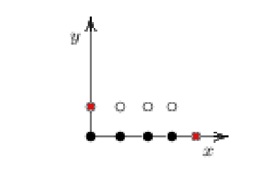

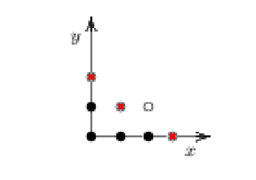

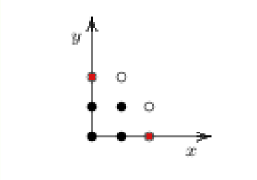

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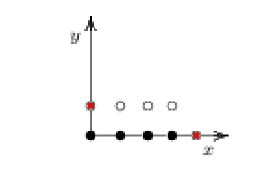

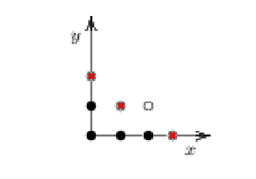

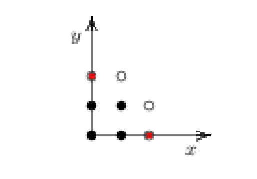

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1. The generic $\mathcal{O}$-border prebasis is the set of polynomials $G = \{g_1, \ldots, g_\nu\}$ in $Q = K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ given by $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i$.
2. For $k = 1, \ldots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the k^{th} formal multiplication matrix associated to G. Then the affine scheme $\mathbb{B}_{\mathcal{O}} \subseteq K^{\mu\nu}$ defined by the ideal $I(\mathbb{B}_{\mathcal{O}})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_\ell - \mathcal{A}_\ell\mathcal{A}_k$ with $1 \le k < \ell \le n$ is called the $\mathcal{O}$-border basis scheme.

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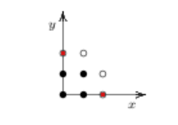

Let $\mathcal{O} = \{1, x, y, xy\}$. We observe that $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = x^2$, $b_2 = y^2$, $b_3 = x^2y$, $b_4 = xy^2$. Let $\sigma = \texttt{DegRevLex}$, so that $x >_\sigma y$.

$$g_4 = xy^2 - c_{14}1 - c_{24}x - c_{34}y - c_{44}xy$$

- If we do the Gröbner computation via critical pairs, then necessarily $c_{42} = 0$, so that g_2 is replaced by

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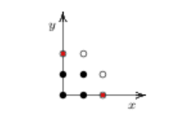

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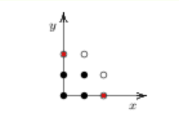

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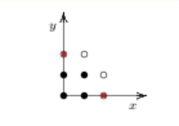

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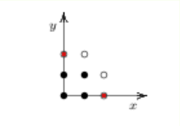

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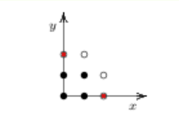

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- The border basis scheme is in general reducible (see the well-known example by Iarrobino).
- In the case $n = 2$ more precise information is available: for instance, it is known that $\mathbb{B}_{\mathcal{O}}$ is reduced, irreducible and smooth of dimension 2μ. Recently M. Huibregtse showed that it is a complete intersection.

Border Bases and Hilbert Schemes

Here we collect some basic observations about border basis schemes in relation with Hilbert schemes.
A good reference is
E. Miller, B. Sturmfels: Combinatorial Commutative Algebra, Graduate Texts in Mathematics 277, Springer 2005.

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An Open Problem

The scheme [illegible] are connected to the origin.

However, the problem of the connectedness of $\mathbb{B}_{\mathcal{O}}$ is still open.

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The scheme $\mathbb{G}_{\mathcal{O},\sigma}$ is connected since it is a quasi-cone, and hence all its points are connected to the origin.

We know the precise relation between the two schemes $\mathbb{G}_{\mathcal{O},\sigma}$ and $\mathbb{B}_{\mathcal{O}}$. However, the problem of the connectedness of $\mathbb{B}_{\mathcal{O}}$ is still open.

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References

A. Kehrein and M. Kreuzer: *Characterizations of border bases*,
J. Pure Appl. Algebra **196** (2005), 251–270.

A. Kehrein and M. Kreuzer: *Computing border bases*,
J. Pure Appl. Algebra **205** (2006), 279–295.

A. Kehrein, M. Kreuzer and L. Robbiano: *An algebraist's view on border bases*, in:
A. Dickenstein and I. Emiris (eds.), **Solving Polynomial Equations: Foundations, Algorithms, and Applications**, Springer, Heidelberg 2005, 169–202.

M. Kreuzer, L. Robbiano: *Deformations of border bases*,
Collectanea Math. **59** (2008)

L. Robbiano: *On border basis and Gröbner basis schemes*,
Collectanea Math. **60** (2009)

M. Kreuzer, L. Robbiano: *The Geometry of Border Bases*,
J. Pure Appl. Algebra **215**, 2005–2018 (2011).

PART 3

POSTERS

The Posters: Presentation

E. Palezzato: *Computing simplicial complexes with CoCoA.*

I. Burke: *Exploiting symmetry in characterizing bases of toric ideals.*

A. Bigatti, M. Caboara: *A statistical package in CoCoA-5.*

D. Pavlov: *Finding the statistical fan of an experimental design.*

Posters I

Given a full design D and an order ideal $\mathcal{O} \subseteq \mathcal{O}_D$, which fractions $F \subseteq D$ have the property that the residue classes of the elements of $\mathcal{O}$ are a K-basis of $P/\mathcal{I}(F)$?

This is called the inverse problem of DoE.

This problem was partially solved in

M. Caboara and L. Robbiano: *Families of Ideals in Statistics*,
Proc. of ISSAC-1997 (Maui, Hawaii, July 1997) (New York, N.Y.), W.W. Küchlin, Ed. (1997), 404–409.

with the use of Gröbner bases,

and totally solved in

M. Caboara and L. Robbiano: *Families of Estimable Terms*
Proc. of ISSAC 2001, (London, Ontario, Canada, July 2001) (New York, N.Y.), B. Mourrain, Ed. 56–63.

with the use of Border bases.

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with the use of Border bases.

Posters II: Fans (of Gröbner?)

Mora, T., Robbiano, L. *The Gröbner Fan of an Ideal*,
J. Symbolic Comput. **6** 183–208 (1988).

Bayer, D., Morrison, I. *Standard bases and geometric invariant theory I. Initial ideals and state polytopes*
J. Symbolic Comput. **6** 209–217 (1988).