

Analytic Isomorphisms of compressed local algebras ^{*}

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Abstract

In this paper we consider Artin local K -algebras with maximal length in the class of Artin algebras with given embedding dimension and socle type. They have been widely studied by several authors, among others by Iarrobino, Fröberg and Laksov. If the local K -algebra is Gorenstein of socle degree 3, then the authors proved that it is canonically graded, i.e. analytically isomorphic to its associated graded ring, see [6]. This unexpected result has been extended to compressed level K -algebras of socle degree 3 in [4]. In this paper we end the investigation proving that extremal Artin Gorenstein local K -algebras of socle degree $s \leq 4$ are canonically graded, but the result does not extend to extremal Artin Gorenstein local rings of socle degree 5 or to compressed level local rings of socle degree 4 and type > 1 . As a consequence we present results on Artin compressed local K -algebras having a specified socle type.

1 Compressed algebras and Inverse System

Let $A = R/I$ be an Artin local ring where $R = K[[x_1, \dots, x_n]]$ is the formal power series ring and K is an algebraically closed field of characteristic zero. We denote by $\mathfrak{m} = \mathcal{M}/I$ the maximal ideal of A where $\mathcal{M} = (x_1, \dots, x_n)$ and let $\text{Soc}(A) = 0 : \mathfrak{m}$ be the socle of A . Throughout this paper we denote by s the *socle degree* of A , that is the maximum integer j such that $\mathfrak{m}^j \neq 0$. The *type* of A is $t := \dim_K \text{Soc}(A)$.

Let A be of initial degree v , that is $I \subseteq \mathcal{M}^v \setminus \mathcal{M}^{v+1}$ and socle degree s . The *socle type* $E = E(A) = (0, \dots, e_{v-1}, e_v, \dots, e_s, 0, 0, \dots)$ of A is the sequence E of natural numbers where

$$e_i := \dim_K((0 : \mathfrak{m}) \cap \mathfrak{m}^i / (0 : \mathfrak{m}) \cap \mathfrak{m}^{i+1}).$$

Clearly $e_s > 0$ and $e_j = 0$ for $j > s$, but other conditions are necessary on E for being *permissible* for an Artin K -algebra of given initial degree v and socle degree s (see [10]). Essentially the conditions require room for generators of I of valuation v and assure that there are not generators of valuation $v - 1$.

A *level algebra* A of socle degree s (say also *s-level*), type t and embedding dimension n is an Artin quotient of R whose socle is concentrated in a single degree (i.e. $\text{Soc}(A) = \mathfrak{m}^s$) and satisfying $\dim_K \text{Soc}(A) = t$. Hence $e_j = 0$ for $j \neq s$ and $e_s = t$. The Artin algebra is Gorenstein if $t = 1$.

It make sense to consider maximum length Artin algebras of given socle type E and they can be characterized in terms of their Hilbert function. The Hilbert vector of A is denoted by $HF(A) = \{h_0, h_1, \dots, h_s\}$ where $h_i = h_i(A) = \dim_K \mathfrak{m}^i / \mathfrak{m}^{i+1}$ is the Hilbert function of A . By its definition, the Hilbert function of A coincides with the Hilbert function of the corresponding associated graded ring $gr_{\mathfrak{m}}(A) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$. We say that the Hilbert function $HF(A)$ is maximal in the class of Artin level algebras of given embedding dimension and socle type, if for each integer i , $h_i(A) \geq h_i(A')$ for any other Artin algebra A' in the same class. The existence of a maximal $HF(A)$ was shown for graded algebras by A. Iarrobino [10]. In the general case by Fröberg and Laksov [9], by Emsalem [8], by A. Iarrobino and the first author in [5] in the local case.

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Definition 1.1. An Artin algebra $A = R/I$ of socle type E is compressed if and only if it has maximal length $e(A) = \dim_K A$ among Artin quotients of R having socle type E and embedding dimension n .

The maximality of the Hilbert function characterizes compressed algebras as follows. If A is an Artin algebra (graded or local) of socle type E , it is known that for $i \geq 0$,

$$h_i(A) \leq \min\{\dim_K R_i, e_i \dim_K R_0 + e_{i+1} \dim_K R_1 + \cdots + e_s \dim_K R_{s-i}\}.$$

Accordingly with [10], Definition 2.4. B, we can rephrase the previous definition in terms of the Hilbert function.

Definition 1.2. A local (or graded) K -algebra A of socle degree s , socle type E and initial degree v is compressed if

$$h_i(A) = \begin{cases} \sum_{u=i}^s e_u (\dim_K R_{u-i}) & \text{if } i \geq v \\ \dim_K R_i & \text{otherwise} \end{cases}$$

In particular a s -level algebra A of type t and embedding dimension n is compressed if

$$h_i(A) = \min\left\{\binom{n+i-1}{i}, t \binom{n+s-i-1}{s-i}\right\}.$$

If $t = 1$ and the above equality holds then A is called an extremal Gorenstein algebra or also compressed Gorenstein algebra.

It is clear that compressed algebras impose several restrictive numerical conditions on the socle sequence E (see [10], Definition 2.2). For instance if v is the initial degree of A , then

$$e_{v-1} = \max\{0, \dim_K R_{v-1} - \sum_{u \geq v} (e_u \dim_K R_{u-(v-1)})\}. \quad (1)$$

If $s \geq 2(v-1)$, then it is easy to see that $e_{v-1} = 0$ because $\dim_K R_{s-(v-1)} \geq \dim_K R_{v-1}$. This is the case if A is Gorenstein.

Y.H. Cho and I. Iarrobino proved that the family of graded compressed level quotients of $P = K[x_1, \dots, x_n]$ is parameterized by an open dense subset of the Grassmanian of codimension c quotients of P_s and has dimension $c(\dim P_s - c)$ (see [3]). Boij studied the minimal free resolutions for compressed graded level algebras [1], Stanley and Hibi have studied connections of these algebras with combinatorics.

Accordingly with Emsalem [8, Pag. 408] we give the following definition.

Definition 1.3. A local algebra (A, \mathfrak{m}) is canonically graded if there exists a K -algebra isomorphism between A and its associated graded ring $gr_{\mathfrak{m}}(A)$.

