A linear-algebraic criterion for indecomposable generalized permutohedra

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Agenda

- Motivation: conditional independence
- Generalized permutohedra
- Indecomposable GP
- Supermodular games
- Criterion

Motivation: geometry of conditional independence

The talk concerns the geometry of *conditional independence* (CI).

- J.R. Morton. Geometry of conditional independence. PhD thesis, University of California Berkeley, 2007.
- M. Studený. *Probabilistic Conditional Independence Structures*. Springer, 2005.

Morton in his thesis established a one-to-one correspondence between structural CI models (Studený, 2005) and certain polytopes, namely Minkowski summands of the permutohedron.

These polytopes are known as *generalized permutohedra*.

Generalized permutohedra

The generalized permutohedra (GP) were introduced by Postnikov et al. as the polytopes obtainable by moving vertices of the usual permutohedron while the directions of edges are preserved.

- A. Postnikov. Permutohedra, associahedra, and beyond. *International Mathematics Research Notices* 6 (2009) 1026–1106; see also `arxiv.org/abs/math/0507163`.
- A. Postnikov, V. Reiner, L. Williams. Faces of generalized permutohedra. *Documenta Mathematica* 13 (2008) 207–273.

Generalized permutohedron: definition

Υ ... the class of enumerations of an unordered set N

$|N| = n \geq 2$, a bijection $\pi : \{1, \ldots, n\} \to N$

Definition (generalized permutohedron)

Let $\{v_\pi\}_{\pi \in \Upsilon}$ be a collection of vectors in $\mathbb{R}^N$ parameterized by enumerations (of N) such that for every $\pi \in \Upsilon$ and for every adjacent transposition $\sigma : \ell \leftrightarrow \ell + 1$, where $1 \leq \ell < n$, a non-negative constant $k_{\pi,\ell} \geq 0$ exists such that

$$v_\pi - v_{\pi\sigma} = k_{\pi,\ell} \cdot (\chi_{\pi(\ell)} - \chi_{\pi(\ell+1)}),$$

where $\pi\sigma$ denotes the composition of π with σ and $\chi_i \in \mathbb{R}^N$ is the zero-one identifier of a variable $i \in N$. The respective *generalized permutohedron* is then the convex hull of that collection of vectors:

$$G(\{v_\pi\}_{\pi \in \Upsilon}) := \mathrm{conv}(\{v_\pi \in \mathbb{R}^N : \pi \in \Upsilon\}).$$

Connection to supermodular/submodular functions

The connection of GP to supermodular/submodular functions was indicated by Doker.

J.S. Doker. Geometry of generalized permutohedra. PhD thesis, University of California Berkeley, 2011.

Definition (lower-standardized supermodular function)

A function $m \in \mathbb{R}^{\mathcal{P}(N)}$ is *supermodular* if

$$\forall A, B \subseteq N \quad m(A) + m(B) \leq m(A \cup B) + m(A \cap B)\,.$$

Moreover, we call m lower-standardized, or briefly *ℓ-standardized*, if

$$m(S) = 0 \qquad \text{for any } S \subseteq N \text{ with } |S| \leq 1.$$

The symbol $\Diamond(N)$ is used to denote the class of supermodular functions on $\mathcal{P}(N)$ satisfying $m(\emptyset) = 0$.

Coalition games and the concept of a core polytope

Supermodular functions m satisfying $m(\emptyset) = 0$ play an important role in coalition game theory, where they are named *convex games*.

Definition

Given a game $m : \mathcal{P}(N) \to \mathbb{R}$, $m(\emptyset) = 0$ its core is the polytope in $\mathbb{R}^N$ defined as follows:

$$\mathcal{C}(m) := \left\{ \mathbf{x} \in \mathbb{R}^N \mid \forall\, S \subseteq N \;\sum_{i \in S} x_i \geq m(S) \;\;\&\;\; \sum_{i \in N} x_i = m(N) \right\}.$$

Cores of supermodular games

In our recent manuscript (Studený, Kroupa, 2014) we showed that the class of GP coincide with the *cores* of supermodular games.

