On the Hilbert function of one-dimensional local complete intersections *

J. Elias † M. E. Rossi G. Valla

July 24, 2013

Abstract

The sequences that occur as Hilbert functions of standard graded algebras A are well understood by Macaulay's theorem; those that occur for graded complete intersections are elementary and were known classically. However, much less is known in the local case, once the dimension of A is greater than zero, or the embedding dimension is three or more.

Using an extension to the power series ring R of Gröbner bases with respect to local degree orderings, we characterize the Hilbert functions H of one-dimensional quadratic complete intersections $A = R/I$, $I = (f, g)$, of type $(2, 2)$ that is, that are quotients of the power series ring R in three variables by a regular sequence f, g whose initial forms are linearly independent and of degree two. We also give a structure theorem up to analytic isomorphism of A for the minimal system of generators of I, given the Hilbert function.

More generally, when the type of I is $(2, b)$ we are able to give some restrictions on the Hilbert function. In this case we can also prove that the associated graded algebra of A is Cohen Macaulay if and only if the Hilbert function of A is strictly increasing.

1 Introduction and preliminaries

Let G be a standard graded K-algebra; by this we mean $G = P/I$ where $P = \mathrm{K}[x_1, \dots x_n]$ is a polynomial ring over the field K and I a homogeneous ideal. It is clear that for every $t \geq 0$ the set I_t of the forms of degree t in P is a K-vector space of finite dimension. For every positive integer t the Hilbert function of G is defined as follows:

$$HF_G(t) = dim_{\mathrm{K}} G_t = dim_{\mathrm{K}} P_t - dim_{\mathrm{K}} I_t = \binom{n+t-1}{t} - \ dim_{\mathrm{K}} I_t.$$

Its generating function $HS_G(\theta) = \sum_{t \in \mathbb{N}} HF_G(t)\theta^t$ is the Hilbert Series of G.

The relevance of this notion comes from the fact that in the case I is the defining ideal of a projective variety V, the dimension, the degree and the arithmetic genus of V can be immediately computed from the Hilbert Series of P/I.

A fundamental theorem by Macaulay describes exactly those numerical functions which occur as the Hilbert functions of a standard graded K-algebra. Macaulay's Theorem says that for each t there is an upper bound for $HF_G(t+1)$ in terms of $HF_G(t)$, and this bound is sharp in the sense that any numerical function satisfying it can be realized as the Hilbert function of a suitable homogeneous standard K-algebra. These numerical functions are called "admissible" and will be described in the next section.

It is not surprising that additional properties yield further constraints on the Hilbert function. Thus, for example, the Hilbert function of a Cohen-Macaulay standard graded algebra is completely described

*2010 *Mathematics Subject Classification.* Primary 13H10; Secondary 13H15
Key words and Phrases: one dimensional local rings, Hilbert functions, complete intersections.
†Partially supported by MTM2010-20279-C02-01

by another theorem of Macaulay which says that the Hilbert series admissible for a Cohen-Macaulay standard graded algebra of dimension d, are of the type

$$\frac{1 + h_1\theta + \dots h_s\theta^s}{(1-z)^d}$$

where $1 + h_1\theta + \dots h_s\theta^s$ is admissible.

The Hilbert function of a local ring A with maximal ideal $\mathfrak{m}$ and residue field K is defined as follows: for every $t \geq 0$

$$HF_A(t) = dim_{\mathrm{K}}\left(\frac{\mathfrak{m}^t}{\mathfrak{m}^{t+1}}\right).$$

It is clear that $HF_A(t)$ is equal to the minimal number of generators of the ideal $\mathfrak{m}^t$ and we can see that the Hilbert function of the local ring A is the Hilbert function of the following standard graded algebra

$$gr_{\mathfrak{m}}(A) = \oplus_{t\geq 0}\ \mathfrak{m}^t/\mathfrak{m}^{t+1}.$$

This algebra is called the associated graded ring of the local ring $(A, \mathfrak{m})$ and corresponds to a relevant geometric construction in the case A is the localization at the origin O of the coordinate ring of an affine variety V passing through O. It turns out that $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *Tangent Cone* of V at O, which is the cone composed of all lines that are the limiting positions of secant lines to V in O.

Despite the fact that the Hilbert function of a standard graded K-algebra G is so well understood in the case G is Cohen-Macaulay, very little is known in the local case. This is mainly because, in passing from the local ring A to its associated graded ring, many of the properties of A can be lost. This is the reason why we are very far from a description of the admissible Hilbert functions for a Cohen-Macaulay local ring when $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay.

An example by Herzog and Waldi (see [12]) shows that the Hilbert function of a one dimensional Cohen-Macaulay local ring can be decreasing, even the number of generators of the square of the maximal ideal can be less than the number of generators of the maximal ideal itself. Further, without restrictions on the embedding dimension, the Hilbert function of a one dimensional Cohen-Macaulay local ring can present arbitrarily many "valleys" (see [7]).

Even if we restrict ourselves to the case of a complete intersection, very little is known when $dim(A) \geq 1$ or when the embedding dimensions of A is at least three. In [19] it has been proved that the Hilbert function of a positive dimensional codimension two complete intersection $R/(f, g)$ is non decreasing, but we have no answer to the question asked by Rossi (see [20]) whether the same is true for every one dimensional Gorenstein local ring.

In the case that the embedding dimension of the local ring is at most three, the first author gave a positive answer to a question stated by J. Sally, by proving that the Hilbert function of a one dimensional Cohen-Macaulay local ring is increasing (see [6]). But examples show that this is not longer true if the embedding dimension is bigger than three.

All this shows that without strong assumptions, the Hilbert function of a one-dimensional Cohen-Macaulay local ring could be very wild. This is the reason why, in this paper, we restrict ourselves to the case $A = \mathrm{K}[\![x, y, z]\!]/I$, where the ideal $I \subseteq (x, y, z)^2$ is generated by a regular sequence $\{f, g\}$ of elements of R. We will see that even with this assumption, the problem of determining the admissible Hilbert functions is not so easy, possibly because it is strictly related to the study of curve singularities in $\mathbb{A}^3$.

If we consider the corresponding Artinian problem, then we deal with a pair of plane curves. Several papers have been written in which the Hilbert function of an Artinian complete intersection ring $A = \mathrm{K}[\![x, y]\!]/(f, g)$ has been studied in terms of the invariants of the curves $f = g = 0$ (see Iarrobino [14], Goto, Heinzer, Kim [10], Kothari [15], ...).

It is a result of Macaulay [16] that the Hilbert function of such a ring A verifies for every positive integer n the following inequalities, :

$$|HF_A(n+1) - HF_A(n)| \leq 1, \quad 0 \leq HF_A(n) \leq n+1.$$

Given such a numerical function there exists a complete intersection $I = (f, g) \subseteq \mathrm{K}[\![x, y]\!]$ with that Hilbert function, [1], [10]. Hence the problem is solved in the Artinian case and when $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay.

Conditions on the Cohen-Macaulayness of $gr_{\mathfrak{m}}(A)$ have been studied by Goto, Heinzer and Kim in [8], [9].

Classical results concerning Cohen-Macaulay local rings of dimension one will be useful in this paper. For example it is well known, see [17],[6], [22], that there exists an integer $e \geq 1$, the multiplicity of A, such that

(i) $HF_A(n) \leq e$ for all n,

(ii) If $HF_A(j) = e$ for some j, then $HF_A(n) = e$ for all $n \geq j$,

(iii) For every $j \geq 0$ we have $HF_A(j) \geq \min\{j+1, e\}$. In particular $HF_A(e-1) = e$.

The least integer r such that $HF(r) = e$ coincides with the reduction number of $\mathfrak{m}$, which is the least integer r such that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$ for some (hence any) superficial element $x \in \mathfrak{m}$. We say that the Hilbert function of A is *increasing* (resp. *strictly increasing*) if $HF(n) \leq HF_A(n+1)$ (resp. $HF(n) < HF_A(n+1)$) for all $n = 0, \cdots, r-1$.

Throughout the whole paper K denotes an algebraically closed field. Even if most of the results do not need this assumption it is useful when we use Hensel Lemma, for instance in Theorem 3.4.

Let $R = \mathrm{K}[\![x_1, \ldots x_n]\!]$ be the ring of formal power series in the indeterminates $\{x_1, \cdots, x_n\}$ with coefficients in K and maximal ideal $\mathcal{M} = (x_1, \cdots, x_n)$. We denote by $\mathbb{U}(R)$ the group of units of R. Let I be an ideal of R and consider the local ring $A = R/I$ whose maximal ideal is $\mathfrak{m} := \mathcal{M}/I$.

We have seen that the Hilbert function of a local ring A is the same as that of the associated graded ring $gr_{\mathfrak{m}}(A)$. Hence it will be useful to recall the presentation of this standard graded algebra. For every power series $f \in R \setminus \{0\}$ we can write $f = f_v + f_{v+1} + \cdots$, where f_v is not zero and f_j is a homogeneous polynomial of degree j in P for every $j \geq v$. We say that v is the order of f, denote f_v by f^* and call it the initial form of f. If $f = 0$ we agree that its order is ∞. It is well known that $gr_{\mathfrak{m}}(A) = P/I^*$, where I^* is the homogeneous ideal of the polynomial ring P generated by the initial forms of the elements of I. A set of power series $f_1, \cdots, f_r \in I$ is a standard basis of I if $I^* = (f_1^*, \cdots, f_r^*)$, (see [13]). It is clear that every ideal I has a standard basis and that every standard basis is a basis. However not every basis is a standard basis. To determine a standard basis of a given ideal of R is difficult, even in the very special case we are involved with in this paper.

In order to determine the Hilbert function of such local complete intersections, we use an extension to power series of the theory of Gröbner bases introduced by Buchberger for ideals in the polynomial ring. Our paper focuses on this technique. Our strategy seems unusual in this context, although it has been used in singularity theory.

We recall that the notion of Gröbner basis is defined by considering a term ordering on the terms of P (i.e. a monomial ordering where all the terms are bigger than 1). Instead, we need here to consider the so called *local degree ordering,* see [11], Chapter 6, a monomial ordering on the terms of P which is not a term ordering.

We denote by $\mathbb{T}^n$ the set of terms or monomials of P; let τ be a term ordering in $\mathbb{T}^n$, and we assume that $x_1 > \cdots > x_n$. We define a new total order $\overline{\tau}$ on $\mathbb{T}^n$ in the following way: given $m_1, m_2 \in \mathbb{T}^n$ we let $m_1 >_{\overline{\tau}} m_2$ if and only if $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and $m_1 >_\tau m_2$. Given $f \in R$ we denote by $\mathrm{Supp}(f)$ the support of f, i.e. if $f = \sum_{\underline{i} \in \mathbb{N}^n} a_{\underline{i}} x^{\underline{i}}$ then $\mathrm{Supp}(f)$ is the set of terms $x^{\underline{i}}$ such that $a_{\underline{i}} \neq 0$. We remark that, given f in R, there is a monomial which is the maximum of the monomials in $\mathrm{Supp}(f)$ with respect to $\overline{\tau}$: namely, since the support of f^* is a finite set, we can take the maximum with respect to τ of the elements of this set. This monomial is called the leading monomial of f with respect to $\overline{\tau}$ and is denoted by $Lt_{\overline{\tau}}(f)$. By definition we have

$$Lt_{\overline{\tau}}(f) = Lt_\tau(f^*).$$

As usual we define the leading term ideal associated to an ideal $I \subset R$ as the monomial ideal $\mathrm{Lt}_{\overline{\tau}}(I)$ generated in R by $\mathrm{Lt}_{\overline{\tau}}(f)$ with f running in I.