Notice that, as a consequence, A is canonically graded if and only if A is analytically isomorphic to a standard graded K -algebra. Remark that a local K -algebra can be considered graded in a non-standard way, but not canonically graded. For instance, consider $A = K[[t^2, t^3]]$; in this case A is not analytically isomorphic to $gr_{\mathfrak{m}}(A) \cong K[x, y]/(y^2)$ because A is reduced and $gr_{\mathfrak{m}}(A)$ is not reduced, but we remark that $A \cong K[x, y]/(y^2 - x^3)$ is graded setting $deg(x) = 2$ and $deg(y) = 3$.

Actually it is very rare that a local ring is isomorphic to its associated graded ring. If this is the case, several can be the applications, see for example [2], [6], [7].

If A is an Artin local (or graded) ring, the dual module $V = Hom(A, K)$ can be identified with a maximal length submodule of the polynomial ring $P = K[y_1, \dots, y_n]$ which is closed under partial derivation. We recall the main facts and establish notation concerning the inverse system of Macaulay in

the study of Artin local rings (A, \mathfrak{m}) . The reader should refer to [8] and [11] for an extended treatment. It is known that P has an R -module structure under the following action

$$\begin{aligned} \circ : R \times P &\longrightarrow P \\ (f, g) &\longrightarrow f \circ g = f(\partial_{y_1}, \dots, \partial_{y_n})(g) \end{aligned}$$

where ∂_{y_i} denotes the partial derivative with respect to y_i . If we denote by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n}$ then

$$x^\alpha \circ y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} & \text{if } \beta_i \geq \alpha_i \text{ for } i = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

where $\frac{\beta!}{(\beta-\alpha)!} = \prod_{i=1}^n \frac{\beta_i!}{(\beta_i-\alpha_i)!}$.

If $S \subseteq P$, then its annihilator in R is $\text{Ann}_R(S) = \{f \in R : f \circ g = 0 \text{ for all } g \in S\}$. The R -submodule generated by S in P is closed under partial derivation, then $\text{Ann}_R(S)$ is an ideal of R . Conversely, for any ideal $I \subset R$ we define the following R -submodule of P :

$$I^\perp := \{g \in P \mid \langle f, g \rangle = 0 \ \forall f \in I\} = \{g \in P \mid I \circ g = 0\}.$$

Starting from an Artin local algebra R/I with socle-degree s , then I^\perp is generated by polynomials of degree $\leq s$.

If S is generated by a sequence $\underline{G} := G_1, \dots, G_t$ of polynomials of P , then we will write $\text{Ann}_R(\underline{G})$ and

$$A_{\underline{G}} = R/\text{Ann}_R(\underline{G})$$

where $\text{Ann}_R(\underline{G}) = \bigcap_{j=1}^t \text{Ann}_R(G_j)$. If $A_{\underline{G}}$ is compressed, then we may assume that each A_{G_j} is an extremal Gorenstein algebra. Notice that even if each A_{G_j} is canonically graded, then $A_{\underline{G}}$ is not necessarily canonically graded because it could not exist a uniform analytic isomorphism (see Example 3.5).

F. S. Macaulay in [12] proved that there exists a one-to-one correspondence between ideals $I \subseteq R$ such that R/I is an Artin local ring and R -submodules M of P which are finitely generated, see also Emsalem [8, Section B, Proposition 2] and Iarrobino [11, Lemma 1.2]. In this correspondence, Artin K -algebras $A = R/I$ of embedding dimension n , socle degree s and type t correspond to R -submodules of P generated by t polynomials G_1, \dots, G_t of degree d_i (depending on the socle type) such that the leading forms $G_1[d_1], \dots, G_t[d_t]$ of degree d_i (write $G_i = G_i[d_i] + \text{lower terms} \dots$) are linearly independent. If $A = A_{\underline{G}}$ where $\underline{G} := G_1, \dots, G_t$ is s -level local algebra of type t ($d_i = s$ for all $i = 1, \dots, t$), then the associated graded ring $gr_{\mathfrak{m}}(A)$ is not necessarily level and in general $\dim_K \text{Soc}(gr_{\mathfrak{m}}(A)) \geq t$. Actually the graded K -algebra

$$Q = P/\text{Ann}_P(\underline{G}[s])$$

(where $\underline{G}[s] := G_1[s], \dots, G_t[s]$) is a graded s -level algebra of type t and it is the unique quotient of $gr_{\mathfrak{m}}(A)$ with socle degree s and type t . In particular $gr_{\mathfrak{m}}(A)$ is level if and only if $gr_{\mathfrak{m}}(A) \simeq Q$ (see [4]).

The following result was proved in [10, Proposition 3.7 and Corollary 3.8].

Proposition 1.4. *The compressed local algebra A whose dual module is generated by G_1, \dots, G_t of degrees d_1, \dots, d_t has a compressed associated graded ring $gr_{\mathfrak{m}}(A)$ whose dual module is generated by $G_1[d_1], \dots, G_t[d_t]$, the leading forms of G_1, \dots, G_t . Conversely if $gr_{\mathfrak{m}}(A)$ is compressed, then A is compressed and $E(A) = E(gr_{\mathfrak{m}}(A))$.*

It is possible to compute the Hilbert function of $A = R/I$ via the inverse system. We define the following K -vector space:

$$(I^\perp)_i := \frac{I^\perp \cap P_{\leq i} + P_{< i}}{P_{< i}}. \quad (2)$$

Then it is known that

$$h_i(R/I) = \dim_K (I^\perp)_i. \quad (3)$$

Notice that in the graded setting $A = R/Ann(\underline{G})$, then

$$h_i(R/I) = \dim_K(I^\perp)_i = \dim_K\langle \partial_{s-i}\underline{G} \rangle \quad (4)$$

where ∂_{s-i} denotes the partial derivatives of order $s-i$. We will translate the above equality in terms of the rank of suitable matrices associated to \underline{G} .

Let $\Omega = \{\omega_i\}$ be the canonical basis of R/\mathcal{M}^{s+1} as a K -vector space consisting of the standard monomials x^α ordered by the deg-lex order with $x_1 > \dots > x_n$ and, then the dual basis with respect to the action \circ is the basis $\Omega^* = \{\omega_i^*\}$ of $P_{\leq j}$ where

$$(x^\alpha)^* = \frac{1}{\alpha!}y^\alpha,$$

in fact $\omega_i^*(\omega_j) = \langle \omega_j, \omega_i^* \rangle = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Given a form G of degree s and an integer $q \leq s$, we denote by $\Delta^q(G)$ the $\binom{n-1+s-q}{n-1} \times \binom{n-1+q}{n-1}$ matrix whose columns are the coordinates of $\partial_{\underline{i}}(G)$, $|\underline{i}| = q$, with respect $(x^L)^* = \frac{1}{L!}y^L$, $|L| = s-q$ (\underline{i} and L are n -uples of integers). We will denote by (L, \underline{i}) the corresponding position in the matrix $\Delta^q(G)$. In the following $L + \underline{i}$ denotes the sum in \mathbb{N}^n .