M. Studený, T. Kroupa. Core-based criterion for extreme supermodular functions. Submitted to Discrete Applied Mathematics, available at arxiv.org/abs/1410.8395 .

Theorem

A polytope $P \subseteq \mathbb{R}^N$ is a generalized permutohedron iff it is the core of a *supermodular* game m over N, that is, iff $\exists\, m \in \Diamond(N)$ such that $P = \mathcal{C}(m)$.

Note that Doker (2011) gave an ambiguous formulation of the above fact.

Minkowski summands of a polytope

The third possible view on generalized permutohedra is as follows.

Definition (Minkowski summand)

A polytope $P \subseteq \mathbb{R}^N$ is a Minkowski summand of a polytope $Q \subseteq \mathbb{R}^N$ if there exists $\lambda > 0$ and a polytope $R \subseteq \mathbb{R}^N$ such that $\lambda \cdot Q = P \oplus R$.

The following auxiliary fact was also proved in our manuscript.

Theorem

A polytope $P \subseteq \mathbb{R}^N$ is a generalized permutohedron iff it is a Minkowski summand of the classic permutohedron.

Indecomposable generalized permutohedra

We have been interested in the description of those supermodular games that are *extreme* (= generating the extreme rays of the pointed cone $\Diamond_{\ell}(N)$ of ℓ-standardized supermodular games).

It turns out that the cores for these extreme supermodular games are just those generalized permutohedra P that are *indecomposable* in sense of (Meyer 1974).

W.J. Meyer. Indecomposable polytopes. *Transaction of the American Mathematical Society* 190 (1974) 77–86.

Definition (indecomposable polytope)

A polytope P is called *indecomposable* if every Minkowski summand of $P \subseteq \mathbb{R}^N$ is $\alpha \cdot P \oplus \{v\}$, where $\alpha \geq 0$ and $v \in \mathbb{R}^N$.

A linear-algebraic criterion

Motivated by the game-theoretical point of view, we have offered in our 2014 manuscript a simple *linear-algebraic criterion to recognize whether a (standardized) supermodular game is extreme*.

The criterion is based on the vertex-description of the corresponding core polytope achieved by Shapley (1972).

Our criterion leads to solving a linear equation system determined by the combinatorial *core structure*, which is a concept recently pinpointed in the context of game theory (Kuipers *et al.*, 2010).

J. Kuipers, D. Vermeulen, M. Voorneveld. A generalization of the Shapley-Ichiishi result. *International Journal of Game Theory* 39 (2010) 585–602.

Supermodular functions and GP

$\Diamond(N)$ cone of supermodular games

$\Diamond_\ell(N)$ pointed cone of ℓ-standardized supermodular games

From a supermodular game to a GP and conversely

$$m \in \Diamond(N) \mapsto P_m := \mathcal{C}(m) \qquad P \mapsto m_P(S) := \min_{\mathbf{x}\in P} \sum_{i \in S} x_i$$

Theorem

There is a one-to-one correspondence between the (standardized) GP in $\mathbb{R}^N$ and the (ℓ-standardized) supermodular functions. In particular, the indecomposable standardized GP are mapped onto the generators of extreme rays in $\Diamond_\ell(N)$.