In [1] a set $\{f_1, \ldots, f_r\}$ of elements of I is called an *enhanced* standard basis of I if the corresponding leading terms generate $\mathrm{Lt}_{\overline{\tau}}(I)$. Every enhanced standard basis is also a standard basis, but the converse is not true: an example is given by the ideal $I = (x^2 + y^2, xy + y^3)$ in the power serie ring $K[[x, y]]$.

In [11] an enhanced standard basis of I is simply called a standard basis. We have $\mathrm{Lt}_{\overline{\tau}}(I)P = Lt_{\tau}(I^*)$ (see [1] Proposition 1.5.) so that

$$HF_{R/I} = HF_{P/I^*} = HF_{R/\mathrm{Lt}_{\overline{\tau}}(I)}.$$

In the theory of enhanced standard basis a crucial result is the Grauert's Division theorem, [11, Theorem 6.4.1]. It claims the following. Given a set of formal power series $f, f_1, \cdots, f_m \in R$ there exist power series $q_1, \ldots, q_m, r \in R$ such that $f = \sum_{j=1}^{m} q_j f_j + r$ and, for all $j = 1, \ldots, m$,

(1) No monomial of r is divisible by $Lt_{\overline{\tau}}(f_j)$,

(2) $Lt_{\overline{\tau}}(q_j f_j) \leq Lt_{\overline{\tau}}(f)$ if $q_j \neq 0$.

With the above result we can define

$$NF(f|\{f_1, \ldots, f_m\}) := r$$

and obtain in this way a reduced normal form of any power series f with respect to a given finite subset of R. Using the reduced normal form, we can obtain in the formal power series ring all the properties of Gröbner bases analogous to those proves in the classical case for polynomial rings. In particular Buchberger's criterion holds for the power series ring $\mathrm{K}[\![x_1, \ldots x_n]\!]$, see [11, Theorem 1.7.3]. A similar approach was introduced by Briançon (see [2]) for local artinian ideals in $\mathbb{C}\{x, y\}$, and by Mora for ideals in the localization of P at the maximal ideal $(x_1, \ldots, x_n)$ (see [18]). We notice that the work by Briançon is someway related to the main ideas of this paper; in turn we can say that Briançon obtained his approach by Hironaka.

We come now to describe the content of the paper. The main result is the description of all the numerical functions which are the Hilbert functions of what we call a quadratic complete intersection of codimension two in $\mathrm{K}[\![x, y, z]\!]$. By this we mean local rings of type $\mathrm{K}[\![x, y, z]\!]/(f, g)$ where f and g are power series of order two which form a regular sequence in $\mathrm{K}[\![x, y, z]\!]$ with the property that $g^* \notin (f^*)$.

We first prove in Proposition 2.2 that for the Hilbert function H of such local rings with multiplicity e, there are only two possibilities:

(1) either H increases by one until reaching the multiplicity e,

i	0	1	2	3	4	5	...	e-3	e-2	e-1	...
H(i)	1	3	4	5	6	7	...	e-1	e	e	...

(2) or it is increasing by one until reaching e, except for a unique flat in degree n for some n. By this we mean that for some integer $n \leq e - 3$ the sequence H satisfies

i	0	1	2	3	4	...	n-1	n	n+1	n+2	...	e-2	e-1	e	...
H(i)	1	3	4	5	6	...	n+1	n+2	n+2	n+3	...	e-1	e	e	...

It turns out that if the Hilbert function is increasing by one, case (1), there is no restriction on the multiplicity. However, if the Hilbert function has a flat, case (2), the multiplicity satisfies $e \leq 2n$. This is proved in Theorem 3.6 which is the main result of this paper. Examples 2.3 and 3.7 show that the above Hilbert functions are realized.

We present also two more results on the Cohen-Macaulayness of the tangent cone of such complete intersections. First, in Proposition 2.5, we prove that a quadratic complete intersection of codimension two in $\mathrm{K}[\![x, y, z]\!]$ with Hilbert function increasing by one has an associated graded ring which is Cohen-Macaulay. Finally, as a second application of the methods we used in the proof of the main theorem, we are able to prove in Proposition 3.8 that for a quadratic complete intersection $A = \mathrm{K}[\![x, y, z]\!]/I$, the tangent cone is Cohen-Macaulay when the vector space I^*_2 does not contain a square of a linear form.

In Section 4 we give structure theorems, modulo analytic isomorphisms, for the minimal system of generators of quadratic complete intersection ideals I of codimension two in $\mathrm{K}[\![x, y, z]\!]$. Theorem 4.1 and Theorem 4.2 take into account the two cases for the Hilbert function H.

In the last section of the paper we give several examples to illustrate our results, as well as possible extensions.

Most of the computations we made to support the stuff of this last section have been done with the help of CoCoa, using an algorithm written by A.Conca concerning the Hilbert function of local algebra. We would like to thank A. Conca for this support and for the many helpful conversations and suggestions while preparing this paper. We are grateful to the referee for useful suggestions and a careful reading of the paper.

2 Ideals of type $(2, b)$

From now on we assume that $A = \mathrm{K}[\![x, y, z]\!]/I$ where I is a codimension two complete intersection ideal of $R = \mathrm{K}[\![x, y, z]\!]$. Given the integers $b \geq a \geq 2$, we say that A is *of type* (a, b), or I is of type (a, b), if I can be generated by a regular sequence $\{f, g\}$ such that order(f)=a, order(g)=b and $g^* \notin (f^*)$. In the language of [13, Chapter III, Section 1] we write $\nu^*(I) = (a, b)$ with the meaning that I is of type (a, b).

In this paper we will be mainly concerned with local rings of type $(2, 2)$; however in this section properties of local rings of type $(2, b)$ will be considered.

A local ring A of type $(2, b)$ is Cohen-Macaulay of embedding dimension three and dimension one so that we know that the Hilbert function is not decreasing by [6]. We say that HF_A *admits a flat in position* n if

$$HF_A(n) = HF_A(n + 1) < e.$$

For instance, the sequence $\{1, 3, 5, 5, 5, 6, ...\}$ has a flat in position 2 and a flat in position 3; the sequence $\{1, 3, 5, 6, 6, 7, 8, 9, 9, 10, 11, 11, 11, 11, 11, 12, 12, ...\}$ has a flat in position 3, a flat in position 7, a flat in position 10, a flat in position 11, a flat in position 12, a flat in position 13 and a flat in position 14. All toghether it has 7 flats.

The first basic properties of the Hilbert function of a local ring of type $(2, b)$ are collected in the following proposition which is an easy consequence of the classical Macaulay Theorem, [3] Theorem 4.2.10.

We recall that given two positive integer n and c, the n-binomial expansion of c is

$$c = \binom{c_n}{n} + \binom{c_{n-1}}{n-1} + \cdots \binom{c_j}{j}$$

where $c_n > c_{n-1} > \cdots c_j \geq j \geq 1$. We let

$$c^{<n>} = \binom{c_n + 1}{n + 1} + \binom{c_{n-1} + 1}{n} + \cdots \binom{c_j + 1}{j + 1}.$$

The Theorem of Macaulay states that a numerical function $\{h_0, h_1, \cdots, h_i, \cdots,\}$ is the Hilbert function of a standard graded algebra if and only if $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for every $i \geq 1$. We remark that if $n + 1 \leq c \leq 2n$ then the n-binomial expansion of c is

$$c = \binom{n+1}{n} + \binom{n-1}{n-1} + \cdots \binom{2n-c+1}{2n-c+1},$$

so that $c^{<n>} = c + 1$.

Further, if $f_1, \ldots, f_r$ are elements of order $d_1, \ldots, d_r$ in the regular local ring $(R, \mathcal{M})$ and J the ideal they generate, it is known that

$$J_n^* = (J \cap \mathcal{M}^n + \mathcal{M}^{n+1})/\mathcal{M}^{n+1}$$

and

$$(f_1^*, \ldots, f_r^*)_n = (\sum_{i=1}^{r} \mathcal{M}^{n-d_i} f_i + \mathcal{M}^{n+1})/\mathcal{M}^{n+1}$$

for every non negative integer n. With this notation we have the following basic lemma which is possibly well known: we include it for completeness.

Lemma 2.1. *Let $I = (f, g)$ be an ideal of $(R, \mathcal{M})$ with order$(f) = 2 \leq$ order$(g) = b$. Then*

(i) ${I^*}_j = (f^*)_j$ *for every integer* $2 \leq j < b$.

(ii) ${I^*}_b = (f^*, g^*)_b$.

(iii) If $g^* \notin (f^*)$ *then* ${I^*}_{b+1} = (f^*, g^*)_{b+1}$.

Proof. Since $j + 1 \leq b$ we have $g \in \mathcal{M}^b \subseteq \mathcal{M}^{j+1} \subseteq \mathcal{M}^j$, hence

$$(f, g) \cap \mathcal{M}^j + \mathcal{M}^{j+1} = (g) + (f) \cap \mathcal{M}^j + \mathcal{M}^{j+1} = f\mathcal{M}^{j-2} + \mathcal{M}^{j+1}.$$

The first assertion follows. We prove now (ii). We have:

$$(f, g) \cap \mathcal{M}^b = (g) + (f) \cap \mathcal{M}^b = (g) + f\mathcal{M}^{b-2}.$$

As for (iii) we need to prove that if $g^* \notin (f^*)$ then $(f, g) \cap \mathcal{M}^{b+1} = f\mathcal{M}^{b-1} + g\mathcal{M}$. The inclusion $\supseteq$ is clear, so let $\alpha = cf + dg \in \mathcal{M}^{b+1}$. If $d \in \mathcal{M}$ then $cf \in \mathcal{M}^{b+1}$ and this implies $c \in \mathcal{M}^{b-1}$ as required. If $d \notin \mathcal{M}$ then $g \in ((f) + \mathcal{M}^{b+1}) \cap \mathcal{M}^b = \mathcal{M}^{b+1} + f\mathcal{M}^{b-2}$ which implies $g^* \in (f^*)$, a contradiction. $\square$

Proposition 2.2. *Let $A = R/I$ be a local ring of type $(2, b)$ and $I = (f, g)$ with* order$(f) = 2$, order$(g) = b$ *and $g^* \notin (f^*)$. Then the following properties hold.*

(i) $HF_A(j) = 2j + 1$ *if* $j < b$.

(ii) $HF_A(b) = 2b$.

(iii) $HF_A(j - 1) \leq HF_A(j) \leq HF_A(j - 1) + 1$ *if* $j \geq b$.

(iv) HF_A *admits at most* $b - 1$ *flats.*

Proof. By (i) of the above Lemma we have for every $j < b$

$$HF_A(j) = HF_{P/I^*}(j) = HF_{P/(f^*)}(j) = 2j + 1.$$

We prove now the second assertion. By (ii) of the above Lemma we have

$$HF_A(b) = HF_{P/I^*}(b) = HF_{P/(f^*,g^*)}(b).$$

Since $g^* \notin (f^*)$ we get $HF_A(b) = HF_{P/(f^*)}(b) - 1 = 2b + 1 - 1 = 2b$ as required.

As for (iii) we need only to prove that $HF_A(j) \leq HF_A(j - 1) + 1$ if $j \geq b$. We have $HF_A(b) = 2b$, $HF_A(b - 1) = 2b - 1$, hence we can argue by induction on j. Let $j \geq b$ and assuming $HF_A(j) \leq HF_A(j - 1) + 1$ we need to prove that $HF_A(j + 1) \leq HF_A(j) + 1$.

We have $j + 1 \leq HF_A(j) < HF_{P/(f^*)}(j) = 2j + 1$, hence, by the remark before the Lemma, we get

$$HF_A(j + 1) \leq HF_A(j)^{<j>} = HF_A(j) + 1$$

as claimed.