Proposition 1.5. *Let $G \in P = K[y_1, \dots, y_n]$ be a form of degree s . Then*

$$h_{s-i}(A_G) = \text{rank}(\Delta^i(G)) \leq \min\left\{\binom{n-1+s-i}{n-1}, \binom{n-1+i}{n-1}\right\}$$

for $i = 0, \dots, s$. The equality holds if and only if A_G is compressed.

Proof. It is straight consequence of (2) and (3), and the definition of the matrix $\Delta^i(G)$ of size $\binom{n-1+i}{n-1} \times \binom{n-1+s-i}{n-1}$. \square

Proposition 1.6. *Let $G = \sum_{|\underline{j}|=s} \beta_{\underline{j}} \frac{1}{\underline{j}!} y^{\underline{j}}$ be a degree s homogeneous polynomial. Given an integer q , the (L, \underline{i}) -component of $\Delta^q(G)$, $|L| = s-q$, $|\underline{i}| = q$, is*

$$\Delta^q(G)_{(L, \underline{i})} = \beta_{L+\underline{i}}$$

Proof. Let us consider the derivative

$$\partial_{\underline{i}}(G) = \sum_{|\underline{j}|=s, \underline{j} \geq \underline{i}} \beta_{\underline{j}} \frac{1}{(\underline{j}-\underline{i})!} y^{\underline{j}-\underline{i}}$$

$|\underline{i}| = q$, $\underline{j} \geq \underline{i}$ means that $j_w \geq i_w$ for $w = 1, \dots, n$. If we write $L = \underline{j} - \underline{i}$, then

$$\partial_{\underline{i}}(G) = \sum_{|L|=s-q} \beta_{L+\underline{i}} \frac{1}{L!} y^L$$

From this we get the claim. \square

It is easy to deduce the following corollary which gives via (4) an alternative proof of the fact that a graded Gorenstein algebra A_G has symmetric Hilbert function.

Corollary 1.7. *Let G be a form of P of degree s . Given an integer $i \leq s$, then*

$$\Delta^i(G) = {}^t \Delta^{s-i}(G)$$

where t denotes the transpose matrix.

With the previous notation, let $A_{\underline{G}}$ be a graded level algebra. We can define for all integer $i \leq s$,

$$\Delta^i(\underline{G}) = \begin{pmatrix} \frac{\Delta^i(G_1)}{\Delta^i(G_t)} \\ \vdots \\ \frac{\Delta^i(G_t)}{\Delta^i(G_t)} \end{pmatrix} \quad (5)$$

which is a $t \binom{n-1+s-i}{n-1} \times \binom{n-1+i}{n-1}$ matrix. We get the following result.

Proposition 1.8. *Let $A = A_{\underline{G}}$ be a compressed s -level local algebra of type t . Then for every $i = 1, \dots, s$*

$$h_i(A) = \text{rank}(\Delta^i(\underline{G}[s])) = \min\left\{\binom{n-1+i}{n-1}, t \binom{n-1+s-i}{n-1}\right\}.$$

Proof. By Proposition 1.4 we know that $gr_m(A)$ is level compressed of socle degree s and type t . Since $gr_m(A)$ is level if and only if $gr_m(A) \simeq Q = P/Ann(\underline{G}[s])$, the result follows by Proposition 1.5. \square

2 Automorphisms of Artin local algebras and associated matrices

Given a K -algebra C , we will denote by $Aut(C)$ the group of the automorphisms of C as a K -algebra and by $Aut_K(C)$ as a K -vector space. The automorphisms of R as a K -algebra are well known. They act as replacement of x_i by z_i , $i = 1, \dots, n$, such that $\mathcal{M} = (x_1, \dots, x_n) = (z_1, \dots, z_n)$. Actually, since $\mathcal{M}^{s+1} \subseteq I$, we are interested in the automorphisms of R/\mathcal{M}^{s+1} of K -algebras induced by the projection $\pi : R \rightarrow R/\mathcal{M}^{s+1}$. Clearly $Aut(R/\mathcal{M}^{s+1}) \subseteq Aut_K(R/\mathcal{M}^{s+1})$. For all $p \geq 1$, I_p denotes the identity matrix of order $\binom{n+p-1}{p}$.

For any $\varphi \in Aut_K(R/\mathcal{M}^{s+1})$ we may associate a matrix $M(\varphi)$ with respect to the basis Ω of size $r = \dim_K(R/\mathcal{M}^{s+1}) = \binom{n+s}{s}$. Given I and J ideals of R such that $\mathcal{M}^{s+1} \subset I, J$, there exists an isomorphism of K -algebras

$$\varphi : R/I \rightarrow R/J$$

if and only if φ is canonically induced by a K -algebra automorphism of R/\mathcal{M}^{s+1} sending I/\mathcal{M}^{s+1} to J/\mathcal{M}^{s+1} . In particular φ is an isomorphism of K -vector spaces. Passing to $Hom(\ , K)$, φ corresponds to

$$\varphi^* : J^\perp \rightarrow I^\perp$$

where ${}^tM(\varphi)$ is the matrix associated to φ^* with respect to the basis Ω^* of $P_{\leq s}$.

We denote by \mathcal{R} the subgroup of $Aut_K(P_{\leq s})$ (automorphisms of $P_{\leq s}$ as a K -vector space) represented by the matrices ${}^tM(\varphi)$ of $Gl_r(K)$ with $\varphi \in Aut(R/\mathcal{M}^{s+1})$.

By Emsalem, [8, Proposition 15], the classification, up to analytic isomorphism, of the Artin local K -algebras of multiplicity d , socle degree s and embedding dimension n is equivalent to the classification, up to the action of \mathcal{R} , of the K -vector subspaces of $P_{\leq s}$ of dimension d , stable by derivations and containing $P_{\leq 1} = K[y_1, \dots, y_n]_{\leq 1}$.

Our goal is to translate the study of analytic isomorphisms of Artin level K -algebras $A = R/I$ of socle s and type t in terms of the corresponding dual polynomials of degree s in P in an effective computational framework. This section is quite technical and the idea is the generalization of the method used in [6]. The main goal is Corollary 2.3 and, in view of it, the machinery could be presented assuming $s \leq 4$. Actually the presentation does not improve substantially and one loses the control of the general case that could be useful for further investigations.

