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The structure of the inverse system of Gorenstein k-algebras

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In honor of Tito Valla our teacher and friend

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A B S T R A C T

Macaulay's Inverse System gives an effective method to construct Artinian Gorenstein k-algebras. To date a general structure for Gorenstein k-algebras of any dimension (and codimension) is not understood. In this paper we extend Macaulay's correspondence characterizing the submodules of the divided power ring in one-to-one correspondence with Gorenstein d-dimensional k-algebras. We discuss effective methods for constructing Gorenstein graded rings. Several examples illustrating our results are given.

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1. Introduction

Gorenstein rings were introduced by Grothendieck, who named them because of their relation to a duality property of singular plane curves studied by Gorenstein, [13,14]. The zero-dimensional case had previously been studied by Macaulay, [22]. Gorenstein rings are very common and significant in many areas of mathematics, as it can be seen in Bass's paper [2], see also [16]. They have appeared as an important component in a significant number of problems and have proven useful in a wide variety of applications in commutative algebra, singularity theory, number theory and more recently in combinatorics, among other areas.

Gorenstein rings are a generalization of complete intersections, and indeed the two notions coincide in codimension two. Codimension three Gorenstein rings are completely described by Buchsbaum and Eisenbud's structure theorem, [4]. More recently Reid in [25] studied the projective resolution of Gorenstein ideals of codimension 4, aiming to extend the result of Buchsbaum and Eisenbud. Kustin and Miller in a series of papers studied the structure of Gorenstein ideal of higher codimension, see [21] and the references therein.

Notice that the lack of a general structure of homogeneous Gorenstein ideals of higher codimension is the main obstacle to extending the Gorenstein liaison theory in codimension at least three; the codimension two Gorenstein liaision case is well understood, see [20]. See, for instance, [23,21] and [19] for some constructions of particular families of Gorenstein algebras.

Let k be a field and let I be an ideal (not necessarily homogeneous) of the power series ring R (or of the polynomial ring in the homogeneous case). As an effective consequence of Matlis duality, it is known that an Artinian ring R/I is a Gorenstein k-algebra if and only if I is the ideal of a system of polynomial differential operators with constant coefficients having a unique solution. This solution determines an R-submodule of the divided power ring Γ (or its completion) denoted by $I^\perp$ and called the inverse system of I which contains the same information as in the original ideal. Macaulay at the beginning of the 20th century proved that the Artinian Gorenstein k-algebras are in correspondence with the cyclic R-submodules of Γ where the elements of R act as derivatives on Γ, see [10,18]. In the last twenty years several authors have applied this device to several problems, among others: Warings's problem, [12], n-factorial conjecture in combinatorics and geometry, [15], the cactus rank, [24], the geometry of the punctual Hilbert scheme of Gorenstein schemes, [18], Kaplansky–Serre's problem, [26], classification up to analytic isomorphism of Artinian Gorenstein rings, [9].

The aim of this paper is to extend the well-known Macaulay's correspondence characterizing the submodules of Γ in one-to-one correspondence with Gorenstein d-dimensional k-algebras (Theorem 3.8). These submodules are called G_d-admissible (Definition 3.6) and in positive dimension they are not finitely generated. The G_d-admissible submodules of Γ can be described in some coherent manner and we discuss effective methods for constructing Gorenstein k-algebras with a particular emphasis to standard graded k-algebras

(Theorem 3.11). In Section 4 several examples are given, in particular we propose a finite procedure for constructing Gorenstein graded k-algebras of given multiplicity or given Castelnuovo–Mumford regularity (Proposition 4.2). We discuss possible obstructions in the local case corresponding to non-algebraic curves. Our hope is that our results will be successfully applied to give new insights in the above mentioned applications and problems.

The computations are performed in characteristic zero ($k = \mathbb{Q}$) by using the computer program system Singular, [6], and the library [8].

2. Inverse system

Let V be a vector space of dimension n over a field k where, unless specifically stated otherwise, k is a field of any characteristic. Let $R = Sym^k_{\cdot} V = \oplus_{i\geq 0} Sym^k_i V$ be the standard graded polynomial ring in n variables over k and $\Gamma = D^k_{\cdot}(V^*) = \oplus_{i\geq 0} D^k_i(V^*) = \oplus_{i\geq 0}\mathrm{Hom}_k(R_i, k)$ be the graded R-module of graded k-linear homomorphisms from R to k. Through the paper if V denotes the k-vector space $\langle z_1, \dots, z_n\rangle$, then we denote by $V^* = \langle Z_1, \dots, Z_n\rangle$ the dual base and $\Gamma = \Gamma(V^*) \simeq k_{DP}[Z_1, \dots, Z_n]$ the divided power ring. In particular $\Gamma_j = \langle\{Z^{[L]} \ / \ |L| = j\}\rangle$ is the span of the dual generators to $z^L = z_1^{l_1} \cdots z^{l_n}$ where L denotes the multi-index $L = (l_1, \dots, l_n) \in \mathbb{N}^n$ of length $|L| = \sum_i l_i$. If $L \in \mathbb{Z}^n$ then we set $X^{[L]} = 0$ if any component of L is negative. The monomials $Z^{[L]}$ are called divided power monomials (DP-monomials) and the elements $F = \sum_L b_L Z^{[L]}$ of Γ the divided power polynomials (DP-polynomials).

We extend the above setting to the local case considering R as the power series ring on V. If V denotes the k-vector space $\langle z_1, \dots, z_n\rangle$, then $R = k[\![z_1, \dots, z_n]\!]$ will denote the formal power series ring and $\mathfrak{m} = (z_1, \dots, z_n)$ denotes the maximal ideal of R. The injective hull $E_R(k)$ of R is isomorphic to the divided power ring (see [11]). For detailed information see [7,10,18], Appendix A.

We recall that Γ is a R-module acting R on Γ by *contraction* as it follows.

Definition 2.1. If $h = \sum_M a_M z^M \in R$ and $F = \sum_L b_L Z^{[L]} \in \Gamma$, then the contraction of F by h is defined as

$$h \circ F = \sum_{M,L} a_M b_L Z^{[L-M]}$$

The contraction is $Gl_n(k)$-equivariant. If the characteristic of the field k is zero, then there is a natural isomorphism of R-algebras between $(\Gamma, \circ)$ equipped with an internal product and the usual polynomial ring P replacing the contraction with the partial derivatives. In this paper we do not consider the ring structure of Γ, but we always consider Γ as R-module by contraction and k will be a field of any characteristic.

The contraction $\circ$ induces a exact pairing:

$$\begin{array}{rccc} \langle\ ,\ \rangle : & R \times \Gamma & \longrightarrow & k \\ & (f, g) & \to & (f \circ g)(0) \end{array}$$

If $I \subset R$ is an ideal of R then $(R/I)^\vee = \operatorname{Hom}_R(R/I, \Gamma)$ is the R-submodule of Γ

$$I^\perp = \{g \in \Gamma \mid I \circ g = 0\} = \{g \in \Gamma \mid \langle f, g\rangle = 0 \ \ \forall f \in I\}.$$

This submodule of Γ is called *Macaulay's inverse system of I*. If I is a homogeneous ideal of a polynomial ring R, then $I^\perp$ is homogenous (generated by forms in Γ in the standard meaning) and $I^\perp = \oplus I_j^\perp$ where $I_j^\perp = \{F \in \Gamma_j \mid \ h \circ F = 0 \ \text{ for all } \ h \in I_j\}$.

Given a R-submodule W of Γ, then the dual $W^\vee = \operatorname{Hom}_R(W, \Gamma)$ is the ring $R/\operatorname{Ann}_R(W)$ where

$$\operatorname{Ann}_R(W) = \{f \in R \ \mid \ f \circ g = 0 \ \text{ for all } \ g \in W\}.$$

Notice that $\operatorname{Ann}_R(W)$ is an ideal of R. Matlis duality assures that

$$\operatorname{Ann}_R(W)^\perp = W, \ \ \operatorname{Ann}_R(I^\perp) = I.$$

If W is generated by homogeneous DP-polynomials, then $\operatorname{Ann}_R(W)$ is a homogeneous ideal of R.

Macaulay in [22, IV] proved a particular case of Matlis duality, called Macaulay's correspondence, between the ideals $I \subseteq R$ such that R/I is an Artinian local ring and R-submodules $W = I^\perp$ of Γ which are finitely generated. Macaulay's correspondence is an effective method for computing Artinian rings, see [5], Section 1, [17,12] and [18].

If $(A, \mathfrak{n})$ is an Artinian local ring, we denote by $Soc(A) = 0 :_A \mathfrak{n}$ the socle of A. Throughout this paper we denote by s the *socle degree* of A (also called Löwey length), that is the maximum integer j such that $\mathfrak{n}^j \neq 0$. The *type* of A is $t(A) := \dim_k Soc(A)$; A is an Artinian Gorenstein ring if $t(A) = 1$. If R/I is an Artinian local algebra of socle-degree s then $I^\perp$ is generated by DP-polynomials of degree $\leq s$ and $\dim_k(A)(=$ multiplicity of $A) = \dim_k I^\perp$.