Finally we prove (iv). We have $HF_A(b) = 2b$ and at each step HF_A goes up at most by one. Hence, if there are p flats between b and j, we have $HF_A(j) = 2b + j - b - p$. But $HF_A(j) \geq j + 1$, so that $p \leq b - 1$. $\square$

From the above proposition it follows that the Hilbert function of a local ring of type $(2, b)$ either is strictly increasing or it has one or more flats (no more than $b - 1$); if the first is the case, it has the following shape

$$HF_A(j) = \begin{cases} 2j+1 & j = 0, \dots, b-1, \\ j+b & b \le j \le e-b, \\ e & j \ge e-b+1. \end{cases} \tag{1}$$

where e and b are integers, $b \ge 2$ and $e \ge 2b$.

We show with the following example that given a numerical function H as in (1) we can find a local ring of type $(2,b)$ with multiplicity e whose Hilbert function is H.

Example 2.3. Let $b \ge 2$ and $e \ge 2b$. We claim that the above numerical function is the Hilbert function of the following local ring of type $(2,b)$ and multiplicity e.

Let $I = (x^2+y^{e-2b+2}, xy^{b-1})$ and $A = \mathrm{K}[\![x,y,z]\!]/I$. We fix an ordering on the monomials of P with the property that $x > y$. We let $f := x^2 + y^{e-2b+2}$, $g := xy^{b-1}$ and claim that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy^{b-1}, y^{e-b+1})$.

Since $e \ge 2b$ and $x > y$ it is clear that $\mathrm{Lt}_{\overline{\tau}}(f) = x^2$. We have

$$S(f,g) = y^{b-1}f - xg = y^{b-1}(x^2 + y^{e-2b+2}) - xxy^{b-1} = y^{e-b+1}.$$

Let $h := S(f,g) = y^{e-b+1}$, then

$$S(f,h) = y^{e-b+1}f - x^2h = y^{e-b+1}(x^2 + y^{e-2b+2}) - x^2y^{e-b+1} = y^{2e-3b+3} = y^{e-2b+2}h$$

and $S(g,h) = 0$. It follows that

$$\begin{aligned} \mathrm{NF}(S(f,g) \mid \{h\}) &= \mathrm{NF}(h \mid \{h\}) \;=\; 0, \\ \mathrm{NF}(S(f,h) \mid \{h\}) &= \mathrm{NF}(y^{e-2b+2}h \mid \{h\}) \;=\; 0, \\ \mathrm{NF}(S(g,h) \mid \{h\}) &= \mathrm{NF}(0 \mid \{h\}) \;=\; 0. \end{aligned}$$

By the Buchberger criterion we get that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy^{b-1}, y^{e-b+1})$ as claimed. By a simple computation we can prove that $K[x,y,z]/(x^2, xy^{b-1}, y^{e-b+1})$ has the above Hilbert function; clearly the same is true for the local ring $\mathrm{K}[\![x,y,z]\!]/(x^2 + y^{e-2b+2}, xy^{b-1})$.

We end this section by proving that for a local ring of type $(2,b)$ the condition that the Hilbert function is strictly increasing is equivalent to the Cohen-Macaulayness of the tangent cone. First we need to prove that the property of having type (a,b) is preserved by passing to the quotient modulo a suitable superficial element. We recall that an element $\ell \in \mathcal{M}$ is superficial for $\mathcal{M}/I$ if ℓ does not belong to any of the associated primes of I^* different from the homogeneous maximal ideal. Since the residual field is infinite the existence of superficial elements is guaranteed. Moreover, it is easy to prove:

Proposition 2.4. *Let I be an ideal of R of type (a,b) with $2 \le a \le b$. There exists $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ such that*

(i) the coset of ℓ in R/I is superficial for $\mathcal{M}/I$,

(ii) $\bar{I} = I + (\ell)/(\ell)$ is an ideal of $R/(\ell)$ of type (a,b).

Proof. It is well known that ℓ verifies (i) if ℓ^* does not belong to any of the associated prime ideals of I^* (different from the homogeneous maximal ideal). Let $I = (f,g)$ be with $order(f) = a \le order(g) = b$. Then it is easy to see that $\bar{I}$ satisfies (ii) provided:

a) ℓ^* does not divide f^*

b) $g^* \notin (f^*, \ell^*)$.

Namely $\bar{I} = (\bar{f}, \bar{g})$ in R/ℓ and $order(\bar{f}) = a$ by condition a) while $\bar{g}^{\,*} \notin (\bar{f}^{\,*})$ by condition b) Since $\operatorname{depth} \mathrm{K}[x,y,z]/(f^*, g^*) \ge 1$, it is easy to see that for having a) and b) it is enough to choose $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ such that ℓ^* is regular in $P/(f^*, g^*)$. Clearly, if this is the case, ℓ^* does not divide f^* and if $g^* \in (f^*, \ell^*)$, then $g^* = \alpha f^* + \beta \ell^*$ with $\alpha, \beta \in P$. Since ℓ^* is $P/(f^*, g^*)$-regular, then $\beta \in (f^*, g^*)$. Hence $g^* = \alpha f^* + \ell^*(\beta_1 g^* + \beta_2 \ell^*)$, so $g^*(1 - \ell\beta_1) \in (f^*)$, a contradiction because $g^* \notin (f^*)$. Since the residue field is infinite, an element $\ell \in \mathcal{M} \setminus \mathcal{M}^2$ verifying the conditions of the proposition can be selected by avoiding the associated prime ideals to I^* and to (f^*, g^*). $\square$

It is well known that if the associated graded ring $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, then the Hilbert function of A is strictly increasing. However the converse is in general very rare. In the following result we will show a special case where this implication holds true.

Proposition 2.5. *Let $A = R/I$ be a local ring of type $(2, b)$. Then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay if and only if HF_A is strictly increasing.*

Proof. Let $I = (f, g)$ with order$(f) = 2$, order$(g) = b$ and $g^* \notin (f^*)$. If the associated graded ring is Cohen-Macaulay, then its Hilbert function is strictly increasing and thus the Hilbert function of A is strictly increasing as well. By using (1) a simple computation gives

$$\Delta HF_A(n) := HF_A(n+1) - HF_A(n) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \dots, b-1, \\ 1 & n = b, \dots, r-1, \\ 0 & n \geq r \end{cases}$$

with $r = e - b + 1$.

From Proposition 2.4 there exists a superficial element $x \in A$ such that

$$HF_{A/xA}(n) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \dots, b-1, \\ 1 & n = b. \end{cases}$$

From Macaulay's characterization of Hilbert functions and the fact that $e(A/xA) = e(A)$, we get $\Delta HF_A = HF_{A/xA}$. Hence $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay, [23]. $\square$

Notice that the above proposition cannot be extended to local rings of type (a, b) with $a > 2$, as the following example shows. Consider the local ring $A = R/I$ where $I = (x^4, x^2y + z^4) \subseteq R = K[[x, y, z]]$; A is a one-dimensional Gorenstein local ring and

$$HF_A = \{1, 3, 6, 9, 11, 13, 14, 15, 16, 16, \dots, \}$$

is strictly increasing. Now it is clear that $x^4, x^2y \in I^*$ and since $x^2(x^2y + z^4) - yx^4 \in I$, also $x^2z^4 \in I^*$. This implies that $x^3z^3(x, y, z) \subseteq I^*$; since $x^3z^3 \notin I^*$ $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay.

A natural and general problem would be to characterize the Hilbert functions of all the ideals I of type $(2, b)$. If the Hilbert function has one or more flat, the behavior is difficult to control. However if we denote by p the number of flats, by Proposition 2.2 we know that $p \leq b - 1$. With the aid of huge computations made with CoCoa, we ask the following question.

Question 2.6. *Let $A = R/I$ be a local ring of type $(2, b)$ with $b \geq 2$ and multiplicity e. Let $n := \min\{j : HF_{R/I}(j) = HF_{R/I}(j+1) < e\}$ and let p be the number of flats. Then*

$$e \leq (p+1)n \ (\leq bn).$$

The main result of the paper answer the question in the case $a = b = 2$.

3 The main result

In this section we present a complete characterization of the numerical functions which are the Hilbert functions of local rings of type $(2, 2)$. In particular we prove that certain monomial ideals cannot be the initial ideals of a complete intersection, a relevant result even in the graded setting (see for example [5]).

By the definition we gave in the above section, a local ring $A = K[[x, y, z]]/I$ of type $(2, 2)$ is of dimension one and has $HF_A(1) = 3$ and $HF_A(2) = 4$. In particular I can be generated by a regular sequence, say $I = (f, g)$, where f and g are power series of order two such that f^* and g^* are linearly independent in the vector space $K[x, y, z]_2$. We recall that by Lemma 2.1 we have $I_2^* = (f^*, g^*)_2$ and $I_3^* = (f^*, g^*)_3$.

Now it is clear that f^*, g^* form a regular sequence if and only if the Hilbert Series of $K[x, y, z]/(f^*, g^*)$ is $1 + 3z + 4z + 4z^2 + 4z^3 + \dots$

Since $HF_A(2) = 4$, we know that

$$4 = HF_A(2) \leq HF_A(3) \leq HF_A(2) + 1 = 5.$$

If $HF_A(3) = 4$, then $HF_{K[x,y,z]/(f^*,g^*)}(3) = 4$ and this implies that f^* and g^* form a regular sequence in $K[x, y, z]$. As a consequence $I^* = (f^*, g^*)$ and the Hilbert function of A is $\{1, 3, 4, 4, 4, ...\}$ which is as in (1) with $b = 2$, $e = 4$.

We want to study the REMAINING CASE when $A = R/I$ IS A LOCAL RING OF TYPE $(2, 2)$ WITH $HF_A(0) = 1, HF_A(1) = 3, HF_A(2) = 4, HF_A(3) = 5$.

We first remark that in this case f^* and g^* share a common factor, say L, which must be linear because f^* and g^* are linearly independent. Hence we can write

$$f^* = LM, \quad g^* = LN$$

where M, N are linear forms linearly independent in $K[x, y, z]$. In particular ${I^*}_2 =< LM, LN >$.

We have two possibilities, either L, M, N are linearly independent or they are linearly dependent. We remark that this property depends on the ideal I and not on the generators of I. Namely, if we say that ${I^*}_2$ is *square free* with the meaning that it does not contain a square of a linear form, we can prove the following easy result:

Lemma 3.1. *With the above notation the vectors L, M, N are linearly independent if and only if ${I^*}_2$ is square-free.*

Proof. Let us first assume that L, M, N are linearly dependent. Since M, N are linearly independent we have $L = \alpha M + \beta N$ so that $L^2 = \alpha LM + \beta LN \in< {I^*}_2 >$. Hence ${I^*}_2$ is not square-free.

We prove now that if L, M, N are linearly independent then ${I^*}_2$ is square-free. Let P be a linear form such that $P^2 \in {I^*}_2 =< LM, LN >$; then $P \in (L)$ so that $P = \lambda L$. We have $\lambda^2 L^2 = \alpha LM + \beta LN$ hence $\lambda^2 L = \alpha M + \beta N$; since L, M, N are linearly independent this implies $\lambda = 0$ and finally $P = 0$. □

For completeness, we need now to recall the notion of k-algebra isomorphism. Given a set of minimal generators $\underline{y} = \{y_1, y_2, ..., y_n\}$ of the maximal ideal of $R = \mathrm{K}[\![x_1, \dots x_n]\!]$, we let $\phi_{\underline{y}}$ be the automorphism of R which is the result of substituting y_i for x_i in a power series $f(x_1, x_2, ..., x_n) \in R$. Given two ideals I and J in R it is well known that there exist a K-algebra isomorphism $\alpha : R/I \to R/J$ if and only if for some generators $y_1, y_2, ..., y_n$ of the maximal ideal of R, we have $I = \phi_{\underline{y}}(J)$.

