POINCARÉ SERIES AND DEFORMATIONS OF GORENSTEIN LOCAL ALGEBRAS

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Abstract. Let $(A, \mathfrak{m}, K)$ be an Artinian Gorenstein local ring with K an algebraically closed field of characteristic 0. In the present paper we prove a structure theorem describing the Artinian Gorenstein local K-algebras satisfying $\mathfrak{m}^4 = 0$. We use this result in order to prove that such a K-algebra has rational Poincaré series and it is smoothable in any embedding dimension, provided $\dim_K \mathfrak{m}^2/\mathfrak{m}^3 \leq 4$. We also prove that the generic Artinian Gorenstein local K-algebra with $\mathfrak{m}^4 = 0$ has rational Poincaré series.

1. Introduction and preliminary results

In this paper K is an algebraically closed field of characteristic zero and A is a Noetherian commutative local K-algebra with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} \simeq K$. For an Artinian local ring $(A, \mathfrak{m}, K)$ let s be the integer such that $\mathfrak{m}^s \neq 0$ and $\mathfrak{m}^{s+1} = 0$. Hence $\mathfrak{m}^s$ is contained in the socle $Soc(A) = (0 :_A \mathfrak{m})$ and A is Gorenstein if $Soc(A)$ is a K-vector space of dimension one. In particular the socle of a Gorenstein local ring A is concentrated in degree s and we say that s is the socle degree of A.

We study Gorenstein, local and finite-dimensional K-algebras $(A, \mathfrak{m}, K)$ with socle degree $s \leq 3$. The starting point is a structure theorem for this class of K-algebras. The key ingredients are some well–known facts about Macaulay's inverse system and a recent work of J. Elias and M.E. Rossi (see [13]).

Macaulay's inverse system has an important role in the study of Artinian local K-algebras. The reader should refer to [15], [18], [19] and [13] for an extended treatment. We present in this section the main results that will be used in our approach.
Throughout the paper let $R = K[\![x_1, \dots x_h]\!]$ be a power series ring with maximal ideal $\mathfrak{n} = (x_1, \cdots, x_h)$ and let $S = K[x_1, \dots x_h]$ the correspondent polynomial ring. Since A, being Artinian, is complete with respect to the $\mathfrak{m}$-adic topology, we may assume A is a quotient of R.

Date: May 7, 2011.

2000 *Mathematics Subject Classification.* Primary 13D40; Secondary 13H10.

Key words and phrases. Gorenstein, Artinian, Poincaré series, smoothable.

* This work was done in the framework of PRIN "Geometria delle varietà algebriche e dei loro spazi di moduli", cofinanced by M.I.U.R. (cofin 2008).

† Partially supported by MTM2007-67493, Acción Integrada España-Italia 07-09.

‡ Partially supported by M.I.U.R.: PRIN 07-09, Azione Integrata Italia-Spagna 07-09 .

We can endow S with a structure of R-module by means of the following multiplication

$$\begin{array}{rccc} \circ: & R\times S & \longrightarrow & S \\ & (f,g) & \to & f\circ g = f(\partial_{y_1},\dots,\partial_{y_h})(g) \end{array}$$

where ∂_{y_i} denotes the partial derivative with respect to y_i.

Macaulay in [21] characterized Artinian local K–algebras in terms of suitable R-submodules of S which are finitely generated. In particular, they proved that a local ring $A = R/I$ is an Artinian Gorenstein local K–algebra with socle degree s if and only if its dual module is a cyclic R-submodule of S generated by a polynomial $F \in S$ of degree s (see also J. Emsalem in [15], Section B, Proposition 2, and [18], Lemma 1.2.). Given $F \in S$ as above and the ideal of R

$$Ann_R(F) := \{g \in R \mid g\circ F = 0\ \}$$

then $A = R/Ann_R(F)$. To stress the dependence on F, we write $A_F = R/Ann_R(F)$. The polynomial F is not unique, but it is determined up an unit u of R.

In the general case, one can translate in terms of classification and deformation of the associated polynomials, the problems of classification and deformation of an Artinian Gorenstein local K–algebra. For example, when A is an Artinian Gorenstein local K–algebra of socle degree $s = 2$ and embedding dimension h, we have $A \simeq A_F$ with $F = y_1^2 + \dots + y_h^2 \in S$, due to the projective classification of the quadrics. Hence $A \simeq R/I$ where

$$I = (x_i x_j, x_u^2 - x_1^2)_{1\le i<j\le h,\ u=2,\dots,h}.$$

We are interested in the structure of an Artinian Gorenstein local K–algebra of socle degree $s \geq 3$. To this end we need to introduce more invariants.