Let $\underline{F} = F_1, \dots, F_t$, respectively $\underline{G} = G_1, \dots, G_t$, be polynomials of degree s . Let $\varphi \in Aut(R/\mathcal{M}^{s+1})$, from the previous facts we have

$$\varphi(A_{\underline{F}}) = A_{\underline{G}} \quad \text{if and only if} \quad (\varphi^*)^{-1}(\langle \underline{F} \rangle_R) = \langle \underline{G} \rangle_R. \quad (6)$$

If $F_i = b_{i1}w_1^* + \dots + b_{ir}w_r^* \in P_{\leq s}$, then we will denote the *row vector* of the coefficients of the polynomial with respect to the basis Ω^* by

$$[F_i]_{\Omega^*} = (b_{i1}, \dots, b_{ir}).$$

If there exists $\varphi \in \text{Aut}(R/\mathcal{M}^{s+1})$ such that

$$[G_i]_{\Omega^*} M(\varphi) = [F_i]_{\Omega^*}, \text{ for every } i = 1, \dots, t, \text{ then } \varphi(A_{\underline{F}}) = A_{\underline{G}} \quad (7)$$

Let s be a positive integer, the aim of this section is to provide a structure of the matrix $M(\varphi)$ associated to special K -algebra isomorphisms $\varphi \in \text{Aut}(R/\mathcal{M}^{s+1})$. Let φ_{s-p} be an automorphism of R/\mathcal{M}^{s+1} such that $\varphi_{s-p} = \text{Id}$ modulo \mathfrak{m}^{p+1} , with $1 \leq p \leq s$, that is

$$\varphi_{s-p}(x_j) = x_j + \sum_{|\underline{i}|=p+1} a_{\underline{i}}^j x^{\underline{i}} + \text{higher terms} \quad (8)$$

for $j = 1, \dots, n$ and $a_{\underline{i}}^j \in K$ for each n -uple \underline{i} such that $|\underline{i}| = p + 1$. In the following we will denote $\underline{a} := (a_{\underline{i}}^1, |\underline{i}| = p + 1; \dots; a_{\underline{i}}^n, |\underline{i}| = p + 1) \in K^{n \binom{n+p}{n-1}}$.

The matrix associated to φ_{s-p} , say $M(\varphi_{s-p})$, is an element of $Gl_r(K)$, $r = \binom{n+s}{s+1}$, with respect to the basis Ω of R/\mathcal{M}^{s+1} . We write $M(\varphi_{s-p}) = (B_{i,j})_{0 \leq i, j \leq s}$ where $B_{i,j}$ is a $\binom{n+i-1}{i} \times \binom{n+j-1}{j}$ matrix of the coefficients of monomials of degree i appearing in $\varphi(x^{\underline{j}})$ where $\underline{j} = (j_1, \dots, j_n)$ such that $|\underline{j}| = j$.

It is easy to verify that:

$$B_{i,j} = \begin{cases} 0, & 0 \leq i < j \leq s, \text{ or } j = 1, i = 1, \dots, s, \\ I_i, & i = j = 0, \dots, s, \\ 0, & j = s - p, \dots, s - 1, i = j + 1, \dots, s, \text{ and } (i, j) \neq (s, s - p). \end{cases}$$

The matrix $M(\varphi_{s-p})$ has the following structure

$$M(\varphi_{s-p}) = \left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \hline 0 & 0 & I_2 & 0 & 0 & 0 & 0 & \vdots \\ \hline \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & B_{p+1,1} & 0 & \dots & I_{s-p} & 0 & 0 & \vdots \\ \hline 0 & \dots & B_{p+2,2} & 0 & 0 & I_{s-p+1} & 0 & \vdots \\ \hline 0 & \dots & \dots & \ddots & \vdots & 0 & \ddots & 0 \\ \hline 0 & B_{s,1} & B_{s,2} & \dots & B_{s,s-p} & 0 & \dots & I_s \end{array} \right)$$

The entries of $B_{p+1,1}, B_{p+2,2}, \dots, B_{s,s-p}$ are linear forms in the variables $a_{\underline{i}}^j$, with $|\underline{i}| = p + 1, j = 1, \dots, n$. We are mainly interested in $B_{s,s-p}$ which is a $\binom{n+s-1}{s} \times \binom{n+s-p-1}{s-p}$ matrix whose columns correspond to x^W with $|W| = s - p$ and the rows correspond to the coefficients of x^L with $|L| = s$ in $\varphi(x^W)$. One has

$$\varphi(x^W) = x^W + \sum_{j=1}^n w_j \left(\sum_{|\underline{i}|=p+1} a_{\underline{i}}^j x^{W-\delta_j+\underline{i}} \right) + \dots$$

where $W = (w_1, \dots, w_n) \in \mathbb{N}^n$ with $|W| = s - p$, here δ_j is the n -uple with 0-entries but 1 in position j . We remark that $|W - \delta_j + \underline{i}| = s$. Then the entry of $B_{s,s-p}$ corresponding to the L row, $|L| = s$, and W column, $|W| = s - p$, is

$$(B_{s,s-p})_{L,W} = \sum_{W-\delta_j+\underline{i}=L} w_j a_{\underline{i}}^j. \quad (9)$$

Let F, G be polynomials of degree s of P and let φ_{s-p} be a K -algebra isomorphism of type (8) sending A_F to A_G . We denote by $F[j]$ (respectively $G[j]$) the homogeneous component of degree j of F (respectively of G), that is $F = F[s] + F[s-1] + \dots$ ($G = G[s] + G[s-1] + \dots$). By (7) we have

$$[G]_{\Omega^*} M(\varphi_{s-p}) = [F]_{\Omega^*}, \quad (10)$$

in particular we deduce

$$[F[j]]_{\Omega^*} = \begin{cases} [G[s-p]]_{\Omega^*} + [G[s]]_{\Omega^*} B_{s,s-p}, & j = s-p, \\ [G[j]]_{\Omega^*}, & j = s-p+1, \dots, s. \end{cases} \quad (11)$$

We are going to study $[G[s]]_{\Omega^*} B_{s,s-p}$. Let $[\alpha_{\underline{i}}]$ be the vector of the coordinates of $G[s]$ w.r.t. Ω^* , i.e.

