The maximum likelihood degree of rank 2 matrices via Euler characteristics

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Joint work with Botong Wang

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A mixture of independence models

1. Consider a pair of four sided dice: one red die and one blue die R_1, B_1.
2. Consider a second pair of four sided dice: one red die and one blue die R_2, B_2.
3. Consider a biased coin $C = [c_1, c_2]$

- The following map induces a set of probability distributions denoted $\mathscr{M}_{44} \subset \triangle_{15} \subset \mathbb{R}^{16}$ and is called the model.

$$\Delta_1 \times (\Delta_3 \times \Delta_3) \times (\Delta_3 \times \Delta_3) \to \mathscr{M}_{44} \subset \Delta_{15} \subset \mathbb{R}^{16}$$

$$c_1 R_1 B_1^T + c_2 R_2 B_2^T = [p_{ij}]$$

- $\mathscr{M}_{44}$ is the set of 4×4 nonnegative rank at most 2 matrices.
- $\mathscr{M}_{44}$ is a mixture of two independence models.

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Collecting data and the likelihood function

Roll the dice

- Rolling the dice we may observe the following data:

$$u = [u_{ij}] = \begin{bmatrix} 160 & 8 & 16 & 24 \\ 32 & 200 & 16 & 8 \\ 8 & 24 & 176 & 32 \\ 16 & 40 & 8 & 232 \end{bmatrix}$$

- To each p in the set of probability distributions $\mathscr{M}_{44}$ we assign the likelihood of p with respect to u by the likelihood function:

$$\ell_u(p) = \binom{\sum u_{ij}}{u_{11}, \ldots, u_{44}}^{-1} \prod_{ij} p_{ij}^{u_{ij}}.$$

- The probability distribution maximizing $\ell_u(p)$ on the set of distributions $\mathscr{M}_{44}$ is called the maximum likelihood estimate (mle).
- The mle is the best point of $\mathscr{M}_{44}$ to describe the observed data.
- The statistics problem is to determine mle's.

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Applied Algebraic Geometry

The mle can be determined by solving the likelihood equations.

- Instead of $\mathscr{M}_{44}$, we consider its Zariski closure X_{44}.
- The Zariski closure is described by zero sets of homogeneous polynomials.
- The defining polynomials of X_{44} are the 3×3 minors of

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}$$

  and the linear constraint $p_{11} + p_{12} + \cdots + p_{44} - p_s = 0$.
- The equations define a projective variety of $\mathbb{P}^{16}$: rank at ≤ 2 matrices
- We consider the homogenized likelihood function $\ell_u(p) = \prod_{ij} (p_{ij}/p_s)^{u_{ij}}$ on X_{44}.

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Geometric definition of critical points

Critical points can be determined by solving a system of polynomial equations.

- For the models in this talk, the mle is a critical point of the homogenized likelihood function.
- The solutions to the likelihood equations are critical points.
- One way to formulate the likelihood equations is to use Lagrange multipliers.
  - We omit a formal description of the likelihood equations, but instead give a geometric description of critical points.

Geometric definition of critical points (cont.)

Critical points can be determined by solving a system of polynomial equations.

- Let X^o denote the open variety $X \setminus \{\text{coordinate hyperplanes}\}$.
  - X^o is the set of points in X which have nonzero coordinates.
- The gradient of the likelihood function up to scaling equals

$$\nabla \ell_u(p) = \begin{bmatrix} \frac{u_{11}}{p_{11}} & \frac{u_{12}}{p_{12}} & \cdots & \frac{u_{44}}{p_{44}} & \frac{u_s}{p_s} \end{bmatrix}, \quad u_s := -\sum_{ij} u_{ij}.$$

  - The gradient is defined on X^o.
- We say $p \in X^o$ is a complex critical point, whenever $\nabla \ell_u(p)$ is orthogonal to the tangent space of X at p and $p \in X^o_{reg}$.
- The mle is a critical point (in the cases we consider).