From Macaulay's correspondence, Artinian Gorenstein k-algebras $A = R/I$ of socle degree s correspond to cyclic R-submodules of Γ generated by a divided power polynomial $F \neq 0$ of degree s.

We will denote by $\langle F\rangle_R$ the cyclic R-submodule of Γ generated by the divided power polynomial F.

We can compute the Hilbert function of a graded or local k-algebra $A = R/I$ (not necessarily Artinian) in terms of its inverse system. The Hilbert function of $A = R/I$ is by definition

$$\mathrm{HF}_A(i) = \dim_k \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)$$

where $\mathfrak{n} = \mathfrak{m}/I$ is the maximal ideal of A.

We denote by $\Gamma_{\leq i}$ (resp. $\Gamma_{<i}$, resp. Γ_i), $i \in \mathbb{N}$, the k-vector space of DP-polynomials of Γ of degree less or equal (resp. less, resp. equal) to i, and we consider the following k-vector space

$$(I^\perp)_i := \frac{I^\perp \cap \Gamma_{\le i} + \Gamma_{<i}}{\Gamma_{<i}}.$$

Notice that if I is an homogeneous ideal of the polynomial ring R, then $(I^\perp)_i = (I_i)^\perp$.

Proposition 2.2. *With the previous notation and for all $i \ge 0$*

$$\mathrm{HF}_A(i) = \dim_k (I^\perp)_i.$$

Proof. Let's consider the following natural exact sequence of R-modules

$$0 \longrightarrow \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} \longrightarrow \frac{A}{\mathfrak{n}^{i+1}} \longrightarrow \frac{A}{\mathfrak{n}^i} \longrightarrow 0.$$

Dualizing this sequence we obtain

$$0 \longrightarrow (I + \mathfrak{m}^i)^\perp \longrightarrow (I + \mathfrak{m}^{i+1})^\perp \longrightarrow \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^\vee \longrightarrow 0$$

so we get the following sequence of k-vector spaces:

$$\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^\vee \cong \frac{(I + \mathfrak{m}^{i+1})^\perp}{(I + \mathfrak{m}^i)^\perp} = \frac{I^\perp \cap \Gamma_{\le i}}{I^\perp \cap \Gamma_{\le i-1}} \cong \frac{I^\perp \cap \Gamma_{\le i} + \Gamma_{<i}}{\Gamma_{<i}}.$$

Then the result follows since $\dim_k \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^\vee = \dim_k \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} = \mathrm{HF}_A(i)$. □

Example 2.3. Let $I = (xy, y^2 - x^3) \subseteq R = k[[x, y]]$ and let $\Gamma = k_{DP}[X, Y]$. It is easy to see that

$$I^\perp = \langle X^{[3]} + Y^{[2]} \rangle_R$$

and $\langle X^{[3]} + Y^{[2]} \rangle_R = \langle X^{[3]} + Y^{[2]}, X^{[2]}, X, Y, 1 \rangle_k$ as k-vector space. Hence by using Proposition 2.2, one can compute the Hilbert series of $A = R/I$

$$\mathrm{HS}_A(z) = \sum_{i \ge 1} \mathrm{HF}_A(i) z^i = 1 + 2z + z^2 + z^3.$$

3. Structure of the inverse system

In this section R denotes the power series ring and $\mathfrak{m}$ the maximal ideal. Let I be an ideal of R such that $I \subset \mathfrak{m}^2$ and $A = R/I$ is Gorenstein of dimension $d \ge 1$. Where specified $A = R/I$ will be a standard graded algebra and in this case R will be the polynomial ring and $\mathfrak{m}$ the homogeneous maximal ideal.

We assume that the ground field k is infinite. If A is a standard graded k-algebra it is well known that we can pick $\underline{z} := z_1, \dots, z_d$ which are part of a basis of R_1 and they represent a linear system of parameters for R/I. If $A = R/I$ is a local k-algebra we can pick $\underline{z} := z_1, \dots, z_d \in \mathfrak{m} \setminus \mathfrak{m}^2$ which are part of a minimal set of generators of $\mathfrak{m}$ and their cosets represent a system of parameters for R/I. In both cases we will say that $\underline{z} := z_1, \dots, z_d$ is a *regular linear sequence* for R/I. We remark that $z_1, \dots, z_d$ can be extended to a minimal system of generators of $\mathfrak{m}$, say $z_1, \dots, z_d, \dots, z_n$ where $n = \dim R$. If $z_1, \dots z_n$ is a minimal set of generators of $\mathfrak{m}$, we denote by $Z_1, \dots, Z_n$ the corresponding dual basis such that $z_i \circ Z_j = \delta_{ij}$, hence $\Gamma = k_{DP}[Z_1, \dots, Z_n]$.

Assume $\underline{z} := z_1, \dots, z_d$ a regular linear sequence for A. For every $L = (l_1, \dots, l_d) \in \mathbb{N}^d$ we denote by $\underline{z}^L$ the sequence of pure powers $z_1^{l_1}, \dots, z_d^{l_d} \in R$. Consider $L = (l_1, \dots, l_d) \in \mathbb{N}^d_+$, we denote by

$$\Gamma_{\underline{z}^L} = (\underline{z}^L)^\perp$$

the R-submodule of Γ orthogonal to $\underline{z}^L$. Let $W = I^\perp$ be the inverse system of I in Γ and let

$$W_{\underline{z}^L} = W \cap \Gamma_{\underline{z}^L} = (I + (\underline{z}^L))^\perp. \tag{1}$$

Since $A/(\underline{z}^L)A$ is an Artinian Gorenstein local ring for all $L \in \mathbb{N}^d_+$, see for instance [3] Proposition 3.1.19(b), then $W_{\underline{z}^L}$ is a non-zero cyclic R-submodule of Γ for all $L \in \mathbb{N}^d_+$. We are interested in special generators of $W_{\underline{z}^L}$ strictly related to a given generator of $W_{\underline{z}}$.

We consider in $\mathbb{N}^d$, $d \le n$, the componentwise ordering, i.e. given two multi-indexes $L = (l_1 \dots, l_d)$ and $M = (m_1, \dots, m_d) \in \mathbb{N}^d$ then $L \le_d M$ if and only if $l_i \le m_i$ for all $i = 1, \dots, d$. We recall that $|L| = l_1 + \dots + l_d$ is the total degree of L. If $L \in \mathbb{N}^d$, we denote by $\Gamma_{\le L}$, resp. $\Gamma_{<L}$, the set of elements of Γ of multidegree less or equal (resp. less) than L with respect to $Z_1, \dots, Z_d$. We remark that if $\underline{z} = z_1, \dots, z_d$, then

$$(\underline{z}^L)^\perp = \Gamma_{<L}. \tag{2}$$

Lemma 3.1. *If $L \in \mathbb{N}^d_+$, then*

(i) $W_{\underline{z}^L}/\mathfrak{m} \circ W_{\underline{z}^L} \cong Soc(A/\underline{z}^L A)^\vee$,
(ii) $\bigcup_{L \in \mathbb{N}^d_+} W_{\underline{z}^L} = I^\perp$

Proof. (i) see [5], Lemma 1.9 (ii).

(ii) follows from (2) and (1) because

$$\bigcup_{L \in \mathbb{N}^d_+} W_{\underline{z}^L} = \bigcup_{L \in \mathbb{N}^d_+} (I^\perp \cap \Gamma_{<L}) = I^\perp. \quad \square$$

For all $i = 1, \dots, d$ we denote by $\gamma_{\mathbf{i}} = (0, \dots, 1, \dots 0) \in \mathbb{N}^d$ the i-th the coordinate vector. We write $\mathbf{1}_d = (1, \dots, 1) \in \mathbb{N}^d$, more in general, for all positive integer $t \in \mathbb{N}$ we write

$$\mathbf{t}_d = (t, \dots, t) \in \mathbb{N}^d$$

The following result is a consequence of a modified Koszul complex on R/I.

Proposition 3.2.