The Hilbert function HF_A of a local ring $(A, \mathfrak{m}, K)$ is, by definition, the Hilbert function of its associated graded algebra $G = gr_{\mathfrak{m}}(A) := \bigoplus_{i\ge 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ (by definition $\mathfrak{m}^0 = A$), i.e.

$$HF_A(i) = \dim_K\ \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

In the graded case, the Hilbert function of an Artinian Gorenstein algebra is symmetric. Little is known about the Hilbert function in the local case. The problem comes from the fact that the associated graded algebra G is in general no longer Gorenstein.

Nevertheless A. Iarrobino in [18], Section 1.E proved interesting results in the Gorenstein local case. He defined a stratification of G by a descending sequence of ideals

$$G = C(0) \supset C(1) \supset \dots,$$

whose successive quotient $Q(a) = C(a)/C(a + 1)$ are reflexive G-modules. The reflexivity property imposes conditions on the Hilbert function HF_A. At the end, if A has socle degree s, then HF_A is a sum of the Hilbert functions $HF_{Q(a)}$ that are symmetric with respect to $\frac{s-a}{2}$.

The first subquotient $Q_A(0)$ of G is always a graded Gorenstein algebra and it is the unique quotient of G with the same socle degree s. A. Iarrobino proved that if HF_A is symmetric, then $G = Q_A(0)$ and it is Gorenstein. Hence G is Gorenstein

if and only if HF_A is symmetric, equivalently if $G = Q_A(0)$, (see [18], Proposition 1.7 and [15], Proposition 7).
The G-module $Q_A(0)$ will play a crucial role in our investigation. It can be computed in terms of the corresponding polynomial in the inverse system. If $F \in S$ is a polynomial of degree s such that $A = A_F$ and if we denote by F_s the form of highest degree s in F, that is $F = F_s +$ terms of lower degree, then (see [15], Proposition 7 and [18], Lemma 1.10)

$$Q_A(0) \simeq R/Ann_R(F_s).$$

We say that a homogenous form $F \in S$ of degree d is *non-degenerate* if the K-vector space of the derivatives of order $d-1$ has maximal dimension $h = \dim_K[S]_1$.

If A is an Artinian Gorenstein local K–algebra of embedding dimension h ($= HF_A(1)$) and socle degree 3, then, from the above decomposition of the Hilbert function of A, we deduce that $HF_A(2) = n \le h$ and clearly $HF_A(3) = 1$. In this case we will write that A has Hilbert function $(1, h, n, 1)$; notice that $Q_A(0)$ has Hilbert function $(1, n, n, 1)$.

From now on we let $R_j = K[\![x_1, \dots, x_j]\!]$ and $S_j = K[y_1, \dots, y_j]$ for every positive integer $j \le h$. Hence $R_h = R$ and $S_h = S$. In the following h, n will denote positive integers such that $n \le h$. We assume $A = R/I$ of embedding dimension h, that is $I \subseteq \mathfrak{n}^2$.

J. Elias and M.E. Rossi proved the following result that we include here for sake of completeness.

Theorem 1.1 ([13], Theorem 4.1). *Let A be an Artinian Gorenstein local K–algebra with Hilbert function $(1, h, n, 1)$. Then $A \simeq R/Ann_R(F)$ where*

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S$$

with F_3 a non-degenerate degree three form in S_n ($F = F_3$ if $n = h$).

Starting from the above result we can prove a structure theorem for Artinian Gorenstein local K–algebra $A = R/I$ with maximal ideal $\mathfrak{m}$ and socle degree three.

Lemma 1.2. *Let h, n be positive integers such that $h > n$ and let*

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S$$

where F_3 a non-degenerate degree three form in S_n. Then

$$Ann_R(F) = Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \le j \le h}$$

where $\sigma \in R_n$ is any form of degree 3 such that $\sigma \circ F_3 = 1$.