We start now by deforming, up to isomorphism, the generators f and g of the given ideal I.

Lemma 3.2. *Let $A = R/I$ be a local ring of type $(2, 2)$ such that $HF_A(3) = 5$.*

(i) If ${I^*}_2$ is not square-free we may assume, up to isomorphism, that $I = (f, g)$ with $f^* = x^2$ and $g^* = xy$.

(ii) If ${I^*}_2$ is square-free we may assume, up to isomorphism, that $I = (f, g)$ with $f^* = xy$ and $g^* = xz$,

Proof. Let us first assume that I_2^* is not square-free; then $f^* = LM$, $g^* = LN$ with L, M, N linearly dependent; since M and N are linearly independent, we must have $L = \lambda M + \rho N$ for suitable λ and ρ in K with $(\lambda, \rho) \neq (0, 0)$. By symmetry we may assume $\lambda \neq 0$. Then it is easy to see that L and N are linearly independent so that we can consider an automorphism ϕ sending $x \to L, y \to N$. We have $f = LM + a$ and $g = LN + b$ for suitable $a, b \in \mathcal{M}^3$, and further

$$L^2 = \lambda LM + \rho LN = \lambda f + \rho g - \lambda a - \rho b.$$

We get

$$I = (f, g) = (\lambda f, g) = (L^2 - \rho g + \lambda a + \rho b, g) = (L^2 + \lambda a + \rho b, g) =$$
$$= (L^2 + \lambda a + \rho b, LN + b) = \phi((x^2 + \phi^{-1}(\rho b + \lambda a), xy + \phi^{-1}(b)).$$

The conclusion follows.

Now we assume that ${I^*}_2$ is square-free. Then $f^* = LM$ and $g^* = LN$ where L, M, N are linear forms in $K[x, y, z]$ which are linearly independent. As before we have $f = LM + a$ and $g = LN + b$ for suitable $a, b \in \mathcal{M}^3$,

Let us consider the automorphism ϕ sending $x \to L, y \to M, z \to N$. We have

$$I = (f, g) = (LM + a, LN + b) = \phi((xy + \phi^{-1}(a), xz + \phi^{-1}(b)))$$

and the conclusion follows. $\square$

Using Grauert division theorem, we can prove a first useful preparation result in the case $x^2 = \mathrm{Lt}_{\overline{\tau}}(f)$ and $xy = \mathrm{Lt}_{\overline{\tau}}(g)$.

Lemma 3.3. *Let $A = R/I$ be a local ring of type $(2, 2)$ such that $I = (f, g)$, $\mathrm{Lt}_{\overline{\tau}}(f) = x^2$, $\mathrm{Lt}_{\overline{\tau}}(g) = xy$. Then we can write*

$$I = (x^2 + axz^p + F(y, z), xy + bxz^q + G(y, z))$$

where $p, q \geq 1$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $F, G \in \mathrm{K}[\![y, z]\!]_{\geq 2}$.

Proof. By the assumption we have $f = x^2 + F$ with $\mathrm{Lt}_{\overline{\tau}}(F) <_{\overline{\tau}} x^2$ and $g = xy + G$ with $\mathrm{Lt}_{\overline{\tau}}(G) <_{\overline{\tau}} xy$. Using Grauert's division theorem for the power series F, f, g we get

$$F = \alpha f + \beta g + r$$

where $\alpha, \beta, r \in R$, no monomial of $\mathrm{Supp}(r)$ is divisible by x^2 or xy, and

$$\mathrm{Lt}_{\overline{\tau}}(\alpha f), \mathrm{Lt}_{\overline{\tau}}(\beta g) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(F) <_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(f) = x^2.$$

We can write $\alpha = \sum_{i \geq 0} \alpha_i$, where, for every i, α_i is a degree i form in $\mathrm{K}[\![x, y, z]\!]$. It is clear that the initial form of $\alpha f = \alpha(x^2 + F)$ is $\alpha_0 x^2 + \alpha_0 F_2$ so that $\alpha_0 = 0$, otherwise $\mathrm{Lt}_{\overline{\tau}}(\alpha f) = x^2$. In particular $1 - \alpha$ is a unit. Since

$$(1 - \alpha)f = f - \alpha f = x^2 + F - \alpha f = x^2 + r + \beta g$$

we get

$$I = (f, g) = (x^2 + r, g).$$

We apply now Grauert's Division Theorem to the power series $G, x^2 + r, f$ where $G = g - xy$ and $\mathrm{Lt}_{\overline{\tau}}(G) <_{\overline{\tau}} xy$. We get

$$g - xy = G = t(x^2 + r) + sg + r'$$

where no monomial of $\mathrm{Supp}(r')$ is divisible by $\mathrm{Lt}_{\overline{\tau}}(x^2+r) = \mathrm{Lt}_{\overline{\tau}}(x^2+F-\alpha f-\beta g) = x^2$ or by $\mathrm{Lt}_{\overline{\tau}}(g) = xy$.

Since $g = xy + t(x^2 + r) + sg + r'$, we get

$$g(1 - s) = t(x^2 + r) + r' + xy$$

and we claim that $1 - s$ is a unit. Namely, $\mathrm{Lt}_{\overline{\tau}}(sg) \leq \mathrm{Lt}_{\overline{\tau}}(G) < xy$ and, as before,

$$sg = s(xy + G) = s_0(xy + G) + s_1(xy + G) + ...$$

This implies $s_0 = 0$, otherwise $\mathrm{Lt}_{\overline{\tau}}(sg) = xy$. This proves the claim.

Now we have

$$I = (x^2 + r, g) = (x^2 + r, (1 - s)g) = (x^2 + r, t(x^2 + r) + r' + xy) = (x^2 + r, xy + r'),$$

where no monomial of Supp(r) and Supp(r') is divisible by x^2 or xy.

It is easy to see that this implies

$$\begin{array}{rcl} r & = & axz^p + F(y,z) \\ r' & = & bxz^q + G(y,z) \end{array}$$

with $p, q \geq 1$, $F, G \in \mathrm{K}[\![y,z]\!]_{\geq 2}$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, and $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$. $\square$

We can prove now the main preparation result.

Theorem 3.4. *Let $A = R/I$ be a local ring of type $(2,2)$ such that $HF_A(3) = 5$.*

a) If I_2^* is not square-free then, up to isomorphism, we can write

$$I = (x^2 + axz^p + F(y,z), xy + G(y,z))$$

where $p \geq 2$, $a \in \{0,1\}$, and $F, G \in \mathrm{K}[\![y,z]\!]_{\geq 3}$.

b) If I_2^* is square-free then, up to isomorphism, we can write

$$I = (x^2 + xz + F(y,z), xy + dyz + \alpha y^r + \beta z^s)$$

where $F \in \mathrm{K}[\![y,z]\!]_{\geq 3}$, $d \in \mathrm{K}[\![y,z]\!]$, $d(0,0) = 1$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, $\beta = 0$ or $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$.

Proof. By Lemma 3.2, up to isomorphism, we can find generators f and g of I such that either $f^* = x^2$ and $g^* = xy$ or $f^* = xy$ and $g^* = xz$.

Let us first assume that $f^* = x^2$ and $g^* = xy$; then

$$\mathrm{Lt}_{\overline{\tau}}(f) = Lt_\tau(f^*) = Lt_\tau(x^2) = x^2$$

$$\mathrm{Lt}_{\overline{\tau}}(g) = Lt_\tau(g^*) = Lt_\tau(xy) = xy,$$

so that, as remarked at the end of the proof of Lemma 3.3, we have $I = (x^2 + r, xy + r')$ where no monomial of Supp(r) and Supp(r') is divisible by x^2 or xy.

Since $f^* = x^2$ and $g^* = xy$, we also have $f = x^2 + h$, $g = xy + s$ where order(h), order(s) ≥ 3. This implies $I = (x^2 + r, xy + r') = (x^2 + h, xy + s)$ and $I_2^* = < x^2, xy >$, the vector space spanned by x^2 and xy.. Since the degree 2 component of r is a linear combination of the monomials xz, y^2, yz, z^2, it must be zero, otherwise the leading form of $x^2 + r$ cannot be in $I_2^* = < x^2, xy >$. This proves that the order of r is at least 3. Exactly in the same way we can prove that this holds true also for r'.

It is easy to see that this implies

$$r = axz^p + D(y,z), \qquad r' = bxz^q + E(y,z)$$

where $p, q \geq 2$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$, and $D, E \in \mathrm{K}[\![y,z]\!]_{\geq 3}$.

Now let ϕ be the automorphism of $\mathrm{K}[\![x,y,z]\!]$ defined by

$$x \to x, \quad y \to y - bz^q, \quad z \to z$$

and let $S := \phi(D)$ and $T := \phi(E)$. Then $S, T \in \mathrm{K}[\![y,z]\!]_{\geq 3}$ and we have

$$\phi(f) = \phi(x^2 + r) = \phi(x^2 + axz^p + D(y,z)) = x^2 + axz^p + S(y,z))$$

and

$$\phi(g) = \phi(xy + r') = \phi(xy + bxz^q + E(y,z)) = x(y - bz^q) + bxz^q + \phi(E(y,z)) = xy + T(y,z)).$$

This implies that, up to isomorphism, we may assume

$$I = (x^2 + axz^p + S(y,z), xy + T(y,z))$$

with $p \geq 2$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$, and $S, T \in \mathrm{K}[\![y,z]\!]_{\geq 3}$.

Now if $a = 0$ we are done, otherwise let $a \neq 0$. Since the ground field K is algebraically closed and a is invertible in $K[[z]]$, a straightforward application of the Hensel Lemma enables us to find an element $c \in R$ such that $c^p = a$.

Let us consider the automorphism $\phi : \mathrm{K}[\![x,y,z]\!] \to \mathrm{K}[\![x,y,z]\!]$ defined by

$$x \to x, \quad y \to y, \quad z \to cz.$$

If F and G are power series in $K[[z]]$ such $\phi(F) = S$ and $\phi(G) = T$, then

$$\phi(x^2 + xz^p + F) = x^2 + xc^p z^p + S = x^2 + axz^p + S, \qquad \phi(xy + G) = xy + T.$$

The conclusion easily follows.

We need now to consider the other case when $f^* = xy$, $g^* = xz$. As before we choose a monomial order τ such that $x >_\tau z$ and let ϕ be the automorphism of $\mathrm{K}[\![x,y,z]\!]$ defined by

$$x \to x + z, \quad y \to x, \quad z \to y.$$

We have $f = xy + d$, $g = xz + e$ where d and e have order at least 3. Hence

$$\phi(f) = (x+z)x + \phi(d) = x^2 + xz + h, \qquad \phi(g) = (x+z)y + \phi(e) = xy + yz + s$$

where $h := \phi(d)$ and $s := \phi(e)$ have order ≥ 3. Thus, up to isomorphism, we may assume that I is generated by the power series $x^2 + xz + h$ and $xy + yz + s$; this implies that $I_2^* =< x^2 + xz, xy + yz >$.

Since $x^2 >_\tau xz$ and $xy >_\tau yz$, we get

$$\mathrm{Lt}_{\overline{\tau}}(x^2 + xz + h) = Lt_\tau((x^2 + xz + h)^*) = Lt_\tau(x^2 + xz) = x^2$$

$$\mathrm{Lt}_{\overline{\tau}}(xy + yz + s) = Lt_\tau((xy + yz + s)^*) = Lt_\tau(xy + yz) = xy$$

and we may use Lemma 3.3 to get

$$I = (x^2 + axz^p + S(y,z), xy + bxz^q + M(y,z))$$

where $p, q \geq 1$, $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $b = 0$ or $b \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $S, M \in \mathrm{K}[\![y,z]\!]_{\geq 2}$.