Two experiments and ML degree

Two experiments

- Consider *vectorized* datasets u for likelihood function $\ell_u(p)$ on X_{44}.
  - $u = \{160, 8, 16, 24, 32, 200, 16, 8, 8, 24, 176, 32, 16, 40, 8, 232\}$
    - 191 complex including 25 real
  - $u = \{292, 45, 62, 41, 142, 51, 44, 42, 213, 75, 67, 63, 119, 85, 58, 70\}$
    - 191 complex including 3 real
- The # of complex solutions was always 191 (this is the ML degree).
- For general choices of u we get the same number of complex critical points.
  - This number is called the *ML degree* of a variety.

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Previous Computational Results

- Consider the mixture model $\mathscr{M}_{mn}$ for m-sided red dice and n-sided blue dice. Denote its Zariski closure by X_{mn}.

Theorem

The ML-degrees of X_{mn} include the following:

(m,n)	3	4	5	6	7	8	9	10	11	12
3	10	26	**58**	**122**	**250**	**506**	**1018**	**2042**	*4090*	*8186*
4	26	**191**	**843**	**3119**	**6776**	?	?	?	?	?

- Reference: "Maximum likelihood for matrices with rank constraints"
  - J. Hauenstein, [], and B. Sturmfels using Bertini.
- Any conjectures? (Hint add 6.)
- "Maximum likelihood geometry in the presence of sampling and model zeros" gave supporting evidence for up to $n = 15$.
  - E. Gross and [] using Macaulay2.

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Euler characteristics and ML degrees

Huh proves that the ML degrees are an Euler characteristic in the smooth case.

- Let X be a smooth variety of $\mathbb{P}^{n+1}$ defined by homogeneous polynomials and the linear constraint

$$p_0 + p_1 + \cdots + p_n - p_s = 0.$$

- Let X^o denote the open variety $X \setminus \{\text{coordinate hyperplanes}\}$.

Theorem [Huh]

The ML degree of the *smooth* variety X equals the signed Euler characteristic of X^o, i.e.

$$\chi(X^o) = (-1)^{\dim X}\, \text{MLdegree}(X).$$

- The independence model (one sided coin) is smooth but the mixture model is not.

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Independence model ML degree

Use Huh's result to give a topological proof.

- Let Z denote the Zariski closure of the independence model, a variety of $\mathbb{P}^{16}$.
- The following map gives an algebraic geometry parameterization of Z.

$$\mathbb{P}^3 \times \mathbb{P}^3 \to Z$$

$$([r_1, r_2, r_3, r_4], [b_1, b_2, b_3, b_4]) \to \left[r_i b_j, \sum_{ij} r_i b_j \right] \text{ where } i, j \in \{1, 2, 3, 4\}.$$

- Let $\mathscr{O}$ denote $\mathbb{P}^3 \setminus \mathbf{V}(x_0 x_1 x_2 x_3 (x_0 + x_1 + x_2 + x_3))$. Then we have a parameterization of X^o given by

$$\mathscr{O} \times \mathscr{O} \to X^o$$

because $\sum_{ij} r_i b_j = (\sum_i r_i)(\sum_j b_j)$.

- Using inclusion-exclusion and the additive properties of Euler characteristics we see that $\chi(\mathscr{O}) = -1$.
- By the product property $\chi(\mathscr{O} \times \mathscr{O}) = 1$.
- This parameterization is a homeomorphism thus $\chi(\mathscr{O} \times \mathscr{O}) = \chi(X^o)$.

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ML degrees of singular models

The ML degree is a stratified topological invariant.

- Let $(S_1, S_2, \ldots, S_k)$ denote a Whitney stratification of X^o.
  - When X^o is smooth the Whitney stratification is (X^o).
  - When $k = 2$, $S_1 = X^o_{reg}$ and $S_2 = X^o_{sing}$.

Theorem

Given reduced irreducible X^o with Whitney stratification $(S_1, \ldots, S_k)$, we have

$$\chi\left(X^o_{reg}\right) = e_{11}\mathrm{MLdegree}\left(\bar{S}_1\right) + e_{21}\mathrm{MLdegree}\left(\bar{S}_2\right) + \cdots + e_{k1}\mathrm{MLdegree}\left(\bar{S}_k\right).$$

- The e_{ij} are topological invariants called Euler obstructions, which can be considered as the topological multiplicity of the singularities.
- This theorem is a corollary of Botong Wang and Nero Budur's result that relates ML degrees to Gaussian degrees.
- The Euler obstruction e_{11} always equals $(-1)^{\dim X^o}$.

