Cohen-Macaulay local rings of embedding dimension $e + d - 3$ *

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Abstract

In this paper we determine the possible Hilbert functions of a Cohen-Macaulay local ring of dimension d and multiplicity e, in the case the embedding dimension v satisfies $v = e + d - 3$ and the Cohen-Macaulay type is smaller or equal then $e - 3$.

Introduction

The Hilbert function $H_A(n)$ of a local ring $(A, \mathfrak{m})$ is a good measure of the singularity at $(A, \mathfrak{m})$. We can say that this numerical function describes the degree to which A deviates from a regular local ring, or, equivalently, the associated graded ring $G = \sum_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ deviates from a polynomial ring over the residue field $A/\mathfrak{m}$.

For a given local ring $(A, \mathfrak{m})$ it is often very difficult to compute its Hilbert function. The main idea is to reduce the computation of the Hilbert function of A to the computation of the Hilbert function of a lower dimensional local ring which is a quotient of A modulo an ideal generated by a superficial sequence whose initial forms are a regular sequence in G. Unfortunately, even if A is Cohen-Macaulay, G can have depth zero, which means that its irrelevant maximal ideal $\sum_{n \geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ may consist entirely of zero divisors.

Sometimes the key to whether G has high enough depth lies in the embedding dimension of the local ring. If $(A, \mathfrak{m})$ is a d-dimensional local Cohen-Macaulay ring of multiplicity e, then the embedding dimension v of A satisfies $v \leq e + d - 1$ (see [1]). Here and in the following we let h be the embedding codimension of A, which is

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by definition $h = v - d$. In [12] it was shown that there are precisely three embedding dimensions $v = d, d+1, e+d-1$ which guarantee that G is Cohen-Macaulay for an arbitrary Cohen-Macaulay local ring $(A, \mathfrak{m})$. In each case only a specific Hilbert function is allowed and easily described.

In the case $v = e + d - 2$, in [15] it was shown that G is not necessarily Cohen-Macaulay, the possible exceptions being the local rings of maximal type $e - 2$. In the same paper Sally made the conjecture that in the critical case the depth of G is always at least $d - 1$. This conjecture has been positively solved in [11] and [16] by using deep properties of the Ratliff-Rush filtration of the maximal ideal of A.

Further, in [11], all the possible Hilbert series have been described as follows

$$P_A(z) := \sum_{n \geq 0} H_A(n) z^n = \frac{1 + hz + z^s}{(1-z)^d}$$

where $2 \leq s \leq h + 1$.

Finally in [14] it is shown that for Gorenstein local rings with embedding dimension $v = e + d - 3$, the associated graded ring G is Cohen-Macaulay and its Hilbert series is

$$P_A(z) = \frac{1 + hz + z^2 + z^3}{(1-z)^d}.$$

The proof of this far reaching result is achieved first by reducing, via a standard trick, to the one dimensional case, and then by a very clever analysis of the possible minimal sets of generators of the powers of the maximal ideal of A. Even if this proof is quite computational, we could extract from it some fundamental ideas which are now used in this paper, which can be considered as an extension and completion of Sally's work.

Here we are dealing with a local Cohen-Macaulay ring $(A, \mathfrak{m})$ with embedding dimension $v = e + d - 3$. If we pass to an artinian reduction $B = A/J$, where J is the ideal generated by a maximal superficial sequence in A, we have two possible Hilbert series for B, namely

$$P_B(z) = 1 + hz + 2z^2$$

or

$$P_B(z) = 1 + hz + z^2 + z^3.$$

In the first case we say that A is short, while, in the second case, we say that A is stretched.

If the type $\tau(A)$ of A is bigger than 1, then G is no more Cohen-Macaulay, but if $\tau(A) < h$ we first prove in Theorem 2.3 that the length of the A-module $\mathfrak{m}^2/J\mathfrak{m}$ is at most one. This last condition implies $depth(G) \geq d - 1$, another far reaching result essentially contained in [11] but concretely formulated, at the same time, by Huckaba, Elias, Corso-Polini-Vaz Pinto and Rossi in a series of papers (see [7], [5], [4], [10],) recently appeared.