(i) *Assume $d = 1$. For all $l \geq 2$ there is an exact sequence of finitely generated R-submodules of Γ*

$$0 \longrightarrow W_{z_1} \longrightarrow W_{z_1^l} \xrightarrow{z_1 \circ} W_{z_1^{l-1}} \longrightarrow 0.$$

(ii) *Assume $d \geq 2$. If $\underline{z} = z_1, \dots, z_d$, then for all $L \in \mathbb{N}^r$ such that $L \geq \mathbf{2}_d$ there is an exact sequence of finitely generated R-submodules of Γ*

$$0 \longrightarrow W_{\underline{z}} \longrightarrow W_{\underline{z}^L} \longrightarrow \bigoplus_{k=1}^{d} W_{\underline{z}^{L-\gamma_k}} \longrightarrow \bigoplus_{1 \leq i < j \leq d} W_{\underline{z}^{L-\gamma_i-\gamma_j}}$$

Proof. (i) Assume $d = 1$. Since z_1 is regular on R/I, for all $l \geq 2$, the following sequence of Artinian Gorenstein rings is exact

$$0 \longrightarrow \frac{R}{I + (z_1^{l-1})} \xrightarrow{\cdot z_1} \frac{R}{I + (z_1^l)} \longrightarrow \frac{R}{I + (z_1)} \longrightarrow 0$$

This induces by duality the following exact sequence of finitely generated R-submodules of Γ

$$0 \longrightarrow W_{z_1} \longrightarrow W_{z_1^l} \xrightarrow{z_1 \circ} W_{z_1^{l-1}} \longrightarrow 0.$$

(ii) Assume $d \geq 2$. If $\underline{z} = z_1, \dots, z_d$, then we prove that for all $L \in \mathbb{N}^d$ such that $L \geq \mathbf{2}_d$ the following sequence of R-modules is exact

$$\bigoplus_{1 \leq i < j \leq d} \frac{R}{I + (\underline{z}^{L-\gamma_i-\gamma_j})} \xrightarrow{\phi_L} \bigoplus_{k=1}^{d} \frac{R}{I + (\underline{z}^{L-\gamma_k})} \xrightarrow{\varphi_L} \frac{R}{I + (\underline{z}^L)} \longrightarrow \frac{R}{I + (\underline{z})} \longrightarrow 0$$

where: $\varphi_L(\overline{v_1}, \dots, \overline{v_d}) = \sum_{k=1}^{d} z_k \overline{v_k}$, and
$\phi_L(\overline{v_{i,j}}; 1 \leq i < j \leq d) = \sum_{1 \leq i < j \leq r} (0, \dots, z_j, \dots, -z_i, \dots, 0)\overline{v_{i,j}}$;
for short we denote by $\overline{v}$ the class of an element $v \in R$ in the above different quotients. Since $L \geq \mathbf{2}_d$ for all $1 \leq i < j \leq d$ we have $L - \gamma_i - \gamma_j \in \mathbb{N}^d$. It is easy to prove that $\varphi_L \phi_L = 0$, so we have to prove that $\mathrm{Ker}(\varphi_L) \subset \mathrm{Im}(\phi_L)$. Given $(\overline{v_1}, \dots, \overline{v_d}) \in \mathrm{Ker}(\varphi_L)$ we have that $\sum_{k=1}^{d} z_k v_k \in I + (\underline{z}^L)$, so there are $\lambda_1, \dots, \lambda_r \in R$ such that

$$\sum_{k=1}^{d} z_k (v_k - \lambda_i z_k^{l_k - 1}) \in I.$$

Since $\underline{z}$ is a regular sequence on $A = R/I$ we deduce that, modulo I,

$$((v_k - \lambda_i z_k^{l_k-1})_{k=1,\dots,d}) \equiv \sum_{1\le i<j\le d} \mu_{i,j}(0,\dots,z_j,\dots,-z_i,\dots,0)$$

for some $\mu_{i,j} \in R$, $1 \le i < j \le d$. From this we deduce that $(\overline{v_1},\dots,\overline{v_d}) \in \mathrm{Im}(\phi_L)$.

Now the exact sequence of Artinian Gorenstein rings induces by Matlis duality the following exact sequence of Γ-modules

$$0 \longrightarrow W_{\underline{z}} \longrightarrow W_{\underline{z}^L} \xrightarrow{\varphi_L^*} \bigoplus_{k=1}^{d} W_{\underline{z}^{L-\gamma_k}} \xrightarrow{\phi_L^*} \bigoplus_{1\le i<j\le d} W_{\underline{z}^{L-\gamma_i-\gamma_j}}$$

with $\varphi_L^*(v) = (z_1 \circ v,\dots,z_d \circ v)$ and $\phi_L^*(v_1,\dots,v_d) = (z_j \circ v_i - z_i \circ v_j; 1 \le i < j \le d)$. This proves (ii). □

We recall the following basic fact that will be useful in the following.

Lemma 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and let $f : M \longrightarrow N$ be an epimorphism between two non-zero cyclic R-modules. Let $m \in M$ be an element such that $f(m)$ is a generator of N. Then m is a generator of M.*

Proof. We remark that f induces an isomorphism of one-dimensional k-vector spaces $\bar{f} : M/\mathfrak{m}M \longrightarrow N/\mathfrak{m}N$. Since the coset of $f(m)$ in $N/\mathfrak{m}N$ is non-zero the coset of m in $M/\mathfrak{m}M$ is non-zero. Hence m is a generator of M. □

We remark that if $L = \mathbf{1}_d$, then $W_{\underline{z}^{\mathbf{1}_d}} = W_{\underline{z}} = (I + (z_1,\dots,z_d))^\perp$. Then $W_{\underline{z}}$ is a non-zero cyclic R-submodule of Γ and denote by $H_{\mathbf{1}_d}$ a generator:

$$W_{\mathbf{1}_d} = \langle H_{\mathbf{1}_d} \rangle.$$

In particular $W_{\underline{z}}$ is the dual of the Artinian reduction $R/I + (\underline{z})$. It is clear that $H_{\mathbf{1}_d}$ depends from the regular sequence $\underline{z}$ we consider. Our goal is to lift the generator $H_{\mathbf{1}_d}$ of $W_{\mathbf{1}_d}$ to a suitable generator H_L of $W_{\underline{z}^L} = (I + (\underline{z}^L))^\perp$, for all $L = (l_1,\dots,l_d) \in \mathbb{N}_+^d$.

Proposition 3.4. *For all $L = (l_1,\dots,l_d) \in \mathbb{N}_+^d$ and for all $i = 1,\dots,d$ such that $l_i \ge 2$, let $H_{L-\gamma_i}$ be a generator of $W_{\underline{z}^{L-\gamma_i}}$. There exists a generator H_L of W_L satisfying*

$$z_i \circ H_L = H_{L-\gamma_i}$$

for all $i = 1,\dots,d$ such that $l_i \ge 2$.

Proof. For every $L \in \mathbb{N}_+^d$ we define $|L|_+$ as the number of positions $i \in \{1,\dots,d\}$ such that $l_i \ge 2$. We proceed by recurrence on the pair $(|L|_+, |L|-(d-|L|_+)) \in \{1,\dots,d\} \times \mathbb{N}$. Notice that $|L| - (d - |L|_+) \ge 2|L|_+$.

Assume that $|L|_+ = 1$. After a permutation, we may assume that $L = (l, 1, \dots, 1)$ with $l \geq 2$. Consider the ideal $J = I + (z_2, \dots, z_d)$; from Proposition 3.2 (i) we get an exact sequence

$$0 \longrightarrow W_{\mathbf{1}_d} \longrightarrow W_L = (J + (z_1^l))^\perp \xrightarrow{z_1 \circ} W_{L-\gamma_1} = (J + (z_1^{l-1}))^\perp \longrightarrow 0.$$

Notice that $|L-\gamma_1|_+ \leq 1$ and that if $|L-\gamma_1|_+ = 1$ then $|L-\gamma_1|-(d-1) = |L|-(d-1)-1$. Hence by induction we know that there exists $H_L \in W_L$ satisfying $z_1 \circ H_L = H_{L-\gamma_1}$ and $W_{L-\gamma_1} = \langle H_{L-\gamma_1} \rangle$. Lemma 3.3 applied to the epimorphism

$$W_L = (J + (z_1^l))^\perp \xrightarrow{x_1 \circ} W_{L-\gamma_1} = (J + (z_1^{l-1}))^\perp \longrightarrow 0$$

with $m = H_L$ gives that H_L is a generator of W_L.