Now let $\alpha := x^2 + axz^p + S(y,z)$; if $p \geq 2$ then $\alpha^* = x^2 + S(y,z)_2$ is an element of the vector space $I_2^* =< x^2 + xz, xy + yz >$, a contradiction. Hence $p = 1$ and thus we get $\alpha^* = x^2 + a_0 xz + S(y,z)_2 \in < x^2 + xz, xy + yz >$. This clearly implies $a_0 = 1$ and $S(y,z)_2 = 0$, so that the order of $S(y,z)$ is at least 3.

Now let $\beta := xy + bxz^q + M(y,z)$; if $b \neq 0$ and $q = 1$ then $b_0 \neq 0$ and we have

$$\beta^* = xy + b_0 xz + M(y,z)_2 \in I_2^* =< x^2 + xz, xy + yz >,$$

a contradiction. Hence it must be either $b = 0$ or $q \geq 2$; in both cases we have

$$\beta^* = xy + M(y,z)_2 \in < x^2 + xz, xy + yz >$$

which implies $M(y,z) = yz + H(y,z)$ where $H(y,z)$ is a power series in $K[[y,z]]$ with order at least 3.

At this point we have $I = (x^2 + axz + S(y,z), xy + bxz^q + yz + H(y,z))$ with $a_0 = 1$, S and H $\in K[[y,z]]_{\geq 3}$ and either $b = 0$ or $q \geq 2$.

Let us consider the automorphism ϕ given by

$$x \to x, \quad y \to y - bz^q, \quad z \to z.$$

We get

$$\phi(x^2 + axz + S(y,z)) = x^2 + axz + B(y,z),$$

and

$$\phi(xy + bxz^q + yz + H(y,z)) = x(y - bz^q) + bxz^q + (y - bz^q)z + \phi(H) = xy + yz + L(y,z)$$

where $B(y,z) := \phi(S)$ and $L(y,z) := -bz^{q+1} + \phi(H) \in K[[y,z]]_{\geq 3}$.

Hence, up to isomorphism, we may assume $I = (x^2 + axz + B(y,z), xy + yz + L(y,z))$ with $a_0 = 1$ and $B, L \in K[[y,z]]_{\geq 3}$. Now it is clear that since $L(y,z)$ has order at least 3, we can write $L(y,z) = cyz + \alpha y^r + \beta z^s$ with $c \in K[y,z]_{\geq 1}$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$. Hence we get $I = (x^2 + axz + B(y,z), xy + yz + cyz + \alpha y^r + \beta z^s)$. We let $d := 1 + c$ so that $d \in K[[y,z]]$, $d(0,0) = 1 + c(0,0) = 1$ and

$$I = (x^2 + axz + B(y,z), xy + dyz + \alpha y^r + \beta z^s).$$

Finally let us consider the automorphism ϕ given by

$$x \to x, \quad y \to y, \quad z \to az.$$

Let $F(y,z) := \phi^{-1}(B(y,z))$ and

$$f := x^2 + xz + F(y,z) \qquad g := xy + \phi^{-1}(d/a)yz + \alpha y^r + \phi^{-1}(\beta/a^s)z^s.$$

Then we get

$$\phi(f) = x^2 + axz + B(y,z)$$

$$\phi(g) = xy + (d/a)yaz + \alpha y^r + (\beta/a^s)(a^s z^s) = xy + dyz + \alpha y^r + \beta z^s.$$

We remark that the constant term of the power series d/a is 1 and the power series β/a^s is invertible if not zero. Hence the same holds for $\phi^{-1}(d/a)$ and $\phi^{-1}(\beta/a^s)$. The conclusion follows. $\square$

We recall that in this section we are assuming that $A = \mathrm{K}[\![x,y,z]\!]/I$ is a local ring of type (2,2) such that $HF_A(1) = 3$, $HF_A(2) = 4$ and $HF_A(3) = 5$. This implies that if we let n be **the least integer such that** $HF_A(n) = HF_A(n+1)$, then $n \geq 3$. Also it is easy to see that $n \leq r$, the reduction number of A. The integer n plays a relevant work in this paper. As consequence of Proposition 2.2, the Hilbert function of a local ring A of type (2,2) and multiplicity e has the following shape:

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-1, \\ e & t \geq e. \end{cases} \tag{2}$$

for some integer $n \leq e-2$. We say that HF_A has a **flat** in position n.

It is clear that we have two possibilities, either $e = n+2$ or $e \geq n+3$. In the first case the Hilbert function increases by one up to the multiplicity, while in the second case the Hilbert function has a unique flat at n, and otherwise increases by one up to the multiplicity.

We are ready to prove the main result of this paper. It says that, quite unexpectedly, if the Hilbert function of a local ring of type $(2,2)$ has a flat in position n then the multiplicity satisfies $e \leq 2n$.

First we need this easy Lemma.

Lemma 3.5. *Let $J \subset P = k[x,y,z]$ be a monomial ideal such that $x^2, xy \in J$. If for some $n \geq 2$ we have $HF_{P/J}(n+1) = n+2$ and $HF_{P/J}(n+2) = n+3$, then xz^n is the unique monomial of degree $n+1$ which is in J and not in (x^2, xy).*

If we have also $HF_{P/J}(n) = n+2$, then $J_d = (x^2, xy)_d$ for all $2 \leq d \leq n$.

Proof. Since $HF_{P/J}(n+1) = n+2 < HF_{P/(x^2,xy)}(n+1) = n+3$ there is a monomial m of degree n+1 which is in J and not in (x^2, xy). If $m \neq xz^n$ it should be $m = y^{n+1-j}z^j$ for some $j = 0, ..., n+1$. But then the monomials of the vector space $(x^2, xy)_{n+2}$ and my, mz would be linearly independent. This implies that

$$n+3 = HF_{P/J}(n+2) \leq HF_{P/(x^2,xy,my,mz)}(n+2) = HF_{P/(x^2,xy)}(n+2) - 2 = n+2,$$

a contradiction. Hence $m = xz^n$.

Let us assume that also $HF_{P/J}(n) = n+2$; if for some $t \le n-1$ we have $HF_{P/J}(t) \le t+1$, then

$$HF_{P/J}(t+1) \le HF_{P/J}(t)^{<t>} \le (t+1)^{<t>} = t+2$$

and going on in this way we would have $HF_{P/J}(n) \le n+1$, a contradiction. It follows that for all $2 \le d \le n$ we have $HF_{P/J}(d) = HF_{P/(x^2,xy)}(d)$ and, as a consequence, $J_d = (x^2, xy)_d$ for the same d's. $\square$

Theorem 3.6. *Let A be a local ring of type (2,2) and multiplicity e. If the Hilbert function of A has a flat in position n, then $e \le 2n$.*

Proof. As usual, we consider a monomial ordering τ on the terms of $K[x,y,z]$ such that $x >_\tau z$. In order to cover both case a) and b) in Theorem 3.4, we may assume $I = (f,g)$ where

$$f := x^2 + axz^p + F(y,z) \qquad g := xy + G(y,z)$$

are power series such that $p \ge 1$ and $a = 0$ or $a \in \mathbb{U}(\mathrm{K}[\![z]\!])$.

Since it is clear that $(x^2, xy) \nsubseteq \mathrm{Lt}_{\overline{\tau}}(I)$, the elements f and g are not a standard basis for I; thus, by Buchberger's criterion, we should have

$$h := \mathrm{NF}(S(f,g), \{f,g\}) \neq 0.$$

It is clear that $h \in I$ and if we let $m := \mathrm{Lt}_{\overline{\tau}}(h) = \mathrm{Lt}_\tau(\mathrm{h}^*)$, then $m \in \mathrm{Lt}_{\overline{\tau}}(I)$ and by 1.6.4 in [23] $m \notin (x^2, xy)$. We claim that m is a monomial of degree $n+1$.

Namely, by the second statement of Lemma 3.5 applied to the monomial ideal $\mathrm{Lt}_{\overline{\tau}}(I)$, it is clear that m has degree at least $n+1$. Let us assume that m has degree $\ge n+2$, so that $\mathrm{order}(h) \ge n+2$. Since for every s and G one can easily prove that $\mathrm{order}(s) \le \mathrm{order}(\mathrm{NF}(s\ |G))$, we get

$$\mathrm{order}(\mathrm{NF}(S(f,h)|\{f,g,h\})) \ge \mathrm{order}(S(f,h)) \ge max\{\mathrm{order}(f), \mathrm{order}(h)\} \ge n+2.$$

In the same way we can also prove that $\mathrm{order}(\mathrm{NF}(S(g,h)|\{f,g,h\})) \ge n+2$. Now recall that, accordingly to the Buchberger algorithm, in order to determine a standard basis of $\mathrm{Lt}_{\overline{\tau}}(I)$, one has to compute $\mathrm{NF}(S(f,h)|\{f,g,h\})$, $\mathrm{NF}(S(g,h)|\{f,g,h\})$, to add those of them which are not zero to the list and go on in this way up to the end. At each step of this procedure the order of the elements can only increase; hence if m has degree $\ge n+2$ then $\mathrm{NF}(S(f,h)|\{f,g,h\})$ and $\mathrm{NF}(S(g,h)|\{f,g,h\})$ have degree at least $n+2$ and we cannot obtain, as Lemma 3.5 requires, the monomial xz^n which has degree $n+1$. This proves the claim. By Lemma 3.5 the claim implies that $m = \mathrm{Lt}_{\overline{\tau}}(h) = \mathrm{Lt}_\tau(h^*) = xz^n$.

We want now to compute $\mathrm{NF}(S(f,g) \mid \{f,g\})$. First it is clear that we can write $G(y,z) = yH(y,z) + \alpha z^c$ with $\alpha = 0$ or invertible in $K[[z]]$ and $c \ge 0$. Hence $I = (f,g)$ where

$$f = x^2 + axz^p + F(y,z), \quad g = xy + yH(y,z) + \alpha z^c$$

with $c \ge 0$, $p \ge 1$ and a and α either zero or invertible in $K[[z]]$. We have

$$S(f,g) = yf - xg = axyz^p + yF(y,z) - xyH(y,z) - \alpha xz^c = g(az^p - H(y,z)) - \alpha xz^c + M(y,z)$$

where $M(y,z) = yF(y,z) - (yH(y,z) + \alpha z^c)(az^p - H(y,z)) \in K[[y,z]]$.

We claim that $\mathrm{NF}(S(f,g) \mid \{f,g\}) = M(y,z) - \alpha xz^c$. Namely we have

$$S(f,g) = 0 \cdot f + (az^p - H(y,z))g + M(y,z) - \alpha xz^c$$

and we need to prove:

a) no monomial in the support of $M(y,z) - \alpha xz^c$ is divisible by x^2 or xy

b) $\mathrm{Lt}_{\overline{\tau}}(g(az^p - H(y,z))) \le \mathrm{Lt}_{\overline{\tau}}(S(f,g))$.

Now a) is true because α is zero or invertible in $K[[z]]$. As for b) it is clear that we have $\mathrm{Lt}_{\overline{\tau}}(g(az^p - H(y,z))) = xy \cdot \mathrm{Lt}_{\overline{\tau}}(az^p - H(y,z))$. This monomial is not in the support of $M(y,z) - \alpha xz^c$, hence it is in the support of $S(f,g)$. This implies b) and the claim

$$h = \mathrm{NF}(S(f,g) \mid \{f,g\}) = M(y,z) - \alpha xz^c \tag{3}$$

is proved.
Since $\mathrm{Lt}_{\overline{\tau}}(h) = xz^n$ it follows $\alpha \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $c = n$, and $\mathrm{order}(M) \geq n+1$. In particular we deduce

$$g = xy + \alpha z^n + yH(y,z). \tag{4}$$

Let $J := I + (y) = (x^2 + axz^p + F(0,z), z^n, y)$; it is clear that $\mathrm{Lt}_{\overline{\tau}}(J) \supseteq (x^2, z^n, y)$. Since R/J is Artinian, $\overline{y}$ is a parameter in $A = R/I$; hence

$$e = e(R/I) \leq \mathit{length}(R/J) = \mathit{length}(R/\mathrm{Lt}_{\overline{\tau}}(J) \leq \mathit{length}(R/(x^2, z^n, y)) = 2n.$$

The conclusion follows. $\square$

In example 2.3, we have seen that however we fix an integer $e \geq 4$, there is a local ring of type $(2,2)$ with multiplicity e and strictly increasing Hilbert function. For each pair of integers (n,e) such that $n \geq 3$ and $n+3 \leq e \leq 2n$, we exhibit now local rings of type $(2,2)$ and multiplicity e whose Hilbert function has a flat in position n.