This reduces the computation of the Hilbert function of A to the computation of the Hilbert function of a local Cohen-Macaulay ring of dimension one and embedding dimension v which satisfies $v = e-2$. Under these assumptions, we prove in Theorem 2.4 that the Hilbert series of A is

$$P_A(z) = \frac{1 + hz + z^t + z^s}{1 - z}$$

where $2 \le t \le s \le \tau + 2$.

Further, if A is stretched, then $t < s$. This gives a fresh and short proof of the result of Sally because, if A is Gorenstein, then A is stretched and $\tau = 1$ so that the theorem implies that

$$P_A(z)(1 - z) = P_B(z),$$

a condition under which G is Cohen-Macaulay.

These results point out the main open question. It is not known for $\tau \ge h$ whether G has depth at least $d - 1$.

The methods we develop here are quite similar to those used in the proof of Sally's conjecture in [11]. The properties of the Ratliff-Rush filtration and the relationships between certain numerical characters of a Cohen-Macaulay local ring of dimension one are illustrated in the first section of the paper and used in the second section where the main result is proved. In the last section we collect some examples to show that the upper bound we have found in Theorem 2.3 is sharp. Other examples are inserted to disprove several conjectures one can made on possible extensions of the results proved here.

Some of the results of this paper have been conjectured after explicit computations performed by the computer algebra system CoCoa ([2]).

1 Preliminaries

Let $(A, \mathfrak{m})$ be a local ring of dimension d, multiplicity e and residue field $k = A/\mathfrak{m}$. The Hilbert function of A is by definition the Hilbert function of the associated graded ring of A which is the homogeneous k-algebra

$$G := gr_{\mathfrak{m}}(A) = \oplus_{n \ge 0}(\mathfrak{m}^n/\mathfrak{m}^{n+1}).$$

Hence

$$H_A(n) = H_G(n) = dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}).$$

The generating function of this numerical function is the power series

$$P_A(z) = \sum_{n \in \mathbf{N}} H_A(n) z^n$$

which is called the Hilbert Series of A. This series is rational and there exists a polynomial $h(z) \in \mathbf{Z}[z]$ such that

$$P_A(z) = \frac{h(z)}{(1-z)^d}$$

where $h(1) = e \geq 1$. We remark that if $dim(A) = 0$, then $e(A) = \lambda(A)$ where λ denotes length.

The polynomial $h(z) = h_0 + h_1 z + \cdots + h_s z^s$ is called the h-polynomial of A.

For every $i \geq 0$, we let

$$e_i := \frac{h^{(i)}(1)}{i!}$$

and

$$\binom{X+i}{i} := \frac{(X+i)\cdots(X+1)}{i!}.$$

Then

$$e_0 = e$$

and the polynomial

$$p_A(X) := \sum_{i=0}^{d-1} (-1)^i e_i \binom{X+d-i-1}{d-i-1}$$

has rational coefficients and degree $d-1$; further for every $n >> 0$

$$p_A(n) = H_A(n).$$

The polynomial $p_A(X)$ is called the Hilbert polynomial of A.

The embedding codimension of A is the integer

$$h := embcod(A) := H_A(1) - d.$$

It is clear that $h = h_1$, the coefficient of z in the h-polynomial of A. Further $embcod(A) = 0$ if and only if A is a regular local ring.

We recall that if A has positive dimension, an element x in $\mathfrak{m}$ is called superficial for A if there exists an integer $c > 0$ such that

$$(\mathfrak{m}^n : x) \cap \mathfrak{m}^c = \mathfrak{m}^{n-1}$$

for every $n > c$.

It is easy to see that a superficial element x is not in $\mathfrak{m}^2$ and that x is superficial for A if and only if $x^* := \overline{x} \in \mathfrak{m}/\mathfrak{m}^2$ does not belong to the relevant associated primes of G. Hence, if the residue field is infinite, superficial elements always exist.

Further if A has positive depth, every superficial element is also a regular element in A.