We may assume that $r = |L|_+ \geq 2$. After a permutation we may assume that $L = (l_1, \dots, l_r, 1 \dots, 1)$ with $l_i \geq 2$ for $i = 1, \dots, r$. We set $\underline{z}' = z_1, \dots, z_r$ and $L' = (l_1, \dots, l_r) \in \mathbb{N}_+^r$. Consider the ideal $J = I + (z_{r+1}, \dots, z_d)$; from Proposition 3.2 (ii) we get an exact sequence

$$0 \longrightarrow T_{\mathbf{1}_r} \longrightarrow T_{L'} \xrightarrow{\varphi_{L'}^*} \bigoplus_{k=1}^{r} T_{L'-\gamma_k} \xrightarrow{\phi_{L'}^*} \bigoplus_{1 \leq i < j \leq r} T_{L'-\gamma_i-\gamma_j}$$

where $T_{\mathbf{1}_r} = W_{\mathbf{1}_d}$, $T_{L'} = (J + \underline{z}'^{L'})^\perp = (I + \underline{z}^L)^\perp = W_L$, $T_{L'-\gamma_k} = (J + \underline{z}'^{L'-\gamma_k})^\perp = (I + \underline{z}^{L-\gamma_k})^\perp = W_{L-\gamma_k}$ and $T_{L'-\gamma_i-\gamma_j} = (J + \underline{z}'^{L'-\gamma_i-\gamma_j})^\perp = (I + \underline{z}^{L-\gamma_i-\gamma j})^\perp = W_{L-\gamma_i-\gamma j}$. Hence, by induction, we know that for all $k = 1, \dots, r$ there exists $H_{L-\gamma_k} \in \Gamma$ such that $W_{L-\gamma_k} = \langle H_{L-\gamma_k} \rangle$ and $z_i \circ H_{L-\gamma_k} = H_{L-\gamma_k-\gamma_i}$ for all $i \in \{1, \dots, r\}, i \neq k$. From this we deduce that

$$(H_{L-\gamma_1}, \dots, H_{L-\gamma_r}) \in \operatorname{Ker}(\phi_{L'}^*),$$

from the above exact sequence there exists $H_L \in W_L$ such that $z_k \circ H_L = H_{L-\gamma_k}$ for all $k = 1, \dots, r$. The same argument as before proves that H_L is a generator of W_L. $\square$

Remark 3.5. With the above notation, given two DP-polynomials $H, G \in \Gamma$, we say that G is a primitive of H with respect to $z_1 \in R$ if $z_1 \circ G = H$. From the definition of $\circ$, we will get

$$G = Z_1 H + C$$

for some $C \in \Gamma$ such that $z_1 \circ C = 0$. Remark that $Z_1 H$ denotes the usual multiplication in a polynomial ring and we do not use the internal multiplication in Γ as DP-polynomials. Hence in Proposition 3.4, we will say that H_L is a primitive of $H_{L-\gamma_i}$ with respect to z_i for all $i = 1, \dots, d$.

We prove now the main result of this paper which is an extension to the d-dimensional case of Macaulay's Inverse System correspondence. We give a complete description of the R-submodules of Γ whose annihilator is a d-dimensional Gorenstein local ring. In the Artinian case they are cyclic generated by a polynomial of Γ, in positive dimension the dual modules are not finitely generated and further conditions will be required.

Definition 3.6. Let $d \leq n$ be a positive integer. An R-submodule M of Γ is called G_d-admissible if it admits a system of generators $\{H_L\}_{L\in\mathbb{N}^d_+}$ in $\Gamma = k_{DP}[Z_1,\dots,Z_n]$ satisfying the following conditions

(1) for all $L \in \mathbb{N}^d_+$ and for all $i = 1,\dots,d$

$$z_i \circ H_L = \begin{cases} H_{L-\gamma_i} & \text{if } L-\gamma_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\operatorname{Ann}_R(H_L) \circ H_{L+\gamma_i} = \langle H_{L-(l_i-1)\gamma_i}\rangle$ for all $i = 1,\dots,d$ and $L = (l_1,\cdots,l_d) \in \mathbb{N}^d_+$.

If this is the case, we also say that M is G_d-admissible with respect to the elements $z_1,\dots,z_d \in R$.

Remark 3.7. Given a G_d-admissible set $\{H_L\}_{L\in\mathbb{N}^d_+}$ in Γ, the condition $H_{\mathbf{1}_d} = 0$ is equivalent to the vanishing of the R-module $M = \langle H_L, L \in \mathbb{N}^d_+\rangle$. In fact, assume that $H_{\mathbf{1}_d} = 0$. We proceed by induction on $t = |L|$ where $L \in \mathbb{N}^d_+$. If $t = d$ then $L = \mathbf{1}_d$ and $H_L = 0$ by hypothesis. Assume that $H_L = 0$ for all L with $|L| \leq t$. We only need to prove that $H_{L+\gamma_i} = 0$ for all $i = 1,\cdots,n$. Since $H_L = 0$ we get that $\operatorname{Ann}_R(H_L) = R$. From the condition (ii) of the above definition we get $R \circ H_{L+\gamma_i} = \langle H_{L-(l_i-1)\gamma_i}\rangle = 0$, so $H_{L+\gamma_i} = 0$. The converse is trivial.

Theorem 3.8. *Let $(R,\mathfrak{m})$ be the power series ring and let $d \leq \dim R$ be a positive integer. There is a one-to-one correspondence $\mathcal{C}$ between the following sets:*

(i) *d-dimensional Gorenstein quotients of R,*
(ii) *non-zero G_d-admissible R-submodules $M = \langle H_L, L \in \mathbb{N}^d_+\rangle$ of Γ.*

In particular, given an ideal $I \subset R$ with $A = R/I$ satisfying (i), then

$$\mathcal{C}(A) = I^{\perp} = \langle H_L, L \in \mathbb{N}^d_+\rangle \subset \Gamma \quad \textit{with} \quad \langle H_L\rangle = (I + (\underline{z}^L))^{\perp}$$

is G_d-admissible with respect to a regular linear sequence $\underline{z} = z_1,\dots,z_d$ for R/I. Conversely, given a R-submodule M of Γ satisfying (ii), then

$$\mathcal{C}^{-1}(M) = R/I \quad \textit{with} \quad I = \operatorname{Ann}_R(M) = \bigcap_{L\in\mathbb{N}^d_+} \operatorname{Ann}_R(H_L)$$

is a d-dimensional Gorenstein ring.

Proof. Let $A = R/I$ be a quotient of R satisfying (i) and consider $\underline{z} = z_1, \dots, z_d$ in R a linear regular sequence modulo I. Let $z_1, \dots, z_d, \dots, z_n$ be a minimal system of generators of $\mathfrak{m}$ and let $Z_1, \dots, Z_n$ be the dual base. Let $H_{\mathbf{1}_d}$ be a generator of $W_{\mathbf{1}_d} = (I + (\underline{z}))^\perp$ in $\Gamma = k_{DP}[Z_1, \dots, Z_n]$. Since $d \geq 1$ we have $H_{\mathbf{1}_d} \neq 0$. By Proposition 3.4 there exist elements H_L, $L \in \mathbb{N}^d_+$ in Γ such that $W_{\underline{z}^L} = (I + (\underline{z}^L))^\perp = \langle H_L \rangle$ and by Lemma 3.1 $M = I^\perp = \langle H_L, L \in \mathbb{N}^d_+ \rangle$ satisfies Definition 3.6 (1).

Since $W_{\underline{z}^{L+\gamma_i}} = \langle H_{L+\gamma_i} \rangle$ we have $(I + (\underline{z}^{L+\gamma_i})) \circ H_{L+\gamma_i} = 0$. In particular $I \circ H_{L+\gamma_i} = z_j^{l_j} \circ H_{L+\gamma_i} = 0$ for all $j \in \{1, \dots, d\}$ and $j \neq i$. Hence we get

$$\operatorname{Ann}_R(H_L) \circ H_{L+\gamma_i} = (I + (\underline{z}^L)) \circ H_{L+\gamma_i} = (z_i^{l_i}) \circ H_{L+\gamma_i} = \langle H_{L-(l_i-1)\gamma_i} \rangle.$$

It follows that $M = \langle H_L, L \in \mathbb{N}^d_+ \rangle$ is a R-submodule of Γ which is G_d-admissible and we set $\mathcal{C}(A) = M$. Since $H_{\mathbf{1}_d} \neq 0$ we have that $M \neq 0$.

Conversely, let $M \neq 0$ be a R-submodule of $\Gamma = k_{DP}[Z_1, \dots, Z_n]$ which is G_d-admissible. Hence M admits a system of generators $\{H_L\}_{L \in \mathbb{N}^d_+}$ satisfying the following conditions

(1) For all $L \in \mathbb{N}^d_+$ and for all $i = 1, \dots, d$

$$z_i \circ H_L = \begin{cases} H_{L-\gamma_i} & \text{if } L - \gamma_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\operatorname{Ann}_R(H_L) \circ H_{L+\gamma_i} = \langle H_{L-(l_i-1)\gamma_i} \rangle$ for all $i = 1, \dots, d$ and $L \in \mathbb{N}^d_+$, and $H_{\mathbf{1}_d} \neq 0$, Remark 3.7.

For all $L \in \mathbb{N}^d_+$ we set $W_L := \langle H_L \rangle$ and $I_L := \operatorname{Ann}_R(W_L)$. We define the following ideal of R

$$I := \bigcap_{L \in \mathbb{N}^d_+} I_L,$$

and we prove that R/I is Gorenstein of dimension d.