$$G[s] = \sum_{|\underline{i}|=s} \alpha_{\underline{i}} \frac{1}{\underline{i}!} y^{\underline{i}};$$

the entries of $[G[s]]_{\Omega^*} B_{s,s-p}$ are bi-homogeneous forms in the components of $[\alpha_{\underline{i}}]$ and $\underline{a} = (a_{\underline{i}}^1, \dots, a_{\underline{i}}^n)$ such that $|\underline{i}| = p+1$ of bi-degree $(1, 1)$. Hence there exists a matrix $M^{[s-p]}(G[s])$ of size $\binom{n-1+s-p}{n-1} \times n \binom{n+p}{n-1}$ and entries in the $K[\alpha_{\underline{i}}]$ such that

$$\tau([\alpha_{\underline{i}}] B_{s,s-p}) = M^{[s-p]}(G[s]) \tau \underline{a} \quad (12)$$

where $\tau \underline{a}$ denotes the transpose of the row-vector \underline{a} . We are going to describe the entries of $M^{[s-p]}(G[s])$. We label the columns of $M^{[s-p]}(G[s])$ with the set of indexes (j, \underline{i}) , $j = 1, \dots, n$, $|\underline{i}| = p+1$, corresponding to the entries of $\underline{a} = (a_{\underline{i}}^1, |\underline{i}| = p+1; \dots; a_{\underline{i}}^n, |\underline{i}| = p+1) \in K^{n \binom{n+p}{n-1}}$.

Lemma 2.1. *The entry of $M^{[s-p]}(G[s])$ corresponding to the W -row, $|W| = s-p$, and column $(j, \underline{i}) \in \{1, \dots, n\} \times \{\underline{i}; |\underline{i}| = p+1\}$ is*

$$M^{[s-p]}(G[s])_{W, (j, \underline{i})} = w_j \alpha_{W - \delta_j + \underline{i}}.$$

Proof. Given W , $|W| = s-p$, the coordinate of $[G[s]]_{\Omega^*} B_{s,s-p}$ with respect $(x^W)^*$ is, by (9),

$$\begin{aligned} ([G[s]]_{\Omega^*} B_{s,s-p})_{(x^W)^*} &= \sum_{|L|=s} (B_{s,s-p})_{W,L} \alpha_L = \\ &= \sum_{|L|=s} \left(\sum_{W - \delta_j + \underline{i} = L} w_j a_{\underline{i}}^j \right) \alpha_L = \sum_{j, |\underline{i}|=p+1} w_j \alpha_{W - \delta_j + \underline{i}} a_{\underline{i}}^j \end{aligned}$$

so the entry of $M^{[s-p]}(G[s])$ corresponding to the W row, $|W| = s-p$, and column $(j, \underline{i}) \in \{1, \dots, n\} \times \{\underline{i}; |\underline{i}| = p+1\}$ is $w_j \alpha_{W - \delta_j + \underline{i}}$. \square

The goal is now to present a structure of $M^{[s-p]}(G[s])$ in terms of $G[s]$. In particular we will prove that the rank of $M^{[s-p]}(G[s])$ can be expressed in terms of the Hilbert function of $A_{G[s]}$. We need further notations.

For every $i = 1, \dots, n$, we denote S_p^i the set of monomials x^α of degree p such that $x^\alpha \in x_i(x_i, \dots, x_n)^{p-1}$, hence $\#(S_p^i) = \binom{p-1+n-i}{p-1}$. By definition $S_p^1 \cup \dots \cup S_p^n$ is the set of monomials of degree p and

$$\text{the last } \binom{p-2+n-i}{p-1} \text{ elements of } S_p^i \text{ correspond to } \frac{x_i}{x_{i+1}} S_p^{i+1}, \quad (13)$$

For instance $S_3^1 = \{x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^2, x_1 x_2 x_3, x_1 x_3^2\}$, $S_3^2 = \{x_2^3, x_2^2 x_3, x_2 x_3^2\}$, $S_3^3 = \{x_3^3\}$. We write $\log(x^\alpha) = \alpha$ for all $\alpha \in \mathbb{N}^n$.

Lemma 2.2. *The matrix $M^{[s-p]}(G[s])$ has the following upper-diagonal structure*

$$M^{[s-p]}(G[s]) = \begin{pmatrix} M_1 & * & \cdots & * & * \\ 0 & M_2 & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & M_{n-1} & * \\ 0 & 0 & 0 & 0 & M_n \end{pmatrix}$$

where M_j is a matrix of size $\binom{s-p-1+n-j}{s-p-1} \times \binom{n+p}{n-1}$, $j = 1, \dots, n$, defined as follows: the entries of M_j are the entries of $M^{[s-p]}(G[s])$ corresponding to the rows $W \in \log(S_{s-p}^j)$ and columns (j, \underline{i}) , $|\underline{i}| = p+1$. We label the entries of M_j with respect to these multi-indexes. Then it holds:

(i) for all $W = (w_1, \dots, w_n) \in \log(S_{s-p}^1)$ and \underline{i} , $|\underline{i}| = p+1$,

$$w_1 \Delta^{p+1}(G[s])_{(W-\delta_1, \underline{i})} = M_{1(W, (1, \underline{i}))},$$

(ii) for all $j = 1, \dots, n-1$, $W \in \log(S_{s-p}^{j+1})$,

$$M_{j+1, (W, (j+1, *))} = w_{j+1} M_{j, (L, (j, *))}$$

with $L = \delta_j + W - \delta_{j+1}$,

Proof. First we prove that $M^{[s-p]}(G[s])$ has the upper-diagonal structure as in the claim. Since the entry of $M^{[s-p]}(G[s])$ corresponding to the W row, $|W| = s-p$, and column $(j, \underline{i}) \in \{1, \dots, n\} \times \{\underline{i}; |\underline{i}| = p+1\}$ is $w_j \alpha_{W-\delta_j+\underline{i}}$, this entry is zero if $W \in \log(S_{s-p}^t)$ and $j < t \leq n$.

(i) Notice that the set of multi-indexes $L = W - \delta_1$, $|L| = s-p$, agrees with the set of multi-indexes of degree $s-p-1$, see (13). Hence by Proposition 1.6 and Proposition 2.1 we have

$$w_1 \Delta^{p+1}(G[s])_{(W-\delta_1, \underline{i})} = w_1 \alpha_{W-\delta_1+\underline{i}} = M_{1(W, (1, \underline{i}))}.$$

(ii) If $|\underline{i}| = p+1$ and $L = \delta_j + W - \delta_{j+1}$, then, Proposition 2.1,

$$M_{j+1, (W, (j+1, *))} = w_{j+1} \alpha_{W-\delta_{j+1}+\underline{i}} = w_{j+1} \alpha_{L-\delta_j+\underline{i}} = w_{j+1} M_{j, (L, (j, *))}.$$

□

As a consequence of (i) and (ii), the matrix $\Delta^{p+1}(G[s])$ is strongly involved in the computation of the rank of $M^{[s-p]}(G[s])$.