A sequence $x_1, \dots, x_r$ in the local ring $(A, \mathfrak{m})$ is called a superficial sequence for A, if x_1 is superficial for A and $\overline{x_i}$ is superficial for $A/(x_1, \dots, x_{i-1})$ for $2 \le i \le r$.

It is clear that if J is generated by a superficial sequence, then

$$J \cap \mathfrak{m}^2 = J\mathfrak{m}.$$

By passing, if needed, to the local ring $A[X]_{(\mathfrak{m},X)}$ we may assume that the residue field is infinite. Hence if $depth(A) \ge r$, every superficial sequence $x_1, \dots, x_r$ is also a regular sequence in A. Such a sequence has the right properties for a good behaviour of the numerical invariants under reduction modulo the ideal it generates.

In particular if $J = (x_1, \dots, x_r)$, and $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$, then B is a local ring with

- $dim(B) = d - r$,
- If $depth(A) \ge r$, then $depth(B) = depth(A) - r$,
- $embcod(A) = embcod(B)$,
- $e_i(A) = e_i(B)$ for $i = 0, \dots, d - r$.
- $e(A) = e(B) = \lambda(A/J)$.

The following relevant property of superficial sequences will also be needed.

- $depth(gr_{\mathfrak{m}}(A)) \ge r \iff x_1^*, \dots, x_r^*$ is a regular sequence in $gr_{\mathfrak{m}}(A) \iff$
$P_A(z) = \frac{P_B(z)}{(1-z)^r} \iff \mathfrak{m}^j \cap J = J\mathfrak{m}^{j-1}$ for every $j \ge 1$.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. For every n we consider the chain of ideals

$$\mathfrak{m}^n \subseteq \mathfrak{m}^{n+1} : \mathfrak{m} \subseteq \mathfrak{m}^{n+2} : \mathfrak{m}^2 \subseteq \cdots \subseteq \mathfrak{m}^{n+k} : \mathfrak{m}^k \subseteq \cdots$$

This chain stabilizes at an ideal which was denoted by Ratliff and Rush in [9] as

$$\widetilde{\mathfrak{m}^n} := \bigcup_{k \ge 1} (\mathfrak{m}^{n+k} : \mathfrak{m}^k).$$

We have

$$\widetilde{\mathfrak{m}} = \mathfrak{m},$$

and for every i, j

$$\mathfrak{m}^i \subseteq \widetilde{\mathfrak{m}^i},$$
$$\widetilde{\mathfrak{m}^i}\widetilde{\mathfrak{m}^j} \subseteq \widetilde{\mathfrak{m}^{i+j}}.$$

Further, if x is superficial for A,

$$\widetilde{\mathfrak{m}^{n+1}} : x = \widetilde{\mathfrak{m}^n}$$

for every $n \geq 0$.

In the following we let J be the ideal generated by a maximal superficial sequence $(x_1, \ldots, x_d)$ and define for every $n \geq 0$

$$\rho_n := \lambda(\widetilde{\mathfrak{m}^{n+1}}/J\widetilde{\mathfrak{m}^n}).$$

For example we have

$$\rho_0 = \lambda(\tilde{\mathfrak{m}}/J) = \lambda(\mathfrak{m}/J) = \lambda(A/J) - \lambda(A/\mathfrak{m}) = e - 1. \tag{1}$$

It is natural to introduce a new set of numerical invariants, namely to let for every $n \geq 0$

$$a_n := \lambda(\widetilde{\mathfrak{m}^n}/\mathfrak{m}^n).$$

In particular we have

$$a_0 = a_1 = 0.$$

Finally we define also for every $n \geq 0$ the integers

$$v_n := \lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n).$$

It is clear that

$$v_0 = \lambda(\mathfrak{m}/J) = e - 1 \tag{2}$$

and

$$v_1 = \lambda(\mathfrak{m}^2/J\mathfrak{m}) = \lambda(A/J\mathfrak{m}) - \lambda(A/\mathfrak{m}^2) = e + d - (1 + h + d) = e - h - 1. \tag{3}$$