Proof. It is easy to check that

$$I := Ann_R(F) \supseteq J := Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \le j \le h}$$

Since R/I and R/J are finitely generated K-vector spaces ($\mathfrak{n}^4 \subseteq I, J$) and there is a surjection between R/J and R/I, the equality $I = J$ follows if they have the same colength. In particular we prove that $HF_{R/I}(i) = HF_{R/J}(i)$ for every $i \geq 0$. If we denote by H^* the homogeneous ideal generated by the initial forms of an ideal H of R, we have

$$L = (Ann_{R_n}(F_3))^* + (x_i x_j, x_j^2)_{i<j,\, n+1 \leq j \leq h} \subseteq J^* \subseteq I^*$$

thus

$$HF_{R/L}(i) \geq HF_{R/J}(i) \geq HF_{R/I}(i)$$

for every $i \geq 0$. In order to prove the assertion we prove that the Hilbert function of R/L is $(1, h, n, 1)$, i.e. equal to the Hilbert function of R/I (see Theorem 1.1). The equality easily follows since $R_n/Ann_{R_n}(F_3)$ has Hilbert function $(1, n, n, 1)$, being F_3 a non-degenerate degree three form in S_n and $L_2 = [Ann_R(F_3)^*]_2$. □

Theorem 1.3. *Let A be an Artinian local K–algebra of embedding dimension h and let $n = HF_A(2)$.*

A is Gorenstein of socle degree three if and only if $n \leq h$ and there exists a non-degenerate cubic form $F_3 \in S_n$ such that $A \simeq R/I$ where

$$I = \begin{cases} Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \leq j \leq h} & \text{if } n < h \\ \\ Ann_R(F_3) & \text{if } n = h \end{cases}$$

being $\sigma \in R_n$ any form of degree 3 such that $\sigma \circ F_3 = 1$.

Proof. Assume A is an Artinian Gorenstein local K–algebra A with socle degree three. The Hilbert function of A is equal to $(1, h, n, 1)$ with $h \geq n$. Theorem 1.1 yields that $A \simeq R/Ann_R(F)$ where

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S$$

if $h > n$ and $F = F_3$ if $h = n$. Then the result follows by Lemma 1.2.
Conversely, the result follows by Macaulay's correspondence and again by Lemma 1.2. □

2. Poincaré series of Gorenstein local algebras with socle degree three

As a consequence of the above results, we get applications to the rationality of the Poincaré series

$$P_A(z) := \sum_{i \geq 0} \dim_K Tor_i^A(K, K) z^i.$$

Over the last 30 years the problem of the rationality of $P_A(z)$ has been a topic of much current interest, see among others [1], [3], [2], [24], [23], [14], [8]. In particular, G. Sjödin proved that all Artinian Gorenstein rings satisfying $\mathfrak{m}^3 = 0$ have a rational Poincaré series (see [25]), while R. Bøgvad provided examples of Artinian Gorenstein local rings with $\mathfrak{m}^4 = 0$ and transcendental Poincaré series (see [4], Theorem of pag.

12). For this last class of K-algebras, we reduce the problem of the rationality $P_A(z)$ to the rationality of $P_{Q_A(0)}(z)$, Theorem 2.1.

We recall that if $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is an element in the socle $Soc(A) = (0 :_A \mathfrak{m})$ of A, then, by [17] Proposition 3.4.4,

$$P_A(z) = \frac{P_{A/(x)}(z)}{1 - z\ P_{A/(x)}(z)}. \tag{1}$$

Moreover if A is an Artinian Gorenstein local ring, then, by [3], Theorem 2 of Section 2,

$$P_A(z) = \frac{P_{A/Soc(A)}(z)}{1 + z^2\ P_{A/Soc(A)}(z)} \tag{2}$$

If $A = A_F$ with $F = F_3 + \dots$ we will write the Poincaré series of A in terms of the one of the Artinian Gorenstein graded K-algebra $Q_A(0) \simeq R/Ann_R(F_3)$.

Theorem 2.1. *Let A be an Artinian Gorenstein local K–algebra with Hilbert function $(1, h, n, 1)$. Then*

$$P_A(z) = \frac{P_{Q_A(0)}(z)}{1 - (h - n)z\ P_{Q_A(0)}(z)}.$$

In particular $P_A(z)$ is rational if and only if $P_{Q_A(0)}(z)$ is rational.