Corollary 2.3. *If $s \leq 4$ then $\text{rank}(M^{[s-p]}(G[s]))$ is maximal if and only if $\text{rank}(\Delta^{p+1}(G[s]))$ is maximal.*

Proof. Notice that $M^{[s-p]}(G[s])$ has an upper-diagonal structure where the rows of the diagonal blocks M_j are a subset of the rows of the first block matrix M_1 . Let us assume that the number of rows of M_1 is not bigger than the number of columns of M_1 , as a consequence the same holds for M_j with $j > 1$. Then we can compute the rank of $M^{[s-p]}(G[s])$ by rows, so $\text{rank}(M^{[s-p]}(G[s]))$ is maximal if and only if $\text{rank}(\Delta^{p+1}(G[s]))$ is maximal. Since M_1 is a $\binom{s-p-2+n}{s-p-1} \times \binom{n+p}{n-1}$ matrix, if $\binom{n+s-p-2}{s-p-1} = \binom{n+s-p-2}{n-1} \leq \binom{n+p}{n-1}$ we get the result. This inequality is equivalent to $n+s-p-2 \leq n+p$, i.e. $s \leq 2p+2$, since $p \geq 1$ we get that $s \leq 4$. □

Example 3.4 shows that Corollary 2.3 fails for $s = 5$.

We may generalize the previous facts to a sequence $\underline{G} = G_1, \dots, G_t$ of polynomials of degree s of P . Let φ_{s-p} be a K -algebra isomorphism of type (8) sending $A_{\underline{F}}$ to $A_{\underline{G}}$ where $\underline{F} = F_1, \dots, F_t$. In particular we assume that, as in (10),

$$[G_r]_{\Omega^*} M(\varphi_{s-p}) = [F_r]_{\Omega^*},$$

for every $r = 1, \dots, t$. We deduce the analogous of (11) and we restrict our interest to

$$[\underline{G}[s]]_{\Omega^*} B_{s,s-p}^{\oplus t}$$

where

$$B_{s,s-p}^{\oplus t} := \begin{pmatrix} B_{s,s-p} \\ \vdots \\ B_{s,s-p} \end{pmatrix}$$

obtained by gluing t times the matrix $B_{s,s-p}$ and where $[\underline{G}[s]]_{\Omega^*}$ is the row $([G_r[s]]_{\Omega^*} : r = 1, \dots, t)$. Accordingly with (12), it is defined the matrix $M^{[s-p]}(G_r[s])$ of size $\binom{n-1+s-p}{n-1} \times n \binom{n+p}{n-1}$ and entries depending on $[\underline{G}[s]]_{\Omega^*}$ such that

$${}^t([G_r[s]]_{\Omega^*} B_{s,s-p}) = M^{[s-p]}(G_r[s]) {}^t \underline{a}$$

If we define

$$M^{[s-p]}(\underline{G}[s]) := \begin{pmatrix} M^{[s-p]}(G_1[s]) \\ \vdots \\ M^{[s-p]}(G_t[s]) \end{pmatrix} \quad (14)$$

which is a $t \binom{n-1+s-p}{n-1} \times n \binom{n+p}{n-1}$ matrix, we get

$$\tau([\underline{G}[s]]_{\Omega^*} B_{s,s-p}^{\oplus t}) = M^{[s-p]}(\underline{G}[s]) \tau \underline{a}. \quad (15)$$

The matrix $M^{[s-p]}(\underline{G}[s])$ has the same shape of $M^{[s-p]}(G[s])$, already described in Lemma 2.2 and its blocks correspond to suitable submatrices of $(\Delta^{p+1}(\underline{G}[s]))$ (see (5)). Hence we have an analogous to (11) for the level case

$$[F_r[j]]_{\Omega^*} = \begin{cases} [G_r[s-p]]_{\Omega^*} + \underline{a} \tau(M^{[s-p]}(\underline{G}[s])), & j = s-p, \\ [G_r[j]]_{\Omega^*}, & j = s-p+1, \dots, s. \end{cases} \quad (16)$$

for all $r = 1, \dots, t$.

3 Compressed Gorenstein algebras

Let $A = A_G$ be an compressed Artin Gorenstein local K -algebra A of socle degree s . We recall that, by Proposition 1.8, if $G = G[s] + G[s-1] + \dots$, then

$$h_i(A) = \text{rank}(\Delta^i(G[s])) = \min\left\{\binom{n-1+i}{n-1}, \binom{n-1+s-i}{n-1}\right\}$$

for every $i = 1, \dots, s$.

Theorem 3.1. *Let A be an extremal Artin Gorenstein local K -algebra. If $s \leq 4$ then A is canonically graded.*

Proof. Let A be a extremal Artin Gorenstein local K -algebra of socle degree $s \geq 2$ and embedding dimension n . Then $A = A_G$ with $G \in P = K[y_1, \dots, y_n]$ a polynomial of degree s and $gr_m(A) = P/\text{Ann}(G[s])$ is an extremal Gorenstein graded algebra of socle degree $s \geq 2$ and embedding dimension n (see Proposition 1.4).

The main result of [6] shows that if $s \leq 3$ then A is canonically graded. Let assume $s = 4$, then the Hilbert function is $\{1, n, \binom{n+1}{2}, n, 1\}$. Because $A_{G[4]}$ is an extremal Gorenstein algebra with the same

Hilbert function of A , we may assume $G = G[4] + G[3]$. In fact $P_1, P_2 \subseteq \langle G[4] \rangle_R$ because of (2) and, as a consequence, it is easy to see that $\langle G[4] + G[3] \rangle_R = \langle G[4] + G[3] + G[2] + \dots \rangle_R$.