It is clear that if $v_j = 0$ for some positive integer j, then $v_t = 0$ for every $t \geq j$. On the other hand, if $v_j = 1$ for some positive integer j, then $\mathfrak{m}^{j+1}/J\mathfrak{m}^j$ is a k-vector space of dimension one , so that

$$\mathfrak{m}^{j+2} \subseteq J\mathfrak{m}^j$$

and

$$\mathfrak{m}^{j+1} = J\mathfrak{m}^j + (ab)$$

for some $a \in \mathfrak{m}$ and $b \in \mathfrak{m}^j$, $ab \notin J\mathfrak{m}^j$. It follows that

$$\mathfrak{m}^{j+2} = J\mathfrak{m}^{j+1} + ab\mathfrak{m} \subseteq J\mathfrak{m}^{j+1} + a\mathfrak{m}^{j+1} = J\mathfrak{m}^{j+1} + (a^2 b) \subseteq \mathfrak{m}^{j+2}.$$

Hence, if $v_j = 1$, for every $t \geq j$ we can find suitable elements $a \in \mathfrak{m}$, $b \in \mathfrak{m}^j$ such that

$$\mathfrak{m}^{t+1} = J\mathfrak{m}^t + (a^{t+1-j} b) \tag{4}$$

It follows that if $v_j = 1$, then $v_t \leq 1$ for every $t \geq j$.

When the ring A has dimension one, we have more relevant properties of the integers already introduced. Hence, from now on, we are assuming $d = 1$ and we let $J = (x)$ where x is a superficial element of the local Cohen-Macaulay ring A. Further we let

$$h_A(z) = h_0 + h_1 z + \cdots + h_s z^s$$

where $h_0 \neq 0$. Hence s is the degree of the h-polynomial of A. We clearly have $h_0 = 1$, $h_1 = h$ and

$$h_i = H_A(i) - H_A(i-1) \tag{5}$$

for every $i \geq 1$ so that

$$e = \sum_{i=0}^{s} h_i = H_A(s). \tag{6}$$

It is well known that for every $j \geq 0$

$$e = H_A(j) + v_j. \tag{7}$$

Further, since x is a regular element in A, from the diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{m}^{j+1}} & \supset & x\widetilde{\mathfrak{m}^j} \\ \cup & & \cup \\ \mathfrak{m}^{j+1} & \supset & x\mathfrak{m}^j \end{array}$$

we easily get for every $j \geq 0$,

$$\rho_j + a_j = v_j + a_{j+1}. \tag{8}$$

Since we have seen that $e = H_A(s)$, from (7) we get $v_s = 0$. On the other hand from (5) and (7) we get

$$h_i = v_{i-1} - v_i \tag{9}$$

for every $i \geq 1$, hence $h_s = v_{s-1} \neq 0$. It follows that

$$\begin{cases} v_j > 0 & \text{if } \ j \leq s-1 \\ v_j = 0 & \text{if } \ j \geq s \end{cases} \tag{10}$$

and

$$e_1 = \sum_{j=0}^{s} j h_j = \sum_{j=0}^{s-1} v_j. \tag{11}$$

A final property which will be used later is the following upper bound for the degree of the h-polynomial in the one-dimensional case, bound which is a consequence of a well known result of Herzog and Waldi (see [6]):

$$s \leq e - 1. \tag{12}$$

We want now to describe the components of the Ratliff-Rush filtration in the one dimensional case.

Lemma 1.1 *Let $d = 1$. For every $t \geq 0$ and $j \leq s$ we have*

$$\mathfrak{m}^{s+t} = x^t \mathfrak{m}^s$$

and

$$\mathfrak{m}^{j+t} : x^t \subseteq \mathfrak{m}^s : \mathfrak{m}^{s-j}.$$

Proof. Since $H_A(s) = e$, by (7) we get $v_s = 0$, hence $\mathfrak{m}^{s+1} = x\mathfrak{m}^s$ and the first assertion follows by multiplication by $\mathfrak{m}$.