Example 3.7. Given the integers n and e such that $n \geq 3$, $n+3 \leq e \leq 2n$, the ideal

$$I = (x^2 - y^{e-2}, xy - z^n)$$

is a complete intersection ideal of $R = \mathrm{K}[\![x,y,z]\!]$ of type $(2,2)$ with multiplicity e, whose Hilbert function has a flat in position n.

Proof. Let us consider a monomial ordering τ such that $x > y > z$; we are going to prove that

$$\{f = x^2 - y^{e-2}, \quad g = xy - z^n, \quad h = -y^{e-1} + xz^n, \quad k = y^e - z^{2n}\}$$

is a standard basis for I. Namely, if this is the case, we get $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, xz^n, y^e)$ and from this an easy computation shows that the local ring $K[x,y,z]]/(x^2 - y^{e-2}, xy - z^n)$has multiplicity e and Hilbert function with a flat in position n.

We have:

$$S(f,g) = yf - xg = y(x^2 - y^{e-2}) - x(xy - z^n) = xz^n - y^{e-1}$$

and since $e \geq n+3$ implies $e-1 \geq n+2 > n+1$, we get $\mathrm{Lt}_{\overline{\tau}}(S(f,g)) = xz^n$.

We let

$$h := S(f,g) = xz^n - y^{e-1}.$$

Now

$$S(f,h) = z^n f - xh = z^n(x^2 - y^{e-2}) - x(xz^n - y^{e-1}) = xy^{e-1} - z^n y^{e-2} = y^{e-2} g$$

so that $\mathrm{Lt}_{\overline{\tau}}(S(f,h)) = y^{e-2}\mathrm{Lt}_{\overline{\tau}}(g) = xy^{e-1}$.

Further

$$S(g,h) = z^n g - yh = z^n(xy - z^n) - y(xz^n - y^{e-1}) = y^e - z^{2n}$$

and since $e \leq 2n$ and $y > z$, we have $\mathrm{Lt}_{\overline{\tau}}(S(g,h)) = y^e$.

We let

$$k := S(g,h) = y^e - z^{2n}$$

with $\mathrm{Lt}_{\overline{\tau}}(S(g,h)) = \mathrm{Lt}_{\overline{\tau}}(k) = y^e$.

Now

$$S(f,k) = y^e f - x^2 k = y^e(x^2 - y^{e-2}) - x^2(y^2 - z^{2n}) = x^2 z^{2n} - y^{2e-2} = z^{2n} f - y^{e-2} k$$

and since $2e - 2 \geq 2(n+3) - 2 = 2n + 4 > 2n + 2$, we have $\mathrm{Lt}_{\overline{\tau}}(S(f,k)) = x^2 z^{2n}$.

Also

$$S(g,k) = y^{e-1} g - xk = y^{e-1}(xy - z^n) - x(y^2 - z^{2n}) = xz^{2n} - y^{e-1} z^n = z^n h$$

so that $\mathrm{Lt}_{\overline{\tau}}(S(g,k)) = z^n \mathrm{Lt}_{\overline{\tau}}(h) = xz^{2n}$.

Finally

$$S(h,k) = y^2 h - xz^n k = y^e(xz^n - y^{e-1}) - xz^n(y^e - z^{2n}) = xz^{3n} - y^{2e-1} = z^{2n} h - y^{e-1} k.$$

Here we can only remark that $\mathrm{Lt}_{\overline{\tau}}(S(h,k)) = \max\{xz^{3n}, y^{2e-1}\}$.

From these computations we get

$$\mathrm{NF}(S(f,g) \mid \{h\}) = \mathrm{NF}(h \mid \{h\}) = 0$$

$$\mathrm{NF}(S(f,h) \mid \{g\}) = \mathrm{NF}(y^{e-2} g \mid \{g\}) = 0$$

$$\mathrm{NF}(S(g,h) \mid \{k\}) = \mathrm{NF}(k \mid \{k\}) = 0$$

$$\mathrm{NF}(S(f,k) \mid \{f,k\}) = \mathrm{NF}(z^{2n} f - y^{e-2} k \mid \{f,k\}) = 0$$

because $\mathrm{Lt}_{\overline{\tau}}(z^{2n} f - y^{e-2} k) = x^2 z^{2n} \geq \mathrm{Lt}_{\overline{\tau}}(z^{2n} f) = x^2 z^{2n}, \mathrm{Lt}_{\overline{\tau}}(y^{e-2} k) = y^{2e-2}$.

$$\mathrm{NF}(S(g,k) \mid \{h\}) = \mathrm{NF}(z^n h \mid \{h\}) = 0$$

$$\mathrm{NF}(S(h,k) \mid \{h,k\}) = \mathrm{NF}(z^{2n} h - y^{e-1} k \mid \{h,k\}) = 0$$

because $\mathrm{Lt}_{\overline{\tau}}(z^{2n} h - y^{e-1} k) = \max\{xz^{3n}, y^{2e-1}\} \geq \mathrm{Lt}_{\overline{\tau}}(z^{2n} h) = xz^{3n}, \mathrm{Lt}_{\overline{\tau}}(y^{e-1} k) = y^{2e-1}$.

By Buchberger's criterion the conclusion follows. $\square$

We prove now that if I_2^* is square-free then the Hilbert function is strictly increasing, so that the associated graded ring is Cohen-Macaulay.

Proposition 3.8. *Let $A = R/I$ be a local ring of type $(2,2)$. If ${I^*}_2$ is square-free, then the Hilbert function of A is strictly increasing and thus the associated graded ring $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay.*

Proof. From Proposition 3.4 (ii) we may assume, up to isomorphism of R, that

$$I = (x^2 + xz + F(y,z), xy + byz + \alpha y^r + \beta z^s)$$

where $F \in \mathrm{K}[\![y,z]\!]_{\geq 3}$, $b \in \mathrm{K}[\![y,z]\!]$ with $b(0,0) = 1$, $r, s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$.

If $HF_{R/I}(n+2) = n+2$, then $e = n+2$ and we conclude by Proposition 2.5. Assume that $HF_{R/I}(n+2) = n+3$, then by Lemma 3.5 we have that $xz^n \in \mathrm{Lt}_{\overline{\tau}}(I)$. By Burchberger's criterion we should have $xz^n = \mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g), \{f,g\}))$. The S-polynomial of the pair f, g is

$$h := S(f,g) = z(b-1)A + \alpha y^{r-1} A + yF - \beta xz^s,$$

$A = byz + \alpha y^r + \beta z^s$. We write $L = z(b-1)A + \alpha y^{r-1} A + yF$; notice that $L \in \mathrm{K}[\![y,z]\!]$ and $\beta \in \mathrm{K}[\![z]\!]$ so $h = \mathrm{NF}(h, \{f,g\})$. Since $\mathrm{Lt}_{\overline{\tau}}(h) = xz^n$ we deduce $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$, $s = n$, and $\mathrm{order}(L) \geq n+1$.
Let now consider

$$\begin{aligned} S(h,g) = W &= \beta z^n g + yh \\ &= \beta byz^{n+1} + \alpha\beta y^r z^n + \beta^2 z^{2n} + yL. \end{aligned}$$

Notice that since $b, \beta \neq 0$

$$\text{order}(\alpha\beta y^r z^n), \text{order}(\beta^2 z^{2n}) \geq n+3 > n+2 = \text{order}(\beta b y z^{n+1})$$

and $\text{Lt}_{\overline{\tau}}(\beta b y z^{n+1}) = yz^{n+1}$.

Recall that $\text{order}(L) \geq n+1$, so in order to prove that $\text{Lt}_{\overline{\tau}}(W) = yz^{n+1}$ we should prove that in $\text{Supp}(yL)$ there is not the monomial yz^{n+1}. This is equivalent to prove that in $\text{Supp}(L)$ there is not the monomial z^{n+1}. To this end we set $y = 0$ in L and we get

$$L(0,z) = (b(0,z) - 1)\beta z^{n+1}.$$

recall that $b(0,0) = 1$ so $\text{order}(L(0,z)) \geq n+2$. Hence we have that $\text{Lt}_{\overline{\tau}}(k) = yz^{n+1}$.

Let us consider now the monomial ideal $J = (x^2, xy, xz^n, yz^{n+1}) \subset \text{Lt}_{\overline{\tau}}(I)$. We have

$$HF_{R/I}(n+2) \leq HF_{R/J}(n+2) = n+2,$$

a contradiction. $\square$

Notice that if I_2^* is square-free then by Lemma 3.2 (ii) we may assume, up to isomorphisms, that $f^* = xy, g^* = xz$. Hence Proposition 3.8 is only a very special case of [9, Corollary 4.6] where the same result is proved for every codimension two complete intersecton in $K[[x_1, \ldots, x_n]]$.

The following example shows that the converse of the above theorem does not hold. Let $I = (x^2 - y^2 z, xy - y^3) \subseteq \mathrm{K}[\![x,y,z]\!]$. It is clear that $x^2 \in I_2^*$ so that I_2^* is not square-free. It is easy to see that the Hilbert function of $A = \mathrm{K}[\![x,y,z]\!]/I$ is strictly increasing, namely is $\{1,3,4,5,5,5,5,.......\}$. By Proposition 2.5 the associated graded ring of A is Cohen-Macaulay.

We close this section by describing the possible minimal free resolutions of the associated graded ring of a local ring of type $(2,2)$.

We have seen in (2) that the Hilbert function of a local ring A of type $(2,2)$ has the following shape

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-1, \\ e & t \geq e. \end{cases} \tag{5}$$

where n is the least integer such that $HF_A(n) = HF_A(n+1)$. We have $3 \leq n \leq e-2$ and it is easy to see that the lex-segment ideal with the above Hilbert function is the following ideal $L := (x^2, xy, xz^n, y^e)$. We can compute the minimal free resolution of P/L by using the well known formula of Eliaouh and Kervaire. We get

$$\begin{aligned} 0 \to P(-n-3) \to P(-3) \oplus P(-n-2)^2 \oplus P(-e-1) \to \\ \to P(-2)^2 \oplus P(-n-1) \oplus P(-e) \to P \to P/L \to 0. \end{aligned}$$

Now we recall that,

• The graded Betti numbers of every homogeneous ideal in the polynomial ring can be obtained from those of the corresponding lex-segment ideal by a sequence of consecutive cancellations.

•• If a given homogeneous ideal and its corresponding lex-segment share the same first Betti numbers (which means that the ideal and the lex-segment are minimally generated in the same degrees) then they share all the Betti numbers.

As a consequence it is easy to see that, in the case $e \geq n+3$, the resolution of a homogeneous ideal $J \subseteq P = K[x,y,z]$ with the above Hilbert Function is the following:

$$\begin{aligned} 0 \to P(-n-3) \to P(-3) \oplus P(-n-2)^2 \oplus P(-e-1) \to \\ \to P(-2)^2 \oplus P(-n-1) \oplus P(-e) \to P \to P/J \to 0 \end{aligned}$$

In the case $e = n+2$ we have instead two possibilities:
either

$$0 \to P(-n-3) \to P(-3) \oplus P(-n-2) \oplus P(-n-3) \to P(-2)^2 \oplus P(-n-1) \to P \to P/J \to 0$$

or

$$0 \to P(-3) \oplus P(-n-2) \to P(-2)^2 \oplus P(-n-1) \to P \to P/J \to 0.$$

It is clear that if P/J is Cohen-Macaulay only the last shorter resolution is available.