So we have to prove that, however we fix $G[3]$, there exists an automorphism $\varphi \in \text{Aut}(R/\mathcal{M}^5)$ such that

$$A_G \simeq A_{G[4]}.$$

We consider for every $j = 1, \dots, n$

$$\varphi(x_j) = x_j + \sum_{|i|=2} a_i^j x^i + \text{higher terms}$$

If $A_F = \varphi_3^{-1}(A_G)$, then from (11) and (12) we get

$$\begin{aligned} [F[3]]_{\Omega^*} &= [G[3]]_{\Omega^*} + \underline{a} \tau(M^{[3]}(G[4])) \\ [F[4]]_{\Omega^*} &= [G[4]]_{\Omega^*} \end{aligned} \tag{17}$$

where $\underline{a} = (a_i^1, \dots, a_i^n)$. By Proposition 1.8 and Corollary 2.3, we know that the matrix $M^{[3]}(G[4])$ has maximal rank and it coincides with the number of the rows, so there exists a solution $\underline{a} \in K^n$ of (17) such that $F[3] = 0$ and $F[4] = G[4]$. \square

Let A be a local K -algebra of embedding dimension n and socle degree s . By looking at the dual module, it is clear that if $s \leq 2$, then A is graded because the dual module can be generated by homogeneous polynomials of degree at least two. The aim is now to list the local compressed algebras of embedding dimension n , socle degree s and socle type $E = (0, \dots, e_{v-1}, e_v, \dots, e_s, 0, 0, \dots)$ which are canonically graded. Examples will prove that the following result cannot be extended to higher socle degrees.

Theorem 3.2. *Let A be a compressed Artin K -algebra of embedding dimension n , socle degree s and socle type E . Then A is canonically graded in the following cases:*

- (1) $s \leq 3$,
- (2) $s = 4$ and $e_4 = 1$,
- (3) $s = 4$ and $n = 2$.

Proof. Since a local ring with Hilbert function $\{1, n, t\}$ is always graded (the dual module can be generated by quadratic forms), we may assume $s \geq 3$.

If $s = 3$ and A is level compressed, then A is canonically graded by [4]. If A is not necessarily level, but compressed, then by (1) the socle type is $\{0, 0, e_2, e_3\}$ and the Hilbert function is $\{1, n, h_2, e_3\}$ where $h_2 = \min\{\dim_K R_2, e_2 + e_3 n\}$. Because $\text{gr}_{\mathfrak{m}}(A) = P/I^*$ has embedding dimension n , then $P_{\leq 1} \subseteq (I^*)^\perp$. Then we may assume that in any system of coordinates I^\perp is generated by e_2 quadratic forms and e_3 polynomials G_1, \dots, G_{e_3} of degree 3. Then the result follows because $R/\text{Ann}_R(G_1, \dots, G_{e_3})$ is a 3-level compressed algebra of type e_3 and hence canonically graded.

Assume $s = 4$ and $e_4 = 1$. We recall that if A is Gorenstein, then the result follows by Theorem 3.1. Since A is compressed, then by (1) the socle type is $(0, 0, 0, e_3, 1)$. This means that I^\perp is generated by e_3 polynomial of degree 3 and one polynomial of degree 4. Similarly to the above part, because $P_{\leq 2} \subseteq (I^*)^\perp$, I^\perp can be generated by e_3 forms of degree 3 and one polynomial of degree 4. As before the problem is reduced to the Gorenstein case with $s = 4$ and the result follows.

Assume $s = 4$ and $n = 2$. If $e_4 = 1$, then we are in case (2). If $e_4 > 1$, because A is compressed, the possible socle types are: $E_i = (0, 0, 0, 0, i)$ with $i = 2, \dots, 5$ and since A is compressed, the corresponding Hilbert function is $\{1, 2, 3, 4, i\}$. In each case A is graded because the Hilbert function forces the dual module to be generated by forms of degree four. \square

The following example shows that Theorem 3.1 fails if A is Gorenstein of socle degree $s = 4$, but not compressed, i.e. the Hilbert function is not maximal.

Example 3.3. Let A be an Artin Gorenstein local K -algebra with Hilbert function $HF_A = \{1, 2, 2, 2, 1\}$. The local ring is called almost stretched and a classification can be found in [7]. In this case A is isomorphic to one and only one of the following rings :

- (a) $A = R/I$ with $I = (x_1^4, x_2^2) \subseteq R = K[[x_1, x_2]]$, and $I^\perp = \langle y_1^3 y_2 \rangle$. In this case A is canonically graded,
- (b) $A = R/I$ with $I = (x_1^4, -x_1^3 + x_2^2) \subseteq R = K[[x_1, x_2]]$, and $I^\perp = \langle y_1^3 y_2 + y_2^3 \rangle$. The associated graded ring is of type (a) and it is not isomorphic to R/I . Hence A is not canonically graded.
- (c) $A = R/I$ with $I = (x_1^2 + x_2^2, x_2^4) \subseteq R = K[[x_1, x_2]]$, and $I^\perp = \langle y_1 y_2 (y_1^2 - y_2^2) \rangle$. In this case A is graded.

The following example shows that Theorem 3.1 cannot be extended to extremal Gorenstein algebras of socle degree $s = 5$.

Example 3.4. Let us consider the ideal

$$I = (x_1^4, x_2^3 - 2x_1^3 x_2) \subset R = K[[x_1, x_2]].$$

The quotient $A = R/I$ is an extremal Gorenstein algebra with $HF_A = \{1, 2, 3, 3, 2, 1\}$, $I^* = (x_1^4, x_2^3)$ and $I^\perp = \langle y_1^3 y_2^2 + y_2^4 \rangle$. We will prove that I is not isomorphic to I^* . Assume that there exists an analytic isomorphism φ of R mapping I into I^* . It is easy to see that Jacobian matrix of φ is diagonal because $(I^*)^\perp = \langle y_1^3 y_2^2 \rangle$. We perform the computations modulo $(x_1, x_2)^5$, so we only have to consider the following coefficients of φ

$$\begin{cases} \varphi(x_1) = ax_1 + \dots \\ \varphi(x_2) = bx_2 + ix_1^2 + jx_1 x_2 + kx_2^2 + \dots \end{cases}$$

where a, b are units, $i, j, k \in K$. After the isomorphism $x_1 \rightarrow 1/ax_1$, $x_2 \rightarrow 1/bx_2$, we may assume $a = b = 1$. Then we have

$$I^* = \varphi(I) = (x_1^4, x_2^3 - 2x_1^3 x_2 + 3ix_1^2 x_2^2 + 3jx_1 x_2^3 + 3kx_2^4) \quad \text{modulo } (x_1, x_2)^5.$$

Hence there exist $\alpha \in K, \beta \in R$ such that

$$x_2^3 - 2x_1^3 x_2 + 3ix_1^2 x_2^2 + 3jx_1 x_2^3 + 3kx_2^4 = \alpha x_1^4 + \beta x_2^3 \quad \text{modulo } (x_1, x_2)^5.$$

From this equality we deduce $\alpha = 0$ and

$$2x_1^3 x_2 = x_2^2 (x_2 + 3ix_1^2 + 3jx_1 x_2 + 3kx_2^2 - \beta x_2) \quad \text{modulo } (x_1, x_2)^5,$$

a contradiction, so I is not isomorphic to I^* .