Proof. By Theorem 1.1, we may assume $n < h$, otherwise $A \simeq G \simeq Q_A(0)$ and the result is obvious. Always by Theorem 1.1, $A \simeq R/Ann_R(F)$ where $R = K[\![x_1, \dots x_h]\!]$ and $F = F_3 + y_{n+1}^2 + \dots + y_h^2$, with F_3 a non-degenerate degree three form in S_n. By Lemma 1.2, we know that

$$Ann_R(F) = Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \le j \le h}$$

where $\sigma \in S_n$ is a cubic form such that $\sigma \circ F_3 = 1$. So the coset of σ in A is a generator of $Soc(A)$. Hence by (2) we get that

$$P_A(z) = \frac{P_C(z)}{1 + z^2\ P_C(z)}$$

where

$$C := \frac{A}{Soc(A)} \simeq \frac{R}{Ann_{R_n}(F_3)R + (\sigma) + (x_i x_j, x_j^2)_{i<j,\, n+1 \le j \le h}}.$$

Since $x_h \in Soc(C)$, by (1), we get that

$$P_C(z) = \frac{P_{C/(x_h)}(z)}{1 - z\ P_{C/(x_h)}(z)}.$$

Since for every $n + 1 \le j < h$ we have $\overline{x}_j \in Soc(C/(x_h, \dots, x_{j+1}))$, iterating the process $h - n$ times we deduce that

$$P_C(z) = \frac{P_D(z)}{1 - (h - n)z\ P_D(z)}.$$

where

$$D = \frac{C}{(x_h, \dots, x_{n+1})} \simeq \frac{R}{Ann_{R_n}(F_3)R + (\sigma) + (x_{n+1}, \cdots, x_h)}.$$

Since $F_3, \sigma \in S_n$ we get that

$$D \simeq \frac{R_n}{Ann_{R_n}(F_3) + (\sigma)}.$$

Notice that, by its definition, the coset of σ in

$$B := R_n/Ann_{R_n}(F_3) \simeq R/Ann_R(F_3) = Q_A(0)$$

is a generator of its socle, so

$$D \simeq \frac{R_n}{Ann_{R_n}(F_3) + (\sigma)} \simeq \frac{B}{Soc(B)}.$$

Hence from (2) we deduce

$$P_D(z) = \frac{P_B(z)}{1 - z^2 \, P_B(z)}.$$

From the above information, summing up, we get

$$P_A(z) = \frac{P_C(z)}{1 + z^2 \, P_C(z)} = \frac{P_D(z)}{1 - (h-n)zP_D(z) + z^2 \, P_D(z)} = \frac{P_B(z)}{1 - (h-n)z \, P_B(z)}.$$

Since $P_A(z)$ is a rational function of $P_B(z)$, it is clear that $P_A(z)$ is rational if and only if so $P_B(z)$ is. □

The above result reduces the problem of the rationality of $P_A(z)$ to the rationality of the Poincaré series of a Gorenstein graded K–algebra with socle degree three. This situation has been studied by several authors. By taking advantage of this knowledge we present the following corollaries.

Corollary 2.2. *Let A be an Artinian Gorenstein local K-algebra with socle degree 3. If $HF_A(2) \leq 4$, then $P_A(z)$ is rational.*

Proof. By Theorem 2.1 we may reduce the problem to graded Gorenstein K-algebras with embedding dimension ≤ 4. Hence the result follows by [2], Theorem 6.4 and [20], Corollary 2.3 who proved that Gorenstein graded rings C with $HF_C(1) - 4 \leq \operatorname{depth}(C)$ have rational Poincaré series. □

We recall that a Koszul graded algebra C has rational Poincaré series. In particular $P_C(z)HS_C(-z) = 1$ where $HS_C(z)$ is the Hilbert series of C, i.e. $HS_C(z) = \sum_{j \geq 0} HF_C(j)z^j$.
If A is a local ring, then the Hilbert series of A is the Hilbert series of the associated graded ring G, that is $HS_A(z) = HS_G(z)$. The local ring A is called a generalized Koszul algebra if G is a Koszul algebra. A result by R. Fröberg in [16] Corollary to Theorem 4 says that a generalized Koszul local algebra satisfies $P_A(z)HS_A(-z) = 1$. The same equality holds for Artinian Gorenstein algebras with socle degree three under weaker assumptions.