Payoff-Array Transformation

Definition

The payoff-array transformation assigns to every game m a real array $x^m \in \mathbb{R}^{\Upsilon \times N}$ such that for every $\pi \in \Upsilon$ and every $i \in N$,

$$x^m(\pi, i) = m\left(\bigcup_{k \leq \pi_{-1}(i)} \{\pi(k)\} \right) - m\left(\bigcup_{k < \pi_{-1}(i)} \{\pi(k)\} \right)$$

- the payoff-array transformation is linear invertible
- for every $m \in \Diamond(N)$, $\quad \mathcal{C}(m) = \mathrm{conv}\{x^m(\pi, *) \mid \pi \in \Upsilon\}$

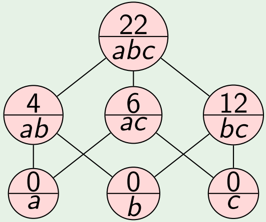
Example 1

Example (Convex measure game)

$N = \{a, b, c\} \equiv \{1, 2, 3\}$

Put $m(S) = \left(\sum_{i \in S} i\right)^2$ and standardize

$$x^m = \begin{pmatrix} 0 & 4 & 18 \\ 4 & 0 & 18 \\ 0 & 16 & 6 \\ 6 & 16 & 0 \\ 10 & 0 & 12 \\ 10 & 12 & 0 \end{pmatrix}$$

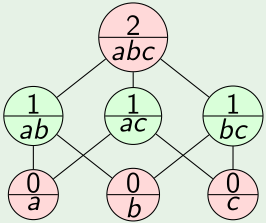


Example 2

Example (Extreme supermodular function)

$N = \{a, b, c\}$
$m(S) = |S| - 1, \quad S \neq \emptyset$

$$x^m = \begin{array}{c} \\ \pi \\ \sigma \\ \tau \\ \pi' \\ \sigma' \\ \tau' \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$



The payoff-array can be reduced by removing the repeated rows!

Null-sets and tightness sets

Let $m \in \Diamond_\ell(N)$ and $x^m \in \mathbb{R}^{\Gamma \times N}$ be its (possibly reduced) payoff-array, where $\Gamma \subseteq \Upsilon$ corresponds to the distinct rows in x^m.

Definition

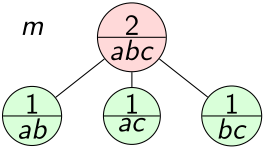
The null-set and the class of tightness sets of $x^m(\tau, *)$ are

$$N_\tau^m := \{i \in N \mid x^m(\tau, i) = 0\}$$

$$\mathcal{S}_\tau^m := \left\{ S \subseteq N \mid m(S) = \sum_{i \in S} x^m(\tau, i) \right\}$$

respectively.

Null-sets and tightness sets: example



$$x^m = \begin{array}{c} \\ \pi \\ \sigma \\ \tau \\ \pi' \\ \sigma' \\ \tau' \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Example

$N^m_\pi = \{a\} \quad \mathcal{S}^m_\pi = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, N\}$

$N^m_\sigma = \{b\} \quad \mathcal{S}^m_\sigma = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, N\}$

$N^m_\tau = \{c\} \quad \mathcal{S}^m_\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c\}, N\}$

Linear constraints on arrays

It is possible to define N_τ^x and $\mathcal{S}_\tau^x$ in terms of any array $x \in \mathbb{R}^{\Gamma \times N}$:

$$N_\tau^x := \{i \in N \mid x(\tau, i) = 0\}$$

$$\mathcal{S}_\tau^x := \left\{ S \subseteq N \mid \forall \pi \in \Gamma \quad \sum_{i \in S} x(\tau, i) \leq \sum_{i \in S} x(\pi, i) \right\}$$

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The linear constraints on arrays $y \in \mathbb{R}^{\Gamma \times N}$ based on $x \in \mathbb{R}^{\Gamma \times N}$

(a) $\forall \tau \in \Gamma \quad$ if $i \in N_\tau^x$, then $y(\tau, i) = 0$

(b) $\forall \tau, \pi \in \Gamma \quad \forall S \subseteq N$ such that $S \in \mathcal{S}_\tau^x \cap \mathcal{S}_\pi^x$

$$\sum_{i \in S} y(\tau, i) = \sum_{i \in S} y(\pi, i)$$

Linear algebraic criterion for extremality in $\Diamond_\ell(N)$

Theorem

Let $m \in \Diamond_\ell(N)$. Then the following are equivalent:

- m is extreme
- every solution $y \in \mathbb{R}^{\Gamma \times N}$ of (a)-(b) is a multiple of $x := x^m$

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The linear constraints on arrays $y \in \mathbb{R}^{\Gamma \times N}$ based on $x \in \mathbb{R}^{\Gamma \times N}$

(a) $\forall\, \tau \in \Gamma \quad$ if $i \in N^x_\tau$, then $y(\tau, i) = 0$

(b) $\forall\, \tau, \pi \in \Gamma \quad \forall\, S \subseteq N$ such that $S \in \mathcal{S}^x_\tau \cap \mathcal{S}^x_\pi$

$$\sum_{i \in S} y(\tau, i) = \sum_{i \in S} y(\pi, i)$$

Linear algebraic criterion for indecomposability of GP

Theorem

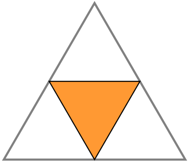
Let P be a GP. Then the following are equivalent:

- P is indecomposable
- every solution $y \in \mathbb{R}^{\Gamma \times N}$ of (a)-(b) is a multiple of

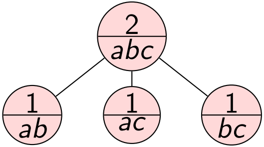
$$x := \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{pmatrix},$$

where $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are the vertices of P

Example: indecomposable standardized GP



$$x = \begin{array}{c} \\ \pi \\ \sigma \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$



Null-sets and tightness classes

$N^x_\pi = \{a\} \quad \mathcal{S}^x_\pi = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, N\}$
$N^x_\sigma = \{b\} \quad \mathcal{S}^x_\sigma = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, N\}$
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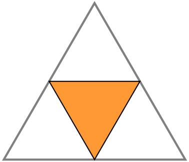

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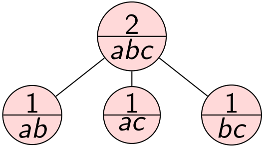

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$N^x_\tau = \{c\} \quad \mathcal{S}^x_\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c\}, N\}$

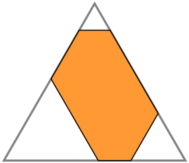
$$y(\pi, a) = 0 \qquad y(\pi,a) + y(\pi,b) = y(\sigma,a) + y(\sigma,b)$$

$$y(\sigma, b) = 0 \qquad y(\pi,a) + y(\pi,c) = y(\tau,a) + y(\tau,c)$$

$$y(\tau, c) = 0 \qquad y(\sigma,b) + y(\sigma,c) = y(\tau,b) + y(\tau,c)$$

$$\sum_{i\in N} y(\pi, i) = \sum_{i\in N} y(\sigma, i) = \sum_{i\in N} y(\tau, i)$$

Example: decomposable standardized GP



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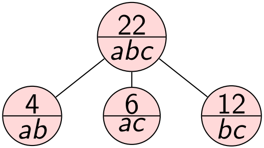
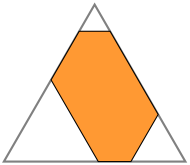


Counterexample:

Example: decomposable standardized GP



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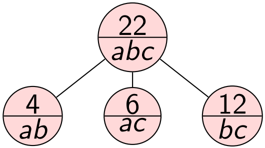

Counterexample:

$$\begin{pmatrix} 0 & 0 & 22 \\ 0 & 0 & 22 \\ 0 & 16 & 6 \\ 6 & 16 & 0 \\ 10 & 0 & 12 \\ 10 & 12 & 0 \end{pmatrix} \neq \alpha \cdot x$$

Conclusions

- Our result for supermodular functions gives, as a by-product, a criterion to recognize whether a given generalized permutohedron is indecomposable.
- The criterion is different (and simpler than) Meyer's general criterion (1974) to recognize indecomposable polytopes based on their facet-description.