Claim 1. *For all $L \in \mathbb{N}^d_+$ it holds $I_L \subset I_{L+\mathbf{1}_d} + (\underline{z}^L)$.*

Proof. Notice that it is enough to prove that $I_L \subset I_{L+\gamma_i} + (z_i^{l_i})$ for $i = 1, \dots, d$. In fact, assume that $I_L \subset I_{L+\gamma_i} + (z_i^{l_i})$ for all $i = 1, \dots, d$. Then

$$I \subset I_{L+\gamma_1} + (z_1^{l_1}) \subset I_{L+\gamma_1+\gamma_2} + (z_1^{l_1}, z_2^{l_2}) \subset \dots$$
$$\dots \subset I_{L+\gamma_1+\dots+\gamma_d} + (z_1^{l_1}, \dots, z_d^{l_d}) = I_L \subset I_{L+\mathbf{1}_d} + (\underline{z}^L).$$

We prove now that $I_L \subset I_{L+\gamma_1} + (z_1^{l_1})$. Given $\beta \in I_L = \operatorname{Ann}_R(H_L)$ by (2) there is $\gamma \in R$ such that

$$\beta \circ H_{L+\gamma_1} = \gamma \circ H_{L-(l_1-1)\gamma_1} = \gamma \circ (z_1^{l_1} \circ H_{L+\gamma_1}).$$

From this identity we deduce $\beta - \gamma z_1^{l_1} \in \operatorname{Ann}_R(H_{L+\gamma_1}) = I_{L+\gamma_1}$, so $\beta \in I_{L+\gamma_1} + (z_1^{l_1})$.

Claim 2. *For all* $L \in \mathbb{N}_+^d$ *it holds* $I_L = I + (\underline{z}^L)$.

Proof. By (1) we get $(\underline{z}^L) \circ H_L = 0$, hence $(\underline{z}^L) \subset I_L$. Since $I \subset I_L$ we get the inclusion $I + (\underline{z}^L) \subset I_L$.

Now we prove that $I_L \subset I + (\underline{z}^L)$. Given $\beta_L \in I_L$, by Claim 1 there are $\beta_{L+\mathbf{1}_d} \in I_{L+\mathbf{1}_d}$ and $\omega^0 \in (\underline{z}^L)$ such that

$$\beta_L = \beta_{L+\mathbf{1}_d} + \omega^0.$$

Since $\beta_{L+\mathbf{1}_d} \in I_{L+\mathbf{1}_d}$, by Claim 1 there are $\beta_{L+\mathbf{2}_d} \in I_{L+\mathbf{2}_d}$ and $\omega^1 \in (\underline{z}^{L+\mathbf{1}_d})$ such that

$$\beta_{L+\mathbf{1}_d} = \beta_{L+\mathbf{2}_d} + \omega^1,$$

so $\beta_L = \beta_{L+\mathbf{2}_d} + \omega^0 + \omega^1$. By recurrence there are sequences $\{\beta_{L+t\mathbf{1}_d}\}_{t\geq 0}$ and $\{\omega^t\}_{t\geq 0}$ such that $\beta_{L+\mathbf{t}_d} \in I_{L+\mathbf{t}_d}$, $\omega^t \in (\underline{z}^{L+\mathbf{t}_d})$. For all $t \geq 0$ it holds

$$\beta_L = \beta_{L+\mathbf{t}_d} + \sum_{i=0}^{t} \omega^t.$$

Since $\omega^t \in (\underline{z}^{L+\mathbf{t}_d})$ for all $t \geq 0$ we get that there exists $\overline{\omega} = \sum_{t\geq 0} \omega^t \in k[\![z_1, \cdots, z_n]\!]$. Hence there exists $\overline{\beta} = \lim_{t\to\infty} \beta_{L+\mathbf{t}_d} \in k[\![z_1, \cdots, z_n]\!]$, so

$$\beta_L = \overline{\beta} + \overline{\omega}.$$

Notice that $\overline{\omega} \in (\underline{z}^L)$ and that $\overline{\beta} \in \bigcap_{t\geq 0} I_{L+\mathbf{t}_d} = I$. From this we get that $\beta_L \in I + (\underline{z}^L)$.

If $R = k[z_1, \cdots, z_n]$ then $\overline{\beta} \in I \subset R$. Since $\beta_L \in R$ we get that $\overline{\omega} = \sum_{t\geq 0} \omega^t \in R = k[z_1, \cdots, z_n]$.

Claim 3. $\underline{z}$ *is a regular sequence of* R/I *and* $\dim(R/I) = d$.

Proof. Since $W_{\mathbf{1}_d} = \langle H_{\mathbf{1}_d} \rangle = I_{\mathbf{1}_d}^{\perp}$, by Claim 2 $I_{\mathbf{1}_d}^{\perp} = (I + (\underline{z}))^{\perp}$. By Remark 3.7 we get that $H_{\mathbf{1}_d} \neq 0$ since $M \neq 0$. Hence $R/I + (\underline{z})$ is Artinian and then $\dim(R/I) \leq d$. Next we prove that $\underline{z}$ is a regular sequence modulo I and hence $\dim(R/I) = d$.

First we prove that z_1 is a non-zero divisor of $A = R/I$. By (1) the derivation by z_1 defines an epimorphism of R-modules

$$W_L = \langle H_L \rangle \xrightarrow{z_1 \circ} W_L = \langle H_{L-\gamma_1} \rangle \longrightarrow 0$$

for all $L \geq_d \mathbf{2}_d$. This sequence induces an exact sequence of R-modules

$$0 \longrightarrow \frac{R}{I + (\underline{z}^{L-\gamma_1})} \xrightarrow{.z_1} \frac{R}{I + (\underline{z}^{L})}.$$

Let $a \in R$ such that $z_1 a \in I$. Since $z_1 a \in I + (\underline{z}^L)$ we deduce that $a \in I + (\underline{z}^{L-\gamma_1})$ for all $L \geq_d \mathbf{2}_d$, and we conclude that $a \in I$.

Assume that $z_1, \dots, z_r$, $r < d$, is a regular sequence of R/I. Given $L' = (l_{r+1}, \dots, l_d) \in \mathbb{N}_+^{d-r}$ such that $l_i \geq 2$, for all $i = r+1, \dots, d$, we write $L = (1, \dots, 1, l_{r+1}, \dots, l_d) \in \mathbb{N}_+^d$. By (1) the derivation by z_{r+1} defines an epimorphism of R-modules

$$W_L = \langle H_L \rangle \overset{z_{r+1}\circ}{\longrightarrow} W_L = \langle H_{L-\gamma_{r+1}} \rangle \longrightarrow 0.$$

This sequence induces an exact sequence of R-modules

$$0 \longrightarrow \frac{R}{I + (z_1, \dots, z_r) + (z_{r+1}^{l_r - 1}, \dots, z_d^{l_d})} \xrightarrow{.z_{r+1}} \frac{R}{I + (z_1, \dots, z_r) + (z_{r+1}^{l_r}, \dots, z_d^{l_d})}.$$

Let $a \in R$ such that $z_{r+1} a \in I + (z_1, \dots, z_r)$. Since $z_{r+1} a \in I + (z_1, \dots, z_r) + (z_{r+1}^{l_r}, \dots, z_d^{l_d})$ we deduce that $a \in I + (z_1, \dots, z_r) + (z_{r+1}^{l_r - 1}, \dots, z_d^{l_d})$ for all $L' \geq_r \mathbf{2}_d$. Hence $a \in I + (z_1, \dots, z_r)$, as wanted.

Claim 4. *R/I is Gorenstein.*

Proof. $(I + (\underline{z}))^\perp$ is cyclic, so $R/I + (\underline{z})$ is Gorenstein. Since R/I is Cohen–Macaulay by Claim 3, we have that R/I is a d-dimensional Gorenstein ring.

We define $\mathcal{C}'(M) = R/I$ where $I = \bigcap_{L \in \mathbb{N}_+^d} I_L$ and we prove that $\mathcal{C}$ and $\mathcal{C}'$ are inverse each other. Let $A = R/I$ be a d-dimensional Gorenstein rings $A = R/I$ such that $z_1, \dots, z_d$ is a regular sequence of A. Then

$$\mathcal{C}'\mathcal{C}(A) = R/J$$

where

$$J = \bigcap_{L \in \mathbb{N}_+^d} \mathrm{Ann}_R((I + (\underline{z}^L))^\perp) = \bigcap_{L \in \mathbb{N}_+^d} I + (\underline{z}^L) = I,$$

because $\mathrm{Ann}_R((I + (\underline{z}^L))^\perp) = I + (\underline{z}^L)$. Hence $\mathcal{C}'\mathcal{C}$ is the identity map in the set of rings satisfying (i).

Let $M = \langle H_L; L \in \mathbb{N}_+^d \rangle$ be an R-module satisfying (ii). Then $\mathcal{C}'(M) = R/I$ where $I = \bigcap_{L \in \mathbb{N}_+^d} \mathrm{Ann}_R(H_L)$. By Claim 2 we know $I + (\underline{z}^L) = \mathrm{Ann}_R(H_L)$ so $\mathcal{C}\mathcal{C}'(M) = M$. $\square$

Remark 3.9. Theorem 3.8 can be also applied to standard graded quotients R/I of R where R is the polynomial ring (and I is an homogeneous ideal). In this case the dual R-submodule M of Γ will be generated by homogeneous DP-polynomials H_L. Hence the correspondence will be between d-dimensional Gorenstein graded k-algebras and G_d-admissible homogeneous R-submodules M of Γ.