We apply this to the associated graded ring of a local ring of type (2,2) and we get the following result.

Proposition 3.9. *Let A be a local ring of type (2,2), e the multiplicity of A and let n be an integer such that $n \le e-2$. If $e = n+2$, then $gr_{\mathfrak{m}}(A)$ is Cohen-Macaulay with minimal free resolution*

$$0 \to P(-3) \oplus P(-n-2) \to P(-2)^2 \oplus P(-n-1) \to P \to gr_{\mathfrak{m}}(A) \to 0.$$

If $e \ge n+3$, then $e \le 2n$ and the associated graded ring $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay with minimal free resolution

$$\begin{gathered} 0 \to P(-n-3) \to P(-3) \oplus P(-n-2)^2 \oplus P(-e-1) \to \\ \to P(-2)^2 \oplus P(-n-1) \oplus P(-e) \to P \to gr_{\mathfrak{m}}(A) \to 0. \end{gathered}$$

Proof. It is enough to remark that by Proposition 2.5 the associated graded ring of a local ring of type (2,2) is Cohen-Macaulay when $e = n+2$. □

4 A structure theorem for quadratic complete intersections of codimension two

The aim of this section is to give a structure, up to analytic isomorphism, of complete intersection ideals I of type $(2,2)$ such that $A = \mathrm{K}[\![x,y,z]\!]/I$ is of multiplicity e. This is a first step towards the problem of the analytic classification of the ideals of type $(2,2)$. In this direction we exhibit in Example **??** two ideals of type $(2,2)$ with the same Hilbert function that are not analytically isomorphic. According to Proposition 2.2, Example 2.3, and Example 3.7 the Hilbert function of A takes one of the following shapes

$$H(e)(t) := \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, e-3, \\ e & t \ge e-2 \end{cases} \tag{6}$$

or

$$H(n,e)(t) := \begin{cases} 1 & t = 0, \\ t+2 & t = 1, \cdots, n, \\ t+1 & t = n+1, \cdots, e-2, \\ e & t \ge e-1. \end{cases} \tag{7}$$

Theorem 4.1. *Let A be a local ring of type $(2,2)$ and multiplicity e. The following conditions are equivalent:*
(i) $HF_A = H(n,e)$ for some integer $n \ge 3$.
(ii) Up to analytic isomorphism, I is generated in $R = \mathrm{K}[\![x,y,z]\!]$ by:

$$\begin{aligned} f &= x^2 + az^p(x+W) - W^2 + L \\ g &= xy + \alpha z^n + yW \end{aligned}$$

where

- $a \in \{0,1\}$, $p \geq 2$, $\alpha \in \mathbb{U}(\mathrm{K}[\![z]\!])$,
- $W, L \in \mathrm{K}[\![y,z]\!]$ *with* $\mathrm{order}(L) \geq n+1$, $\mathrm{order}(W) \geq 2$,
- $n+3 \leq e \leq 2n$,
- $\mathrm{order}(2\alpha z^n W - a\alpha z^{n+p} + yL) \geq e-1$ *and there is equality when* $e < 2n$.

Proof. The proof uses Proposition 3.4 and the computation of $\mathrm{Lt}_{\overline{\tau}}(I)$ according to the Buchberger criterion. As usual assume $x > y, x > z$.
First we prove (i) implies (ii). Since $HF_A = H(n,e)$, then by Proposition 2.5 $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay and, by Theorem 3.6, $n+3 \leq e \leq 2n$. By Proposition 3.8, I_2^* contains a square of a linear form. Hence we may assume that $f^* = x^2$ and $g^* = xy$. Notice that $x^2, xy \in \mathrm{Lt}_{\overline{\tau}}(I)$, hence by the assumption (i) and Lemma 3.5, we have $\mathrm{Lt}_{\overline{\tau}}(I) \supseteq (x^2, xy, xz^n)$. From the particular shape of the Hilbert function it is easy to see that $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, xz^n, m)$ where m is a monomial in $K[y,z]_e$. From Lemma 3.4 we may also assume that

$$\begin{aligned} f &= x^2 + axz^p + F(y,z), \\ g &= xy + G(y,z), \end{aligned}$$

where $a \in \{0,1\}$, $p \geq 2$, $F, G \in \mathrm{K}[\![y,z]\!]$ with both $\mathrm{order}(F), \mathrm{order}(G) \geq 3$. Moreover, from the equation (4) of the proof of Theorem 3.6, (4), we get

$$G(y,z) = yH(y,z) + \alpha z^n$$

where $W \in \mathrm{K}[\![y,z]\!]_{\geq 2}$ and $\alpha \in \mathbb{U}(\mathrm{K}[\![z]\!])$. We recall that $S(f,g) = yf - xg = axyz^p + yF - \alpha xz^n - xyW$. In particular a standard computation gives

$$h := \mathrm{NF}(S(f,g), \{f,g\}) = -\alpha xz^n + yL + \alpha z^n(W - az^p)$$

where $L = F - az^pW + W^2$. Notice that $\mathrm{order}(\alpha z^n(W - az^p)) \geq n+2$. Notice that $xz^n = \mathrm{Lt}_{\overline{\tau}}(h)$.

A simple calculation shows that $\mathrm{NF}(S(h,f), \{h,f,g\}) = 0$. On the other hand

$$S(h,g) = \mathrm{NF}(S(h,g), \{h,f,g\}) = \alpha^2 z^{2n} + y(2\alpha z^n W - a\alpha z^{n+p} + yL) \neq 0$$

because $\alpha \neq 0$ and z^{2n} do not appear in the support of the remaining part. As a consequence $m = \mathrm{Lt}_{\overline{\tau}}(S(h,g))$, and hence $\mathrm{order}(S(h,g)) = e$. It follows that $\mathrm{order}(2\alpha z^n W - a\alpha z^{n+p} + yL) \geq e-1$. In particular $\mathrm{order}(L) \geq n+1$, and $\mathrm{order}(2\alpha z^n W - a\alpha z^{n+p} + yL) = e-1$ if $e < 2n$,.

Conversely, assuming (ii), it is enough to apply Buchberger's criterion for computing $\mathrm{Lt}_{\overline{\tau}}(I)$. By following the previous computations we get $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, xz^n, m)$ where $m = \mathrm{Lt}_{\overline{\tau}}(\alpha^2 z^{2n} + y(2\alpha z^n W - a\alpha z^{n+p} + yL))$ and (i) follows. $\square$

If the Hilbert function is increasing, i.e. of type $H(e)$, we present a structure's theorem under the assumption that I^* does not contain the square of a linear form.

Theorem 4.2. *Let A be a local ring of type $(2,2)$ and multiplicity e. The following conditions are equivalent:*
(i) $HF_A = H(e)$ and I^* does not contain the square of a linear form
(ii) Up to analytic isomorphism, I is generated in $R = \mathrm{K}[\![x,y,z]\!]$ by:

$$\begin{aligned} f &= x^2 + xz + F \\ g &= xy + dyz + \alpha y^r + \beta z^s \end{aligned}$$

where

- $r \geq 3$,

- $F \in \mathrm{K}[\![y,z]\!]$ *and* $\mathrm{order}(F) \geq 3$,
- $d \in \mathbb{U}(\mathrm{K}[\![y,z]\!])$, *with* $d(0,0)=1$,
- $\alpha = 0$ *or* $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, $\beta = 0$ *or* $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$ *and* $s \geq e-1 \geq 3$,
- $\mathrm{order}(F + d(d-1)z^2 + \alpha(2d-1)zy^{r-1} + \alpha^2 y^{2(r-1)}) = e-2$.

Proof. As usual, consider a monomial ordering τ with $x > y, x > z$. We prove (i) implies (ii). By Theorem 3.4 (ii), we may assume that

$$I = (x^2 + xz + F(y,z), xy + dyz + \alpha y^r + \beta z^s)$$

$F \in \mathrm{K}[\![y,z]\!]_{\geq 3}$, $d \in \mathrm{K}[\![y,z]\!]$ with $d(0,0)=1$, $r,s \geq 3$, $\alpha = 0$ or $\alpha \in \mathbb{U}(\mathrm{K}[\![y]\!])$, and $\beta = 0$ or $\beta \in \mathbb{U}(\mathrm{K}[\![z]\!])$. Since the Hilbert function is increasing up to $n = e-2 \geq 2$ and $HF_{R/I}(t) = e$ for all $t \geq e-2$, then $\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, m)$ where $m \in K[y,z]$ is a monomial of degree $e-1$.

By Buchberger's criterion $m = \mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g),\{f,g\}))$. Now

$$S(f,g) = -\beta x z^s + yF(y,z) + xy[(1-d)z - \alpha y^{r-1}]$$

After a computation we get

$$\mathrm{NF}(S(f,g),\{f,g\}) = -\beta x z^s + yW + \alpha\beta y^{r-1}z^s + \beta(d-1)z^{s+1}$$

where $W = F + d(d-1)z^2 + \alpha(2d-1)zy^{r-1} + \alpha^2 y^{2(r-1)}$. Since $yW \in \mathrm{K}[\![y,z]\!]$, $r \geq 3$ and $1 - d \in (y,z)\mathrm{K}[\![y,z]\!]$ we get that if $\beta \neq 0$, then xz^s appears in the support of $\mathrm{NF}(S(f,g),\{f,g\})$. Since $\mathrm{Lt}_{\overline{\tau}}(\mathrm{NF}(S(f,g),\{f,g\})) \in K[y,z]_{e-1}$, it follows that $\mathrm{order}(W) = e-2$ and, if $\beta \neq 0$, then $s \geq e-1$.

Conversely if we assume (ii), then it is easy to see that I^* does not contain the square of a linear form because $I_2^* = (x^2 + xz, xy + yz)$ which is reduced. Moreover by repeating Buchberger's algorithm and imitating the previous computation on $S(f,g)$, we get

$$\mathrm{Lt}_{\overline{\tau}}(I) = (x^2, xy, y\mathrm{Lt}_{\overline{\tau}}(W)),$$

hence $HF_A = H(e)$. □

5 Examples

The aim of this section is to present examples supporting the results of the previous sections or with the goal of detecting the possible extensions to the non quadratic case. All computations are performed by using the CoCoA system ([4]). Here $HS_A(\theta)$ denotes the Hilbert series of A, that is $HS_A(\theta) = \sum_{t\geq 0} HF_A(t)\theta^t$.

We have seen in Proposition 3.9 that the minimal free resolution of the tangent cone of a local ring of type $(2,2)$ cannot have any cancellation, both in the case the Hilbert function is strictly increasing and in the case of a flat. One can ask if this is the case also for local rings of type (a,b) with $3 \leq a \leq b$.

The first two examples that we propose show that the answer is negative.

Example 5.1. Let $A = R/I$ where $I = (x^3, z^5 + xz^3 + x^2y)$. The local ring A has type $(3,3)$ and $I^* = (x^3, x^2y, x^2z^3, -xyz^5 + xz^6, -xz^7, z^{10})$. The resolution of P/I^* is the following

$$0 \to P(-7) \oplus P(-10) \to P(-4) \oplus P^2(-6) \oplus P(-8) \oplus P^2(-9) \oplus P(-11) \to$$
$$\to P^2(-3) \oplus P(-5) \oplus P(-7) \oplus P(-8) \oplus P(-10) \to P \to P/I^* \to 0.$$

hence we can cancel the shift -8 to get the resolution

$$0 \to P(-7) \oplus P(-10) \to P(-4) \oplus P^2(-6) \oplus P^2(-9) \oplus P(-11) \to$$

$$\to P^2(-3) \oplus P(-5) \oplus P(-7) \oplus P(-10) \to P \to P/I^* \to 0.$$

The Hilbert function

$$\{1, 3, 6, 8, 10, 11, 13, 14, 14, 15, 15, ...\}$$

has a flat in degre 7.