If now $ax^t \in \mathfrak{m}^{j+t}$, then

$$ax^t \mathfrak{m}^{s-j} \subseteq \mathfrak{m}^{t+s} = x^t \mathfrak{m}^s$$

so that $a \in \mathfrak{m}^s : \mathfrak{m}^{s-j}$. □

Proposition 1.2 *Let $d = 1$. Then we have:*

$$\widetilde{\mathfrak{m}^j} = \begin{cases} \mathfrak{m}^s : x^{s-j} = \mathfrak{m}^s : \mathfrak{m}^{s-j} & \text{if } j \leq s \\ \mathfrak{m}^j & \text{if } j \geq s \end{cases}$$

Proof. For some $t \geq 0$ we have

$$\widetilde{\mathfrak{m}^j} = \mathfrak{m}^{j+t} : \mathfrak{m}^t \subseteq \mathfrak{m}^{j+t} : x^t.$$

If $j \leq s$ we can use the above lemma and we get

$$\mathfrak{m}^{j+t} : x^t \subseteq \mathfrak{m}^s : \mathfrak{m}^{s-j} \subseteq \mathfrak{m}^s : x^{s-j} \subseteq \mathfrak{m}^{2s} : x^{s-j}\mathfrak{m}^s = \mathfrak{m}^{2s} : \mathfrak{m}^{2s-j} \subseteq \widetilde{\mathfrak{m}^j}.$$

If instead $j \geq s$, we get

$$\mathfrak{m}^{j+t} : x^t = x^{j+t-s}\mathfrak{m}^s : x^t \subseteq x^{j-s}\mathfrak{m}^s \subseteq \mathfrak{m}^j \subseteq \widetilde{\mathfrak{m}^j}.$$

In both cases the conclusion follows. □

As a consequence of the above proposition we get

$$a_j = 0 \text{ for every } j \geq s. \tag{13}$$

By using (7) and (8), we easily get the well known formula (see [8])

$$e_1 = \sum_{j=0}^{s-1} v_j = \sum_{j=0}^{s-1} \rho_j \tag{14}$$

We will need in the following a suitable lower bound for the integers a_j which will be a consequence of the subsequent result.

Lemma 1.3 *Let $d = 1$. For every $k \geq 0$ and $1 \leq p \leq s-1$ we have*

$$\mathfrak{m}^p \not\subset x\mathfrak{m}^{p-1} + (\mathfrak{m}^{p+k+1} : x^k).$$

Proof. If $\mathfrak{m}^p \subset x\mathfrak{m}^{p-1} + (\mathfrak{m}^{p+k+1} : x^k)$, by using the preceding lemma we get

$$\mathfrak{m}^p \subseteq x\mathfrak{m}^{p-1} + (\mathfrak{m}^s : \mathfrak{m}^{s-p-1})$$

so that

$$\mathfrak{m}^s = \mathfrak{m}^{s-p}\mathfrak{m}^p \subseteq \mathfrak{m}^{s-p}(x\mathfrak{m}^{p-1} + (\mathfrak{m}^s : \mathfrak{m}^{s-p-1})) \subseteq x\mathfrak{m}^{s-1} + \mathfrak{m}^{s+1}$$

which, by Nakayama Lemma, implies $\mathfrak{m}^s = x\mathfrak{m}^{s-1}$. This means $v_{s-1} = 0$, which is a contradiction. □

Proposition 1.4 *Let $d = 1$ and j, t be non negative integers such that $j + t \leq s$. Then*

$$a_j \geq H_A(j-1) - H_A(j+t-1) + t.$$

Proof. We have an exact sequence

$$0 \to \mathfrak{m}^{j-1} \cap (\mathfrak{m}^{j+t} : x^t)/\mathfrak{m}^j \to \mathfrak{m}^{j-1}/\mathfrak{m}^j \xrightarrow{x^t} \mathfrak{m}^{j+t-1}/\mathfrak{m}^{j+t} \to \mathfrak{m}^{j+t-1}/x^t\mathfrak{m}^{j-1} + \mathfrak{m}^{j+t} \to 0.$$