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Ternary Cubic Example for Singular Case

We determine the ML degree of a singular X using the previous theorem.

- Let X be defined by

$$p_2(p_1-p_2)^2-(p_0-p_2)^3=p_0+p_1+p_2-p_s=0.$$

- The Whitney stratification of X^o consists of S_1 the regular points (so $\bar{S}_1=X$) and S_2 the singular point which is $[1:1:1:3]$,

$$\chi(S_1)=e_{11}\text{MLdegree}(X)+e_{21}\text{MLdegree}(\bar{S}_2).$$

- S_2 is a point so $S_2=\bar{S}_2$ and $\text{MLdegree}(\bar{S}_2)=1$.
- The Euler obstruction e_{21} is the signed multiplicity of the singular point, i.e. $e_{21}=-2$.
  - In general, the sign depends on the dimension of S_2 and the multiplicity is actually the Euler characteristic of a link [Kashiwara].
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Returning to the mixture model

We apply the Whitney stratification-ML degree theorem to X_{mn}^o.

- The Whitney stratification of $X^o = X_{mn}^o$ is given by (S_1, S_2) where S_1 are the regular points $X_{mn}^o \backslash Z_{mn}^o$ and S_2 are the singular points Z_{mn}^o.
  - Denote the singular points of X_{mn}^o by Z_{mn}^o.
  - Z_{mn}^o should be thought of as the set of rank 1 matrices (Z_{mn} is the Zariski closure of the independence model)
- By the theorem we have

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Determining the Euler characteristic $\chi(X^o_{mn} \backslash Z^o_{mn})$

This is our main theorem.

- If we knew $\chi(X^o_{mn} \backslash Z^o_{mn})$, then we would know $\text{MLdegree}(X_{mn})$.
- Let Λ_m be a sequence of $m-1$ integers $(\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$.

Theorem [- and B. Wang]

Fix m greater than or equal to 2. Then, there exists Λ_m such that

$$\chi(X^o_{mn} \backslash Z^o_{mn}) = (-1)^{n-1} \sum_{1 \le i \le m-1} \frac{\lambda_i}{i+1} - \sum_{1 \le i \le m-1} \frac{\lambda_i}{i+1} \cdot i^{n-1}.$$

- Now we prove the conjecture of Hauenstein, [], Sturmfels.

Using the main theorem

Fix $m = 3$.

$$\chi\left(X^o_{3n} \backslash Z^o_{3n}\right) = (-1)^{n-1}\left(\frac{\lambda_1}{2} + \frac{\lambda_2}{3}\right) - \left(\frac{\lambda_1}{2}\cdot 1^{n-1} + \frac{\lambda_2}{3}\cdot 2^{n-1}\right).$$

$$\chi\left(X^o_{mn} \backslash Z^o_{mn}\right) = -\mathrm{MLdegree}\left(X_{3n}\right) + (-1)^{3+n-1}\left(\min\{3, n\} - 1\right).$$

- $\mathrm{MLdegree}\left(X_{32}\right) = 1$ yields the relation $-\lambda_1 - \lambda_2 = 0$.
- $\mathrm{MLdegree}\left(X_{33}\right) = 10$ yields the relation $-\lambda_2 = -12$.

$$\mathrm{MLdegree}\left(X_{3n}\right) = \left(2^{n+1} - 6\right) + (-1)^n\left(\left(\min\{3, n\} - 3\right)\right)$$

- Main idea: For fixed m, if we knew

$$\mathrm{MLdegree}\left(X_{m2}\right), \mathrm{MLdegree}\left(X_{m3}\right), \ldots, \mathrm{MLdegree}\left(X_{mm}\right)$$

then we can solve for $\Lambda_m = (\lambda_1, \ldots, \lambda_{m-1})$ thereby giving a closed form expression for $\mathrm{MLdegree}\left(X_{mn}\right)$ for all n.