Corollary 2.3. *Let $(A, \mathfrak{m}, K)$ be an Artinian Gorenstein local K–algebra satisfying $\mathfrak{m}^4 = 0$. If $Q_A(0)$ is Koszul, then*

$$P_A(z)HS_A(-z) = 1.$$

Proof. We may assume A of socle degree $s = 3$, because if $s = 2$ A is generalized Koszul and the result is clear. By Theorem 2.1 we reduce the problem to graded Gorenstein K-algebras which are Koszul. Since $Q_A(0)$ is Koszul, then $P_{Q_A(0)}(z)HS_{Q_A(0)}(-z) = 1$, hence $P_{Q_A(0)}(z)(1 - nz + nz^2 - z^3) = 1$, where $n = HF_A(2)$. Then the result follows from Theorem 2.1 by an easy computation. □

Koszul filtrations were introduced for studying the Koszulness of a standard graded algebra C. It has been proved in [12], Proposition 1.2 that, if C has a Koszul filtration, then all the ideals of the filtration have a linear C-free resolution, hence C is Koszul. For Gorenstein graded algebras with Hilbert function $(1, n, n, 1)$, the property of having a Koszul filtration can be detected directly on the dual cubic form (see [11] Theorem 6.3 and [10] Theorem 3.2). Notice that if F_3 is a generic cubic, then the corresponding Gorenstein algebra $R/Ann_R(F_3) = Q_A(0)$ has a Koszul filtration (see [11], Theorem 6.3), hence $P_A(z)$ is rational by Corollary 2.3.

We say that A_F is a generic Artinian Gorenstein local K–algebra if F is a generic polynomial of S. By the previous discussion we get the following result.

Corollary 2.4. *The generic Artinian Gorenstein local K-algebra with socle degree three has rational Poincaré series.*

3. Deformations of Gorenstein local algebras with socle degree three

Starting from the structure theorem for Artinian Gorenstein K–algebras A with $\mathfrak{m}^4 = 0$, proved in Section 1, we achieve some results concerning the smoothability of Spec(A) in the Hilbert scheme $\mathcal{H}ilb_d(\mathbb{A}^h)$ parameterizing punctual subschemes of multiplicity d of the affine space $\mathbb{A}^h$.

In [22] (see also [5]) it is proved that $\mathcal{H}ilb_d(\mathbb{A}^h)$ is irreducible if $d \leq 7$ and, in particular, each Artinian K–algebra A of length $d \leq 7$ is smoothable i.e. it can be flatly deformed to the trivial K–algebra K^d. In [7] and [9] the attention is focused on the problem of the smoothability of Gorenstein algebras in $\mathcal{H}ilb_d(\mathbb{A}^h)$. The locus $\mathcal{H}ilb_d^G(\mathbb{A}^h) \subseteq \mathcal{H}ilb_d(\mathbb{A}^h)$ of such K–algebras contains all the algebras isomorphic to the trivial K–algebra K^d, thus it makes sense to ask when $\mathcal{H}ilb_d^G(\mathbb{A}^h)$ is irreducible and what kind of algebras it contains.

We start by recalling some facts about families of K-algebras of fixed dimension. Let $d \geq 2$ be an integer and let A be an Artinian K-algebra of dimension d, hence $A \simeq K^d$ as K-vector spaces. It is interesting to understand whether such an A is smoothable in the following sense.

Definition 3.1. *An Artinian K-algebra A of length d is smoothable if the scheme* $\mathrm{Spec}(A) \in \mathcal{H}ilb_d(\mathbb{A}^d)$ *is in the closure* $\mathcal{H}ilb_d^{gen}(\mathbb{A}^d)$ *inside* $\mathcal{H}ilb_d(\mathbb{A}^d)$ *of the locus of points representing reduced schemes.*

We focus here our attention to Artinian Gorenstein local K-algebras A with socle degree 3.

Theorem 3.2. *Let A be an Artinian Gorenstein local K-algebra with socle degree 3. Then A is smoothable if $Q_A(0)$ is smoothable.*

Proof. Denote by $(1, h, n, 1)$ the Hilbert function of A. We will extend the methods used in [6] and [7], proving the statement by induction on $h \geq n$. Recall that $A \simeq R/Ann_R(F)$ where (see Theorem 1.1)

$$F = F_3 + y_{n+1}^2 + \cdots + y_h^2 \in S = K[y_1, \ldots, y_h]$$

with F_3 a non-degenerate degree three form in $K[y_1, \ldots, y_h]$. If $h = n$, then $A \simeq R/Ann_R(F_3) \simeq G \simeq Q_A(0)$ and the statement is trivial in this case.