Since by Lemma 1.1

$$\mathfrak{m}^{j+t} : x^t \subseteq \mathfrak{m}^s : \mathfrak{m}^{s-j} \subseteq \widetilde{\mathfrak{m}^j},$$

we get

$$a_j \geq H_A(j-1) - H_A(j+t-1) + \lambda(\mathfrak{m}^{j+t-1}/x^t\mathfrak{m}^{j-1} + \mathfrak{m}^{j+t}).$$

Now, if $t = 0$, we have nothing to prove. If $t \geq 1$, we have

$$\mathfrak{m}^{j+t-1} \supseteq \mathfrak{m}^{j+t} + x\mathfrak{m}^{j+t-2} \supseteq \mathfrak{m}^{j+t} + x^2\mathfrak{m}^{j+t-3} \supseteq \cdots \supseteq \mathfrak{m}^{j+t} + x^t\mathfrak{m}^{j-1}.$$

Since the length of this chain is t, the conclusion follows if we are able to prove that all the inclusions are strict. But if we would have

$$\mathfrak{m}^{j+t} + x^k\mathfrak{m}^{j+t-1-k} = \mathfrak{m}^{j+t} + x^{k+1}\mathfrak{m}^{j+t-2-k}$$

for some $0 \leq k \leq t-1$, then

$$\mathfrak{m}^{j+t-1-k} \subseteq x\mathfrak{m}^{j+t-2-k} + (\mathfrak{m}^{j+t} : x^k).$$

We get a contradiction by using the above lemma after observing that if we let $p = j+t-1-k$, then, since $k+2 \leq j+t \leq s$, we get $1 \leq p = j+t-1-k \leq s-1$. $\square$

By using this result we get a rough but useful upper bound for the degree of the h-polynomial of A in term of the Cohen-Macaulay type $\tau(A)$ of A which is by definition

$$\tau(A) = dim(J : \mathfrak{m}/J).$$

Proposition 1.5 *Let $d = 1$. Then*

$$s \leq 1 + \tau(A) + \lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/x : \mathfrak{m}).$$

Proof. From the diagram

$$\begin{array}{ccccc} x : \mathfrak{m} + \widetilde{\mathfrak{m}^2} & \supset & \widetilde{\mathfrak{m}^2} & \supset & \mathfrak{m}^2 \\ \cup & & & & \cup \\ x : \mathfrak{m} & \supset & xA & \supset & x\mathfrak{m} \end{array}$$

we get

$$1 + \tau(A) + \lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/x : \mathfrak{m}) = \lambda(\mathfrak{m}^2/x\mathfrak{m}) + a_2 + \lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/\widetilde{\mathfrak{m}^2}).$$

By (3)

$$\lambda(\mathfrak{m}^2/x\mathfrak{m}) = e - h - 1$$

while by the above proposition we get

$$a_2 \geq H_A(1) - H_A(s-1) + s - 2 \geq h - e - s.$$

Since $x \in x : \mathfrak{m}$, $x \notin \widetilde{\mathfrak{m}^2}$ we also have

$$\lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/\widetilde{\mathfrak{m}^2}) \geq 1.$$

Hence

$$1 + \tau(A) + \lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/x : \mathfrak{m}) \geq e - h - 1 + h - e - s + 1 = s,$$

as wanted. $\square$

We end this section with a result which will be used in the proof of the main theorem. We need a preliminary Lemma which uses the main idea of the proof of the classical Cayley-Hamilton theorem.