Using the main theorem

Fix $m = 3$.

$$\chi\left(X_{3n}^{o} \setminus Z_{3n}^{o}\right) = (-1)^{n-1}\left(\frac{\lambda_1}{2} + \frac{\lambda_2}{3}\right) - \left(\frac{\lambda_1}{2} \cdot 1^{n-1} + \frac{\lambda_2}{3} \cdot 2^{n-1}\right).$$

$$\chi\left(X_{mn}^{o} \setminus Z_{mn}^{o}\right) = -\mathrm{MLdegree}\left(X_{3n}\right) + (-1)^{3+n-1}\left(\min\{3, n\} - 1\right).$$

- $\mathrm{MLdegree}\left(X_{32}\right) = 1$ yields the relation $-\lambda_1 - \lambda_2 = 0$.
- $\mathrm{MLdegree}\left(X_{33}\right) = 10$ yields the relation $-\lambda_2 = -12$.

$$\mathrm{MLdegree}\left(X_{3n}\right) = \left(2^{n+1} - 6\right) + (-1)^{n}\left(\left(\min\{3, n\} - 3\right)\right)$$

- Main idea: For fixed m, if we knew

$$\mathrm{MLdegree}\left(X_{m2}\right), \mathrm{MLdegree}\left(X_{m3}\right), \ldots, \mathrm{MLdegree}\left(X_{mm}\right)$$

then we can solve for $\Lambda_m = \left(\lambda_1, \ldots, \lambda_{m-1}\right)$ thereby giving a closed form expression for $\mathrm{MLdegree}\left(X_{mn}\right)$ for all n.

Do Better

- Main idea (from previous slide): For fixed m, if we knew

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  then we can solve for $\Lambda_m = (\lambda_1, \ldots, \lambda_{m-1})$ thereby giving a closed form expression for $\mathrm{MLdegree}(X_{mn})$ for all n.
- We can recursively determine Λ_m thereby giving a closed form formula for $\mathrm{MLdegree}(X_{mn})$ for fixed m but *any* n.
  - Note $\mathrm{MLdegree}(X_{mn}) = \mathrm{MLdegree}(X_{nm})$.
  - Prove λ_{m-1} of Λ_m is $(m-1)m!$.
- Closed form expressions for fixed m and $n \geq m$:

$$\mathrm{MLdeg} X_{4n} = 25 \cdot 1^{n-1} - 40 \cdot 2^{n-1} + 23 \cdot 3^{n-1}$$

$$\mathrm{MLdeg} X_{5n} = -90 \cdot 1^{n-1} + 260 \cdot 2^{n-1} - 270 \cdot 3^{n-1} + 96 \cdot 4^{n-1}$$

$$\mathrm{MLdeg} X_{6n} = 301 \cdot 1^{n-1} - 1400 \cdot 2^{n-1} + 2520 \cdot 3^{n-1} - 2016 \cdot 4^{n-1} + 600 \cdot 5^{n-1}$$

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Using Numerical Algebraic Geometry

Witness sets allow us to use parallelizable algorithms.

- Treat the u_{ij} as parameter values that we can adjust,
- If we have a set of critical points for generic data, then we can solve any specific instance of data quickly using a parameter homotopy.
- Critical points of ℓ_u for $u_{general}$ are taken to
  - critical points of ℓ_u for $u_{specific}$
  - by a parameter homotopy

$$\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} ---> \begin{bmatrix} 160 & 8 & 16 & 24 \\ 32 & 200 & 16 & 8 \\ 8 & 24 & 176 & 32 \\ 16 & 40 & 8 & 232 \end{bmatrix}$$

191 points $--->$ 191 points

  - $\square$ denotes a random complex number.

Thank You

- Contact information
  - Jose Israel Rodriguez
  - jo.ro@ND.edu
  - http://www.nd.edu/~jrodri18/
- SIAM: AG15 in Daejeon, Korea, Aug 3-7.
  - Co-organizing a mini-sympoium with Xiaoxian Tang: Maximum Likelihood Degrees and Critical Points
  - http://www.nd.edu/~jrodri18/quickLinks/AG15rt/

Outline

- Statistics
  - Mixture model
- Applied algebraic geometry
  - Critical points
- Topology
  - ML degree
  - Euler obstructions