Remark 3.10. Let $A = R/I$ be a d-dimensional Gorenstein quotient of R and let $\underline{z} = z_1, \dots, z_d \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a linear regular sequence for R/I. By Theorem 3.8, let $M = \langle H_L, L \in \mathbb{N}^d_+ \rangle$ be the dual module. Then by Macaulay's correspondence, the socle degree of the Artinian reduction $A/(\underline{z})A$ coincides with $\deg(H_{\mathbf{1}_d})$. On the other hand we have the following inequality on the multiplicity of A:

$$e_0(A) \leq \mathrm{Length}_R(A/(\underline{z})A) = \dim_k(\langle H_{\mathbf{1}_d} \rangle).$$

The equality holds in the graded case or in the local case if $\underline{z}$ is a superficial sequence of A. If A is a standard graded k-algebra which is Gorenstein (hence Cohen–Macaulay), then the multiplicity and the Castelnuovo Mumford regularity of A coincide with those of any Artinian reduction, in particular $A/(\underline{z})A$. Hence $e(A) = \dim_k(\langle H_{\mathbf{1}_d} \rangle)$ and $\mathrm{reg}(A) = \deg H_{\mathbf{1}_d}$.

According to the previous remark, in the graded case important geometric information such as the multiplicity, the arithmetic genus, or more in general, the Hilbert polynomial of the Gorenstein k-algebra depend only on the choice of $H_{\mathbf{1}_d}$, the first step in the construction of a G_d-admissible dual module, see Example 4.5. We can state the following result which refines Theorem 3.8 in the case of graded k-algebras.

Theorem 3.11. *Let R be the polynomial ring and let $d \leq \dim R$ be a positive integer. There is a one-to-one correspondence $\mathcal{C}$ between the following sets:*

(i) *d-dimensional Gorenstein standard graded k-algebras $A = R/I$ of multiplicity $e = e(A)$ (resp. Castelnuovo–Mumford regularity $r = \mathrm{reg}(A)$)*
(ii) *non-zero G_d-admissible homogeneous R-submodules $M = \langle H_L, L \in \mathbb{N}^d_+ \rangle$ of Γ such that $\dim_k \langle H_{\mathbf{1}_d} \rangle = e$ (resp. $\deg H_{\mathbf{1}_d} = r$)*

The following remarks will be useful in the effective construction of a G_d-admissible R-submodule of Γ.

Remark 3.12. Condition (2) in Definition 3.6 can be replaced by the following:
(2) $\langle H_L \rangle \cap k[Z_1, \dots, \widehat{Z_i}, \dots, Z_n] \subseteq \langle H_{L-(l_i-1)\gamma_i} \rangle$ for all $L \in \mathbb{N}^d_+$ and for all $i = 1, \dots, d$.

Proof. Let $\{H_L\}_{L \in \mathbb{N}^d_+}$ be in Γ satisfying conditions (1) and (2) of Definition 3.6, we prove condition (2) above. Let $\beta_i \in \langle H_L \rangle \cap k[Z_1, \dots, \widehat{Z_i}, \dots, Z_n]$, then there exists $\alpha_i \in R$, such that $\beta_i = \alpha_i \circ H_L \in k[Z_1, \ldots, \widehat{Z_i}, \ldots, Z_n]$ and hence $z_i \circ \beta_i = 0$. Now $\alpha_i \circ H_{L-\gamma_i} = \alpha_i \circ (z_i \circ H_L) = z_i \circ (\alpha_i \circ H_L) = z_i \circ \beta_i = 0$, hence $\alpha_i \in \operatorname{Ann}(H_{L-\gamma_i})$ and by assumption $\beta_i \in \langle H_{L-(l_i-1)\gamma_i} \rangle$.

Conversely assume (2) above. Let $\alpha_i \in \operatorname{Ann}(H_{L-\gamma_i})$ and we prove $\alpha_i \circ H_L \subseteq \langle H_{L-(l_i-1)\gamma_i} \rangle$. We have $z_i \circ (\alpha_i \circ H_L) = \alpha_i \circ (z_i \circ H_L) = \alpha_i \circ H_{L-\gamma_i} = 0$, hence $\alpha_i \circ H_L \in k[Z_1, \ldots, \widehat{Z_i}, \ldots, Z_n]$. It follows $\alpha_i \circ H_L \in \langle H_L \rangle \cap k[Z_1, \ldots, \widehat{Z_i}, \ldots, Z_n] \subseteq \langle H_{L-(l_i-1)\gamma_i} \rangle$, as required. □

Remark 3.13. It is easy to see that condition (2) of Definition 3.6 is equivalent to

$$H_{L+\gamma_i} \in (\operatorname{Ann}_R(H_{L-(l_i-1)\gamma_i}) \operatorname{Ann}_R(H_L))^{\perp}.$$

It is worth mentioning that any G_d-admissible set $\{H_L\}_{L \in \mathbb{N}^d_+}$ with $M = \langle H_L; L \in \mathbb{N}^d_+ \rangle$, satisfies

(i) $M = \langle H_{\mathbf{t}_d}; t \geq n_0 \rangle$ for any integer $n_0 \geq 1$,
(ii) if $L \in \mathbb{N}^d_+$, then $H_{L+\mathbf{1}_d} \in (\operatorname{Ann}_R(H_{\mathbf{1}_d}) \operatorname{Ann}_R(H_L)^d)^{\perp}$.

Proof. (i) Notice that for all $L \in \mathbb{N}^d_+$ there is $t \in \mathbb{N}$ such that $L \leq_d \mathbf{t}_d$, so H_L is determined by $H_{\mathbf{t}_d}$ because $H_L = \underline{z}^{\mathbf{t}_d - L} \circ H_{\mathbf{t}_d}$, see Theorem 3.8.

(ii) Since $L + \mathbf{1}_d = (L + \mathbf{1}_d - \gamma_1) + \gamma_1$, from Theorem 3.8 (ii) we get that

$$\operatorname{Ann}_R(H_L) \circ H_{L+\mathbf{1}_d} = \langle H_{L+\mathbf{1}_d - l_1\gamma_1} \rangle.$$

Since $\sum_{i=1}^d l_i \gamma_i = L$, by recurrence we deduce that

$$\operatorname{Ann}_R(H_L)^d \circ H_{L+\mathbf{1}_d} = \langle H_{L+\mathbf{1}_d - L} \rangle = \langle H_{\mathbf{1}_d} \rangle.$$

From this identity we get (ii). □

Remark 3.14. Let $M = \langle H_L; L \in \mathbb{N}^d_+ \rangle$ be a non-zero R-submodule of Γ which is G_d-admissible with respect to a linear regular sequence $z_1, \ldots, z_d \in R$. Let $Z_1, \ldots, Z_d$ the corresponding dual elements in Γ_1. As a consequence of Definition 3.6, for $t \geq 1$ and in accordance with Remark 3.5, we can write

$$H_{(\mathbf{t+1})_d} = Z_1 \cdots Z_d H_{\mathbf{t}_d} + C_{t+1} = \sum_{i=0}^{t} Z_1^i \cdots Z_d^i C_{t+1-i}$$

where $(z_1 \ldots z_d) \circ C_i = 0$ for all $i = 1, \ldots, t+1$. Notice that by the above remark the diagonal elements $H_{(\mathbf{t+1})_d}$ can describe the module $M = \langle H_L, L \in \mathbb{N}^d_+ \rangle$.

4. Examples and effective constructions

Many interesting questions arise from Theorem 3.8, but the most challenging aim is to deepen the effective aspects. We are interested in the construction of R-submodules M of Γ which are G_d-admissible and to their corresponding Gorenstein k-algebras. We hope that the following examples can suggest interesting refinements of our main result.

The following is a first (trivial) example.

If J is an ideal in $k[\![z_{d+1}, \dots, z_n]\!]$, we say that an ideal $I \subset R = k[\![z_1, \dots, z_d, \dots, z_n]\!]$ is a cone with respect to J if $I = JR$. In the following result denote $S = k[\![z_{d+1}, \dots, z_n]\!]$ and let $Q = k_{DP}[Z_{d+1}, \dots, Z_n]$ be the corresponding dual.

Proposition 4.1. *Given $H \in Q$, consider the following R-submodule of $\Gamma = k_{DP}[z_1, \dots, z_n]$*

$$M = \langle H_L = Z_1^{l_1-1} \dots Z_d^{l_d-1} H \mid L = (l_1, \dots, l_d) \in \mathbb{N}_+^d \rangle.$$

Then $R/\operatorname{Ann}_R(M)$ is a d-dimensional Gorenstein graded k-algebra and $\operatorname{Ann}_R(M)$ is a cone with respect to $J = \operatorname{Ann}_S(H)$.