It is interesting to notice that we can get the same conclusion by following the line of the proof of Theorem 3.1. Let φ as above sending I into I^* . If we denote by $(z_i)_{i=1, \dots, 6}$ the coordinates of an homogeneous form $G[5]$ of degree 5 in y_1, y_2 with respect Ω^* , then by Lemma 2.1, the matrix $M^{[4]}(G[5])$ ($s = 5, p = 1$) has the following shape

$$\begin{pmatrix} 4z_1 & 4z_2 & 4z_3 & 0 & 0 & 0 \\ 3z_2 & 3z_3 & 3z_4 & z_1 & z_2 & z_3 \\ 2z_3 & 2z_4 & 2z_5 & 2z_2 & 2z_3 & 2z_4 \\ z_4 & z_5 & z_6 & 3z_3 & 3z_4 & 3z_5 \\ 0 & 0 & 0 & 4z_4 & 4z_5 & 4z_6 \end{pmatrix}$$

In our case $G[5] = y_1^3 y_2^2$, so all z_i are zero but $z_3 = 12$, hence the above matrix has rank 4 and it has not maximal rank accordingly with Corollary 2.3.. Since all the rows are not zero except the last one, it is easy to see that $F[4] = y_2^4$ is not in the image of $M^{[4]}(G[5])$, as (11) requires.

The following example shows that Theorem 3.2 cannot be extended to compressed type 2 level algebras of socle degree $s = 4$.

Example 3.5. Let us consider the forms $G_1[4] = y_1^2 y_2 y_3$, $G_2[4] = y_1 y_2^2 y_3 + y_2 y_3^3$ in $P = K[y_1, y_2, y_3]$ of degree 4 and define in $R = K[[x_1, x_2, x_3]]$ the ideal

$$I = \text{Ann}(G_1[4] + y_3^3, G_2[4]).$$

Then $A = R/I$ is a compressed level algebra with socle degree 4, type 2 and Hilbert function $HF_A = \{1, 3, 6, 6, 2\}$. We prove that A is not canonically graded.

We know that $I^* = \text{Ann}(G_1[4], G_2[4])$ and we prove that A and $gr_m(A)$ are not isomorphic as K -algebras. Let φ an analytic isomorphism sending I to I^* , then it is easy to see that $\varphi = I_3$ modulo $(x_1, x_2, x_3)^2$. Following the approach of this paper, we compute the matrix $M^{[3]}(G_1[4], G_2[4])$ of size 20×18 and, accordingly with (11), we show that y_3^3 is not in the image of $M^{[3]}(G_1[4], G_2[4])$.

Let $F_1[4], F_2[4]$ be two homogeneous forms of degree 4 of $R = K[y_1, y_2, y_3]$. We denote by $(z_i^j)_{i=1, \dots, 15}$ the coordinates of $F_j[4]$ with respect the basis Ω^* , $j = 1, 2$. Then the 20×18 matrix $M^{[3]}(F_1[4], F_2[4])$ has the following shape, see (14),

$$\left(\begin{array}{cccccc|cccccc|cccccc} 3z_1^1 & 3z_2^1 & 3z_3^1 & 3z_4^1 & 3z_5^1 & 3z_6^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2z_1^1 & 2z_2^1 & 2z_3^1 & 2z_4^1 & 2z_5^1 & 2z_6^1 & z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2z_1^1 & 2z_2^1 & 2z_3^1 & 2z_4^1 & 2z_5^1 & 2z_6^1 & 0 & 0 & 0 & 0 & 0 & 0 & z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 \\ z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 & 2z_1^1 & 2z_2^1 & 2z_3^1 & 2z_4^1 & 2z_5^1 & 2z_6^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 & z_7^1 & z_8^1 & z_9^1 & z_{10}^1 & z_{11}^1 & z_{12}^1 & z_{13}^1 & z_{14}^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3z_1^1 & 3z_2^1 & 3z_3^1 & 3z_4^1 & 3z_5^1 & 3z_6^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2z_1^1 & 2z_2^1 & 2z_3^1 & 2z_4^1 & 2z_5^1 & 2z_6^1 & z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1^1 & z_2^1 & z_3^1 & z_4^1 & z_5^1 & z_6^1 & 2z_1^1 & 2z_2^1 & 2z_3^1 & 2z_4^1 & 2z_5^1 & 2z_6^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3z_1^1 & 3z_2^1 & 3z_3^1 & 3z_4^1 & 3z_5^1 & 3z_6^1 \\ 3z_1^2 & 3z_2^2 & 3z_3^2 & 3z_4^2 & 3z_5^2 & 3z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 & z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 & z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & z_7^2 & z_8^2 & z_9^2 & z_{10}^2 & z_{11}^2 & z_{12}^2 & z_{13}^2 & z_{14}^2 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 & 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3z_1^2 & 3z_2^2 & 3z_3^2 & 3z_4^2 & 3z_5^2 & 3z_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 & z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_1^2 & z_2^2 & z_3^2 & z_4^2 & z_5^2 & z_6^2 & 2z_1^2 & 2z_2^2 & 2z_3^2 & 2z_4^2 & 2z_5^2 & 2z_6^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3z_1^2 & 3z_2^2 & 3z_3^2 & 3z_4^2 & 3z_5^2 & 3z_6^2 \end{array} \right)$$

It is enough to specialize the matrix to our case for proving that y_3^3 is not in the image of $M^{[3]}(G_1[4], G_2[4])$.

Remark 3.6. Let $\mathcal{C}_{s,t}$ be the family of level Artin algebras of socle degree s and type t . This family can be parameterized by a non-empty open Zariski subset $\mathcal{I}_{s,t}$ of the affine space of t -uples of degree s polynomials of P . The previous examples suggested that if s, t are not in the hypothesis of the last Theorem, then a generic element $F \in \mathcal{I}_{s,t}$ defines a **non-canonically graded** level Artin algebra $R/\text{Ann}_R((F))$ of socle degree s and type t .

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