Example 5.2. Let $A = R/I$ where $I = (x^4, z^4 + x^2y)$. The local ring A has type $(3, 4)$ and $I^* = (x^2y, x^4, x^2z^4, z^8)$. The resolution of P/I^* is the following

$$0 \to P(-9) \to P(-5) \oplus P(-7) \oplus P(-8) \oplus P(-10) \to$$

$$\to P(-3) \oplus P(-4) \oplus P(-6) \oplus P(-8) \to P \to P/I^* \to 0.$$

It is clear that we can cancel the shift -8 to get the resolution

$$0 \to P(-9) \to P(-5) \oplus P(-7) \oplus P(-10) \to$$

$$\to P(-3) \oplus P(-4) \oplus P(-6) \to P \to P/I^* \to 0.$$

The Hilbert function

$$\{1, 3, 6, 9, 11, 13, 14, 15, 16, 16, 16, ...\}$$

is strictly increasing.

The following example shows that Proposition 2.5 cannot be extended to local rings of type (a, b) with $a > 2$.

Example 5.3. Let us consider the ideal $I = (x^4, x^2y + z^4) \subseteq R = k[[x, y, z]]$. The Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 3\theta^3 + 2\theta^4 + 2\theta^5 + \theta^6 + \theta^7 + \theta^8)/(1 - \theta).$$

Hence $A = R/I$ has strictly increasing Hilbert function. Nevertheless $I^* = (x^2y, x^4, x^2z^4, z^8)$, so that $gr_{\mathfrak{m}}(A)$ is not Cohen-Macaulay.

The following example, due to T. Shibuta, shows that the Hilbert function of a one-dimensional local domain of type $(2, b)$ can have $b - 1$ flats, the maximum number according to Proposition 1.

Example 5.4. (see [9], example 5.5) Let $b \geq 2$ be an integer. Consider the family of semigroup rings

$$A = k[[t^{3b}, t^{3b+1}, t^{6b+3}]].$$

It is easy to see that $A = k[[x, y, z]]/I_b$ where $I_b = (xz - y^3, z^b - x^{2b+1})$. Thus A is a one-dimensional local domain of type $(2, b)$. For every $b \geq 2$ the Hilbert function of A has $b - 1$ flats. Namely

$$HF_A(t) = \begin{cases} 1 & t = 0, \\ 2t + 2 & t = 1, \cdots, b - 1, \\ 2b & t = b, \\ 2b + 1 & t = b + 1, \\ 2b + k & t = b + 2k, \quad k = 1, \cdots, b - 1, \\ 2b + k + 1 & t = b + 2k + 1, \quad k = 1, \cdots, b - 1, \\ 3b & t \geq 3b - 1. \end{cases} \tag{8}$$

In the above example the Hilbert function of the local ring of type $(2, b)$ presents $b - 1$ flats which are not consecutive. The following example shows that we can also have $b - 1$ consecutive flats, that is a strip like this: $HF(n) = HF(n + 1) = \cdots = HF(n + b - 1) < e$.

Example 5.5. Let us consider the ideal $I = (x^2, xy^2 + z^5 + xy^3z^2) \subseteq R = k[[x, y, z]]$. Then $A = R/I$ is of type $(2, 3)$ and its Hilbert function presents $2 = b - 1$ flats which are consecutive: namely we have $HF(5) = HF(6) = HF(7) = 8 < e = 10$. In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 2\theta^2 + \theta^3 + \theta^4 + \theta^5 + \theta^8 + \theta^9)/(1 - \theta).$$

We are far from being able to determine the Hilbert functions sequences possible for a one dimensional local ring which is a complete intersection of type (a, b), with $3 \leq a \leq b$,. In order to illustrate the difficulties, we give two more examples, the first of type $(3, 3)$ with one very large platform consisting of 13 consecutive flats, the second of type $(4, 4)$ with nine flats and three platforms.

Example 5.6. Let $I = (x^3 - zy^{14}, x^2y + xz^7) \subseteq R = k[[x, y, z]]$. The local ring $A = R/I$ is of type $(3, 3)$ and

$$HF_A(15) = HF_A(16) = \ldots \cdots = HF_A(29) = 31 < e = 32.$$

In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 2\theta^3 + 2\theta^4 + 2\theta^5 + 2\theta^6 + 2\theta^7 + 2\theta^8 + \theta^9 + 2\theta^{10} + 2\theta^{11} + 2\theta^{12} +$$

$$+ 2\theta^{13} + 2\theta^{14} + \theta^{15} + \theta^{16} + \theta^{30} + \theta^{31})/(1 - \theta)$$

and $I^* = (x^3, x^2y, x^2z^7, xz^{14}, xy^{15}z, y^{31}z)$.

Example 5.7. Let $I = (x^4, xy^3 - z^6) \subseteq R = k[[x, y, z]]$. The local ring $A = R/I$ is of type $(4, 4)$ and

$$HF_A(8) = HF_A(9) = HF_A(10) = HF_A(11) = 18;$$

$$HF_A(13) = HF_A(14) = HF_A(15) = HF_A(16) = 20;$$

$$HF_A(18) = HF_A(19) = HF_A(20) = HF_A(21) = 22 < e = 24;$$

In particular the Hilbert series is:

$$HS_A(\theta) = (1 + 2\theta + 3\theta^2 + 4\theta^3 + 3\theta^4 + 2\theta^5 + \theta^6 + \theta^7 + \theta^8 + \theta^{12} + \theta^{13} + \theta^{17} + \theta^{18} + \theta^{22} + \theta^{23})/(1 - \theta)$$

and $I^* = (xy^3, x^4, x^3z^6, x^2z^{12}, xz^{18}, z^{24})$.

It would be very interesting to describe the isomorphism classes of local rings of type $(2, 2)$ which have the same given Hilbert function. But this is a difficult task, as the following examples show.

First we are given the Hilbert function $\{1, 3, 4, 5, 5, 6, 6, ...\}$ which has a flat in position 3 and multiplicity 6. The two ideals which we are going to prove that are not isomorphic are obtained one from the other with very little modifications, namely by adding a monomial to one of the two generators. Let us consider the Hilbert Function $\{1, 3, 4, 5, 5, 6, 6, ...\}$ which has a flat in position 3 and multiplicity 6. The two ideals which are not isomorphic but share the above Hilbert Function are obtained one from the other with very little modifications, namely by adding a monomial to one of the two generators.

Example 5.8. Let us consider the ideals

$$I = (x^2 - y^4, xy + z^3), \quad J = (x^2 + xz^2 - y^4, xy + z^3)$$

in $R = \mathrm{K}[\![x, y, z]\!]$.

They are of type $(2, 2)$, they have the same Hilbert Function $\{1, 3, 4, 5, 5, 6, 6, ...\}$ and the same leading ideal $\mathrm{Lt}_{\overline{\tau}}(I) = \mathrm{Lt}_{\overline{\tau}}(J) = (x^2, xy, xz^3, y^6)$. On the other hand the ideals of initial forms differ in degree 6:

$$I^* = (x^2, xy, xz^3, y^6 - z^6), \quad J^* = (x^2, xy, xz^3, y^6 + yz^5 - z^6).$$

We prove that $\mathrm{K}[\![x, y, z]\!]/I$ and $\mathrm{K}[\![x, y, z]\!]/J$ are not isomorphic.

If there exists an analytic isomorphism ϕ such that $\phi(I) = J$ then we can find power series f, g, h of order 1 such that $\mathcal{M} = (f, g, h)$ and ϕ is the result of substituting f for x, g for y and h for z in any power series of R. We have $f = L_1 + F \quad g = L_1 + G \quad h = L_3 + H$ where L_1, L_2, L_3 are linearly independent linear forms in $K[x, y, z]$ and F, G, H are power series of order ≥ 2.

We let for $i = 1, 2, 3$

$$L_i = \lambda_{i1}x + \lambda_{i2}y + \lambda_{i3}z$$

with $\lambda_{ij} \in \mathrm{K}$.

Since $x^2 - y^4 \in I$ we have $\phi(x^2 - y^4) = f^2 - g^4 \in J$, hence $L_1^2 \in J^*$. Since ${I^*}_2$ is the K-vector space ${I^*}_2 =< x^2, xy >$, we have

$$(\lambda_{11}x + \lambda_{12}y + \lambda_{13}z)^2 = px^2 + qxy$$

with $p, q \in \mathrm{K}$; this clearly implies $\lambda_{12} = \lambda_{13} = 0$.

In the same way, since $xy + z^3 \in I$, we have $\phi(xy + z^3) = fg + h^3 \in J$, hence $L_1L_2 \in J^*$. Thus we get

$$(\lambda_{11}x)(\lambda_{21}x + \lambda_{22}y + \lambda_{23}z) = rx^2 + sxy$$

with $r, s \in \mathrm{K}$. This implies $\lambda_{23} = 0$ because $\lambda_{11} \neq 0$.

Finally we have

$$y^6 - z^6 = -y^2(x^2 - y^4) + (xy + z^3)(xy - z^3) \in I$$

so that $\phi(y^6 - z^6) = g^6 - h^6 \in J$, and, as before, $L_2^6 - L_3^6 \in J^*$. Looking at the generators of the vector space ${J^*}_6$ we get as a consequence

$$(\lambda_{21}x + \lambda_{22}y)^6 - (\lambda_{31}x + \lambda_{32}y + \lambda_{33}z)^6 = Ax^2 + Bxy + Cxz^3 + D(y^6 + yz^5 - z^6)$$

where A, B, C, D are forms of degree $4, 4, 2, 0$ respectively in the polynomial ring $\mathrm{K}[x, y, z]$.

Since L_1, L_2, L_3 are linearly independent, we must have $\lambda_{33} \neq 0$. Hence, looking at the coefficient of the monomial y^5z in the above formula, we get $\lambda_{32} = 0$. But then, looking at the coefficient of the monomial yz^5, we certainly get $D = 0$ and finally, looking at the coefficient of the monomial z^6, we get $\lambda_{33} = 0$. This is a contradiction, so that the algebras R/I and R/J are not in the same isomorphism class.

The case when the Hilbert function is strictly increasing is not more easy to handle. Here we consider the Hilbert function $\{1, 3, 4, 5, 6, 6, 6, ...\}$ which is strictly increasing and we look at the possible isomorphism classes of local rings with that Hilbert function.

Example 5.9. Let us consider the two ideals

$$I := (x^2 + y^4, xy), \qquad J := (x^2 + y^4 + z^4, xy).$$

They have the same Hilbert function $\{1, 3, 4, 5, 6, 6, 6, ...\}$ and

$$I^* = (x^2, xy, y^5) \qquad J^* = (x^2, xy, y^5 + yz^4).$$

Here the tangent cones are not isomorphic, because in I^* there is the pure power y^5 while there are not pure powers of degree 5 in J^*. The calculation in itself shows that the original algebras cannot be isomorphic.

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Juan Elias
Departament d'Àlgebra i Geometria
Universitat de Barcelona
Gran Via 585, 08007 Barcelona, Spain
e-mail: `elias@ub.edu`

Maria Evelina Rossi
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35, 16146 Genova, Italy
e-mail: `rossim@dima.unige.it`

Giuseppe Valla
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35, 16146 Genova, Italy
e-mail: `valla@dima.unige.it`