Proof. We prove that M is G_d-admissible proving that M satisfies Definition 3.6 with respect to the sequence $z_1, \dots, z_d$. It is easy to show that M satisfies condition (1) setting $H_{\mathbf{1}_d} = H$. We prove now that M satisfies also condition (2) of Definition 3.6, that is $\operatorname{Ann}_R(H_L) \circ H_{L+\gamma_i} \subseteq \langle H_{L-(l_i-1)\gamma_i} \rangle$ for all $i = 1, \dots, d$ and $L \in \mathbb{N}_+^d$. First of all we observe that, since $H \in k[Z_{d+1}, \dots, Z_n]$, it is easy to prove that

$$\operatorname{Ann}_R\langle Z_1^{l_1} \dots Z_d^{l_d} H \rangle = \operatorname{Ann}_R\langle Z_1^{l_1} \dots Z_d^{l_d} \rangle + \operatorname{Ann}_S(H)R = (z_1^{l_1}, \dots, z_d^{l_d}) + \operatorname{Ann}_S(H)R$$

Hence

$$\begin{aligned}\operatorname{Ann}_R(H_L) \circ H_{L+\gamma_i} = ((z_1^{l_1}, \dots, z_d^{l_d}) + \operatorname{Ann}_S(H)R) \circ H_{L+\gamma_i} = \\ = (z_1^{l_1}, \dots, z_d^{l_d}) \circ H_{L+\gamma_i} \subseteq \langle H_{L-(l_i-1)\gamma_i} \rangle.\end{aligned}$$

From Theorem 3.8 we know that there exists a d-dimensional Gorenstein local ring $A = R/I$ such that $I^\perp = \langle H_L; L \in \mathbb{N}_+^d \rangle$ and $I = \operatorname{Ann}_R(M) = \bigcap_{L \in \mathbb{N}_+^d} \operatorname{Ann}_R(H_L)$. Hence

$$I = \bigcap_{L \in \mathbb{N}_+^d} ((z_1^{l_1}, \dots, z_d^{l_d}) + \operatorname{Ann}_S(H)R) = \operatorname{Ann}_S(H)R$$

is a cone with respect to $J = \operatorname{Ann}_S(H)$. $\square$

We are interested now in more significant classes of Gorenstein d-dimensional k-algebras. The dual module of a Gorenstein ring of positive dimension is not finitely generated and we would be interested in effective methods determining Gorenstein rings in a finite numbers of steps. This aim motives the following setting.

Let $t_0 \in \mathbb{N}_+$, we say that a family $\mathcal{H} = \{H_L; L \in \mathbb{N}^d_+, |L| \le t_0\}$ of polynomials of Γ is *admissible* if the elements H_L satisfy conditions (1) and (2) of Definition 3.6 up to L such that $|L| \le t_0$.

We recall that, starting from a polynomial $H_{\mathbf{1}_d}$, Remark 3.12 and Remark 3.13 give inductive procedures for constructing an admissible set $\mathcal{H} = \{H_L; L \in \mathbb{N}^d_+, |L| \le t_0\}$ with respect to a sequence of linear elements $\underline{z} = z_1, \cdots, z_d$.

Proposition 4.2. *Let $\mathcal{H} = \{H_L; L \in \mathbb{N}^d_+, |L| \le t_0\}$ be an admissible set of homogenous polynomials with respect to a linear sequence $\underline{z} = z_1, \cdots, z_d$ with $t_0 \ge (r+2)d$ where $r = \deg H_{\mathbf{1}_d}$. Assume there exists a graded Gorenstein k-algebra $A = R/I$ such that $(I + (\underline{z}^L))^\perp = \langle H_L \rangle$ for all $|L| \le t_0$. Then*

$$I = \operatorname{Ann}_R(H_{(\mathbf{r+2})_d})_{\le r+1} R.$$

Proof. It well known that the Castelnuovo–Mumford reg(A) regularity of the Cohen–Macaulay ring $A = R/I$ coincides with regularity of the Artinian reduction $R/\operatorname{Ann}_R(H_{\mathbf{1}_d})$, and hence with its socle degree. Hence we get that

$$\operatorname{reg}(A) = s(R/\operatorname{Ann}_R(H_{\mathbf{1}_d})) = \deg H_{\mathbf{1}_d} = r$$

Now the maximum degree of a minimal system of generators of I is at most $\operatorname{reg}(A) + 1$. From the identity $\operatorname{Ann}_R(H_{(\mathbf{r+2})_d}) = I + (\underline{z}^{r+2})$ we get the claim. □

Remark 4.3. Notice that the last result provide a test for checking if an admissible set of polynomials can be lifted to a full G_d-admissible family in the graded case. We only need to check if the ideal $I = \operatorname{Ann}_R(H_{(\mathbf{r+2})_d})_{\le r+1} R$ is Gorenstein.

It is worth mentioning that if we know the maximum degree of the generators of I (which coincides with those of the $\operatorname{Ann}_R(H_{\mathbf{1_d}})$, say b, then

$$I = \operatorname{Ann}_R(H_{(\mathbf{b+1})_d})_{\le b} R$$

We show now some effective constructions of Gorenstein k-algebras by using the above results and remarks. In the following example we construct a 1-dimensional Gorenstein graded k-algebra of codimension two.

Example 4.4. Consider $R = k[x, y, z]$ and $H_1 = Y^{[3]} - Z^{[3]} \in \Gamma = k_{DP}[X, Y, Z]$. Notice that

$$\operatorname{Ann}_R(H_1) = (x, yz, y^3 + z^3)$$

Our aim is to construct an ideal $I \subseteq R = k[x, y, z]$ such that $A = R/I$ is Gorenstein and $R/I + (x) = R/\operatorname{Ann}_R(H_1)$. We have $r = \deg H_1 = 3$, but from the above equality we deduce the maximum degree of the generators of the ideal I is three. Hence, by Proposition 4.2 and Remark 4.3, a 1-dimensional lift of the Artinian reduction $R/\operatorname{Ann}_R(H_1)$ it is univocally determined by H_4 in an admissible set $\mathcal{H} = \{H_1, \dots, H_4\}$. In particular

$$I = \operatorname{Ann}_R(H_4)_{\leq 3} R.$$

By using Singular one can verify that the following set is admissible:
$\mathcal{H} = \{H_1, H_2 = XH_1 + YZ^{[3]}, H_3 = XH_2 - Y^{[2]}Z^{[3]}, H_4 = XH_3 + Y^{[3]}Z^{[3]} - 4Z^{[6]}\}$ and hence

$$I = (yz + xz, y^3 + z^3 - xy^2 + x^2y - x^3).$$

Notice that $A = R/I$ is a one-dimensional Gorenstein ring of multiplicity 6, x is a linear regular element of A and $R/\operatorname{Ann}_R(H_1)$ is an Artinian reduction of A.

In the following example we construct a two-dimensional Gorenstein k-algebra of codimension three.

Example 4.5. Consider $R = k[x, y, z, t, w]$ and

$$H_{1,1} = X^{[2]} + Y^{[2]} + XZ \in \Gamma = k_{DP}[X, Y, Z, T, W].$$

Our aim is to construct an ideal $I \subseteq R = k[x, y, z, t, w]$ such $A = R/I$ is Gorenstein and $R/I + (t, w) = R/\operatorname{Ann}_R(H_{1,1})$. We have $r = \deg H_{1,1} = 2$. By Proposition 4.2, a 2-dimensional lift of the Artinian reduction $B = R/\operatorname{Ann}_R(H_{1,1})$ it is univocally determined by $H_{4,4}$ in an admissible set $\mathcal{H} = \{H_{1,1}, \dots, H_{4,4}\}$. In particular $I = \operatorname{Ann}_R(H_{4,4})_{\leq 3} R$.