We assume the statement is true for $h - 1$ and we will prove it for $h > n$. Recall that we have (see Theorem 2.1)

$$Ann_R(F) = Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \leq j \leq h}$$

where σ is a cubic form in S_n such that $\sigma \circ F_3 = 1$. For the techniques used in this proof, it will be useful working in the polynomial ring. Notice that in our setting **we may assume** $R = K[x_1, \ldots, x_h]$ because in $R/Ann_R(F)$ we have $\mathfrak{m}^4 = 0$.

For each $b \in \mathbb{A}^1$, let us consider the ideal in R

$$J_b := Ann_{R_n}(F_3)R + (x_i x_j, x_k^2 - 2\sigma(x_1, \ldots, x_n), x_h^2 - b x_h - 2\sigma(x_1, \ldots, x_n))_{i<j,\, n+1 \leq j \leq h, n+1 \leq k < h}.$$

Notice that

$$J_0 = Ann_R(F) = Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \leq j \leq h},$$

hence $A \simeq R/J_0$. If $b \neq 0$, we claim that

$$J_b = (x_1, \ldots, x_{h-1}, x_h - b) \cap (J_b + (x_h^2)) \subseteq R.$$

Since $J_b \subseteq (x_1, \ldots, x_{h-1}, x_h - b)$, by the modular law, it is enough to prove

$$(x_1, \ldots, x_{h-1}, x_h - b) \cap (x_h^2) = (x_1 x_h^2, \ldots, x_{h-1} x_h^2, (x_h - b) x_h^2) \subseteq J_b.$$

This is trivial because $x_i x_h \in J_b$ for every $i = 1, \ldots h-1$ and $x_h^2(x_h - b) \equiv 2x_h \sigma(x_1, \ldots, x_n)) \equiv 0$ modulo J_b.

If $b \neq 0$, then $x_h \in J_b + (x_h^2)$, thus the ideals $(x_1, \ldots, x_{h-1}, x_h - b)$ and $J_b + (x_h^2)$ are coprime, whence

$$A_b := R/J_b \simeq R/(x_1, \ldots, x_{h-1}, x_h - b) \oplus R/(J_b + (x_h^2)).$$

Since $x_h \in J_b + (x_h^2)$ we also have

$$R/(J_b + (x_h^2)) \simeq A' := R_{h-1}/(Ann_{R_n}(F_3)R + (x_i x_j, x_j^2 - 2\sigma(x_1, \cdots, x_n))_{i<j,\, n+1 \leq j \leq h-1}).$$

By induction assumptions A' is smoothable. It follows easily that $A_b \simeq K \oplus A'$ turns out to be smoothable for $b \neq 0$ (for reader's benefit see e.g. Lemma 4.2 of [5]).

Let $\mathcal{R} := K[b] \otimes_K R \simeq K[b, x_1, \dots, x_h]$ and consider the family $\mathcal{A} \simeq \mathcal{R}/J \to \mathbb{A}^1 \simeq \mathrm{Spec}(k[b])$ where

$$J := Ann_{R_n}(F_3)\mathcal{R} + (x_i x_j, x_k^2 - 2\sigma(x_1, \dots, x_n), x_h^2 - bx_h - 2\sigma(x_1, \dots, x_n))_{i<j,\, n+1 \leq j \leq h, n+1 \leq k < h}.$$

Due to the discussion above all the fibres of $\mathcal{A} \to \mathbb{A}^1$ are Artinian K-algebras of degree d, thus the family is flat. The universal property of Hilbert scheme guarantees the existence of a curve inside $\mathcal{H}ilb_d(\mathbb{A}^d)$ whose general point is in $\mathcal{H}ilb_d^{gen}(\mathbb{A}^d)$, thus $\mathrm{Spec}(A) \in \mathcal{H}ilb_d^{gen}(\mathbb{A}^d)$. □

As already noticed above Theorem 3.2 reduces the smoothability of A to the smoothability of a graded Artinian Gorenstein local K–algebra with socle degree three. In analogy to Corollary 2.2, we prove the following result.