Since the Hilbert function of B is $\{1, 3, 1\}$, by [29, Theorem B] we know that B is the quotient of $k[x, y, z]$ by an ideal minimally generated by 5 forms of degree two. Hence I is minimally generated by 5 forms of degree two and then

$$I = \operatorname{Ann}_R(H_{4,4})_{\leq 2} R = \operatorname{Ann}_R(H_{3,3})_{\leq 2} R.$$

By using Singular one can verify that the following collection of polynomials forms an admissible set:

$$\begin{aligned}
H_{1,1} &= X^{[2]} + Y^{[2]} + XZ, \\
H_{2,2} &= 2X^{[4]} + XY^{[3]} + X^{[2]}YZ + 2Z^{[4]} - X^{[3]}T + Y^{[2]}ZT + XZ^{[2]}T + X^{[3]}W \\
&\quad - X^{[2]}YW - Z^{[3]}W - TWH_{1,1}, \\
H_{3,3} &= X^{[5]}Y + X^{[2]}Y^{[4]} - X^{[5]}Z + X^{[3]}Y^{[2]}Z + Y^{[5]}Z + X^{[4]}Z^{[2]} + XY^{[3]}Z^{[2]} \\
&\quad + Y^{[4]}Z^{[2]} + X^{[2]}YZ^{[3]} + 3Y^{[3]}Z^{[3]} - X^{[2]}Z^{[4]} + 3XYZ^{[4]} - 3Z^{[6]} - 3X^{[5]}T
\end{aligned}$$

$$+ Y^{[5]}T + XY^{[3]}ZT + X^{[2]}YZ^{[2]}T + 3Z^{[5]}T + X^{[4]}T^{[2]} + Y^{[2]}Z^{[2]}T^{[2]}$$
$$+ XZ^{[3]}T^{[2]} - X^{[5]}W + X^{[4]}YW - X^{[3]}Y^2W - Y^{[5]}W - 2X^{[4]}ZW - XY^{[3]}ZW$$
$$- Y^{[4]}ZW - X^{[2]}YZ^{[2]}W - 2Y^{[3]}Z^{[2]}W + X^{[2]}Z^{[3]}W - 2XYZ^{[3]}W + 2Z^{[5]}W$$
$$+ X^{[3]}YW^{[2]} + XY^{[3]}W^{[2]} + Y^{[4]}W^{[4]} - X^{[3]}ZW^{[2]} + X^{[2]}YZW^{[2]} + Y^{[3]}ZW^{[2]}$$
$$- X^{[2]}Z^{[2]}W^{[2]} + XYZ^{[2]}W^{[2]} - Z^{[4]}W^{[2]} - TWH_{2,2}.$$

We are ready now to compute $\mathrm{Ann}_R(H_{3,3})_{\leq 2}R$, hence

$$I = (z^2-xt+zt+zw+tw, yz-t^2+yw, -y^2+xz+t^2, -xy+zt+t^2, x^2-xz-yt+zt-xw+tw).$$

Notice that $A = R/I$ is a two-dimensional Gorenstein ring of multiplicity 5, $\{t, w\}$ is a linear regular sequence of A and B is an Artinian reduction of A. The projective scheme C defined by A is a non-singular arithmetically Gorenstein elliptic curve of $\mathbb{P}^4_k$. Notice that the above generators of I are the Pfaffians of the skew matrix

$$\begin{pmatrix} 0 & -x+t & -t & x & -y \\ x-t & 0 & x & -y & z+t \\ t & -x & 0 & z+w & 0 \\ -x & y & -z-w & 0 & -t \\ y & -z-t & 0 & t & 0 \end{pmatrix}$$

Since the Hilbert function of B is $\{1, 3, 1\}$ we get that the arithmetic genus of C, that coincides with its geometric genus, is $e_1(B)-e_0(B)+1 = 5-5+1 = 1$, where $e_0(B), e_1(B)$ are the Hilbert coefficients of B, see [27].

In the following example we construct a 1-dimensional Gorenstein k-algebra of codimension four.

Example 4.6. Consider $R = k[x, y, z, t, v]$ and $H_1 = X^{[2]} + Y^{[2]} + Z^{[2]} + T^{[2]} \in \Gamma = k_{DP}[X, Y, Z, T, V]$. We have $r = \deg H_1 = 2$. Hence a 1-dimensional lift of the Artinian reduction $R/\mathrm{Ann}_R(H_1)$ it is univocally determined by H_4 in an admissible set $\mathcal{H} = \{H_1, \dots, H_4\}$. In particular

$$I = \mathrm{Ann}_R(H_4)_{\leq 3}R.$$

One can verify that the following collection of polynomials forms an admissible set:
$H_1 = X^{[2]} + Y^{[2]} + Z^{[2]} + T^{[2]}$,
$H_2 = VH_1 + X^{[3]} + Z^{[3]}$,
$H_3 = VH_2 + X^{[4]} + Z^{[4]}$,
$H_4 = VH_3 + X^{[5]} + Z^{[5]}$.

Hence

$$I = (x^2 - z^2 - xv + zv, xy, y^2 - z^2 + zv, xz, yz, z^2 - t^2 - zv, xt, yt, zt).$$

Notice that $A = R/I$ is a one-dimensional Gorenstein ring of multiplicity 6, v is a linear regular element of A and $R/\operatorname{Ann}_R(H_1)$ is a minimal reduction of A with Hilbert function $\{1, 4, 1\}$. In this case A is non-reduced and a minimal prime decomposition is

$$I = (x, y, z - v, t) \cap (x - v, y, z, t) \cap (t^3, t^2 + zv, zt, z^2, yt, yz, y^2 + zv, x - z)$$

with minimal primes

$$(x, y, z - v, t), (x - v, y, z, t), (x, y, z, t);$$

the minimal graded R-free resolution of A is

$$0 \longrightarrow R(-6) \longrightarrow R(-4)^9 \longrightarrow R(-3)^{16} \longrightarrow R(-2)^9 \longrightarrow R \longrightarrow A = R/I \longrightarrow 0.$$

We present now examples in the local (non-homogeneous) case. Possible obstacles to a finite procedure could come in particular when R/I is not algebraic. The ring $A = R/I$ is algebraic if there is an ideal $J \subset T = k[z_1, \dots, z_n]_{(z_1,\dots,z_n)}$ such that A is analytically isomorphic to R/JR, completion of T/J with respect to the $(z_1, \dots, z_n)$-adic topology. If the singularity defined by $A = R/I$ is isolated, then A is algebraic. It was proved by Samuel for hypersurfaces, [28], and, in general, by Artin in [1, Theorem 3.8]. But there are singularities of normal surfaces in $\mathbb{C}^3$ which are not algebraic, [30, Section 14, Example 14.2]. Notice that the ideal defining the above singularity is principal, in particular is Gorenstein.

The following example suggests that in the quasi-homogeneous case, Proposition 4.2 could be still true.

Example 4.7. Consider $R = k[[x, y, z]]$ and we construct a non-homogeneous ideal I in R such that R/I is Gorenstein of dimension 1 and multiplicity 5. By Theorem 3.8 we should exhibit a R-submodule M of $\Gamma = k_{DP}[X, Y, Z]$ which is G_1-admissible. Remark 3.10 suggests to consider a polynomial H_1 such that $\dim_k < H_1 >= 5$. Let $H_1 = Z^{[2]} + Y^{[3]}$ (non-homogeneous, but quasi-homogeneous). In this case $\deg H_1 = 3$ and H_1 is a quasi-homogeneous polynomial. One can verify that

$$\mathcal{H} = \{H_1, H_2 = XH_1, H_3 = X^2H_1, H_4 = X^{[3]}H_1 + Y^{[4]}Z + YZ^{[3]}, H_5 = XH_4\}$$

is an admissible set. In this case is still true that

$$I = \operatorname{Ann}_R(H_5)_{\leq 4}R = (yz - x^3, z^2 - y^3).$$

Notice that the ideal I is the defining ideal in R of the semigroup ring $k[[t^5, t^6, t^9]]$. In particular $A = R/I$ is a domain.

The above example suggests the interesting problem to characterize the generators of the dual module in Theorem 3.8 of a Gorenstein domain.

Next example shows an example where A is a local Gorenstein k-algebra, but the corresponding associated graded ring is not longer Gorenstein.

Example 4.8. Consider $R = k[[x, y, z, t, u, v]]$ and we construct an ideal I in R such that R/I is Gorenstein of dimension 2. By Theorem 3.8 we exhibit a R-submodule M of $\Gamma = k_{DP}[X, Y, Z, T, U, V]$ which is G_2-admissible. Let $H = Z^{[5]} + T^{[4]} + U^{[3]} + W^{[3]} + ZTUV$. One can verify that

$$M = \langle H, F = XYH + U^{[2]}T - WTZ, X^iY^jF,\ i, j \in \mathbb{N}\rangle$$

is G_2-admissible. Then

$$\operatorname{Ann}_R(M) = (z^4 - tuw, t^2w, z^2w, t^2u, t^3 - zuw, zt^2, z^2t, w^2 - ztu, u^2 - tu^2 - ztw - xyzuw).$$

In particular $A = R/\operatorname{Ann}_R(M)$ is a Gorenstein local ring of dimension 2 and of codimension 4. Notice that $gr_{\mathfrak{n}}(A)$ is not Gorenstein because the second difference of the Hilbert function (computed by using Proposition 2.2) is not symmetric.

We end this paper with an example showing that Proposition 4.2 cannot be extended to the local case without a suitable modification.

Example 4.9. For all $n \geq 2$ we consider the one-dimensional local ring $A_n = k[\![x, y]\!]/(f_n)$ with $f_n = y^2 - x^n$. Notice that, for all $n \geq 1$, A_n is algebraic and Gorenstein of multiplicity $e(A_n) = 2$.

On the other hand $A_n/(x) = k[y]/(y^2)$ is an Artinian reduction of A_n, so $H_1 = Y$ and hence $\deg H_1 = 1$. If $n \geq 4$, we cannot recover the ideal (f_n) after $\deg H_1 + 2$ steps as Proposition 4.2 could suggest.

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