Corollary 3.3. *Let A be an Artinian Gorenstein local K-algebra with socle degree 3. If $HF_A(2) \leq 4$, then A is smoothable.*

Proof. It follows by Theorem 3.2 and [9], Proposition 3.4. □

We remark that the previous result cannot be generalized to $HF_A(2) \geq 6$. Indeed A.Iarrobino produced an example of non smoothable local K-algebra A with Hilbert function $(1, 6, 6, 1)$ (see [9], Section 4). The case $n = 5$ is still open.

References

[1] D. J. Anick, *A counterexample to a conjecture of Serre*, Ann. of Math. **115** (1982), 1–33.

[2] L. Avramov, A. Kustin and M. Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204

[3] L. Avramov and G. Levin, *Factoring out the socle of a Gorenstein ring*, J. Algebra **55** (1978), no. 1, 74–83.

[4] R. Bøgvad, *Gorenstein rings with transcendental Poincaré series*, Math. Scand. **53** (1983), no. 1, 5–15.

[5] D.A. Cartwright, D. Erman, M. Velasco, B. Viray, *Hilbert schemes of 8 point in $\mathbb{A}^d$*, Algebra Number Theory **3** (2009), no. 7, 763–795.

[6] G. Casnati, R. Notari, *On some Gorenstein loci in $\mathcal{H}ilb_6(\mathbb{P}^4_k)$*, J. Algebra **308** (2008), no. 2, 493–523

[7] G. Casnati, R. Notari, *On the Gorenstein locus of some punctual Hilbert schemes*, J. Pure Appl. Algebra **213** (2009), no. 11, 2055–2074.

[8] G. Casnati, R. Notari, *On the Poincaré seires of a local Gorenstein ring*, P. Indian As-Math. Sci. **119** (2009), 459–468.

[9] G. Casnati, R. Notari, *On the irreducibility and the singularities of the Gorenstein locus of the punctual Hilbert scheme of degree* 10, To appear in J. Pure Appl. Algebra.

[10] A. Conca, *Koszul algebras and Gröbner bases of quadrics*, Arxiv.0903.2397v1

[11] A. Conca, M.E. Rossi, G. Valla, *Gröbner flags and Gorenstein Artin rings*, Compositio Math. **129** (2001), 95–121.

[12] A. Conca, N.V. Trung, G. Valla, *Koszul property for points in projective spaces*, Math. Scand. (1999).
[13] J. Elias, M.E. Rossi, *Isomorphism classes of Artinian local rings via Macaulay's inverse system*, To appear in Trans. A.M.S.
[14] J. Elias, G. Valla, *A family of local rings with rational Poincaré series*, Proc. AMS Vol. 37, N. 4 (2009), 1175–1178.
[15] J. Emsalem, *Géométrie des points épais*, Bull. Soc. Math. France **106** (1978), no. 4, 399–416.
[16] R. Fröberg, *Connections between a local ring and its Associated graded ring*, J. of Algebra **111**, (1987), 300–305.
[17] T.H. Gulliksen and G. Levin, *Homology of local rings*, Queen's University **20**, (1969).
[18] A. Iarrobino, *Associated graded algebra of a Gorenstein Artin algebra*, Mem. Amer. Math. Soc. **107** (1994), no. 514.
[19] A. Iarrobino, V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, L.M.N. **1721** (1999).
[20] C. Jacobsson, A. Kustin, M. Miller, *The Poincaré series of a codimension four Gorenstein ring is rational*, J. Pure Appl. Algebra **38** (1985), 255–275.
[21] F.S. Macaulay, *The algebraic theory of modular systems.*, Cambridge University, 1916.
[22] G. Mazzola, *Generic finite schemes and Hochschild cocycles*, Comment. Math. Helvetici **55** (1980), 267–293.
[23] J. Sally, *The Poincaré series of stretched Cohen-Macaulay rings*, Canad. J. Math. **32** (1980), 1261–1265.
[24] J.P. Serre, Sur la dimension homologique des anneaux et des modules noetheriens, Proc. Intern. Symposium on Algebraic Number Theory, Tokyo (1955), 175–189.
[25] G. Sjödin, *The Poincaré series of modules over Gorenstein rings with* $\mathbf{m^3 = 0}$, Mathematiska Institutionen, Stockholms Universitet, Preprint **2** (1979).

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