Lemma 1.6 *Let $I = (x_1, \ldots, x_v)$ and L be ideals of the local ring $(A, \mathfrak{m})$. Let us assume that there exist elements $a, x \in \mathfrak{m}$ such that*

$$aI \subseteq xI + L.$$

Then there exists $\sigma \in x\mathfrak{m}^{v-1}$ such that for every $b \in I$ we have

$$(a^v - \sigma)b \in L\mathfrak{m}^{v-1}.$$

Proof. For every $j = 1, \dots, v$ we can find elements $\lambda_{ij} \in A$, $c_j \in L$, such that

$$ax_j + \sum_{i=1}^{v} \lambda_{ij} x x_i = c_j.$$

We get a system of v equations in v variables

$$(x_1, \dots, x_v) M = (c_1, \dots, c_v)$$

where M is the matrix $M = (\delta_{ij} a + \lambda_{ij} x)$. If we multiply on the right by the adjoint of M, whose entries are elements of $\mathfrak{m}^{v-1}$, we get

$$det(M) x_i \in L\mathfrak{m}^{v-1}$$

for every $i = 1, \dots, v$. Since clearly $det(M) = a^v - \sigma$ with $\sigma \in x\mathfrak{m}^{v-1}$, the conclusion follows. $\square$

Theorem 1.7 *Let $i \geq 0$ be an integer and I and K ideals of the local ring $(A, \mathfrak{m})$. Let us assume that I can be generated by v elements and there exist $a, x \in \mathfrak{m}$, $c \in A$, $z \in K$ such that*

1. $\mathfrak{m}^{i+1} = x\mathfrak{m}^i + (acz)$.

2. $az \in xK + I\mathfrak{m}$.

3. $aI \subseteq x(I + K)$.

4. $cK \subseteq \mathfrak{m}^i$.

Then

$$\mathfrak{m}^{v+i+1} = x\mathfrak{m}^{v+i}.$$

Proof. By 2. we can write

$$az = xr + s$$

with $r \in K$ and $s = \sum p_i w_i$, $p_i \in \mathfrak{m}$, $w_i \in I$. Since by 4. $cr \in \mathfrak{m}^i$ and by 1. $acz = xcr + cs \in \mathfrak{m}^{i+1}$, we get

$$cs \in \mathfrak{m}^{i+1}.$$

By 3. we have

$$aI \subseteq xI + xK,$$

hence by the above Lemma we can find $\sigma \in x\mathfrak{m}^{v-1}$ such that

$$w(a^v - \sigma) \in xK\mathfrak{m}^{v-1}$$

for every $w \in I$. It follows that there exist $f_i \in xK\mathfrak{m}^{v-1}$ such that

$$a^v s = \sum p_i (a^v w_i) = \sum p_i (\sigma w_i + f_i) = \sigma s + \sum p_i f_i = \sigma s + g$$

with $g \in xK\mathfrak{m}^v$. From this we get

$$a^{v+1} cz = a^v c(az) = a^v c(xr + s) = c(a^v s) + xa^v (cr) = \sigma cs + cg + xa^v cr.$$

Since
$$cs \in \mathfrak{m}^{i+1}, \quad \sigma \in x\mathfrak{m}^{v-1},$$
we get
$$\sigma cs \in x\mathfrak{m}^{v+i}.$$

On the other hand, since
$$cK \subseteq \mathfrak{m}^i, \quad g \in xK\mathfrak{m}^v,$$
we get
$$cg \in x(cK)\mathfrak{m}^v \subseteq x\mathfrak{m}^{v+i}$$
and
$$xa^v cr \in x\mathfrak{m}^{v+i}.$$
This implies
$$a^{v+1}cz \in x\mathfrak{m}^{v+i}.$$
But, as in (4), by 1. we get
$$\mathfrak{m}^{v+i+1} = x\mathfrak{m}^{v+i} + (a^{v+1}cz).$$
The conclusion follows. $\square$

2 The Hilbert Function

In this section $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of multiplicity $e = h + 3$. As usual we denote by J the ideal generated by a maximal superficial sequence $(x_1, \ldots, x_d)$ and we let $B = A/J$. The h-polynomial of the artinian local ring B is either
$$h_B(z) = 1 + hz + z^2 + z^3$$
or
$$h_B(z) = 1 + hz + 2z^2.$$
In the first case, following [13], we say that A is <u>stretched</u> and we have $\mathfrak{m}^3 \not\subseteq J\mathfrak{m}$ and $\mathfrak{m}^4 \subseteq J\mathfrak{m}$; in the second case, following [3] we say that A is <u>short</u> and we have $\mathfrak{m}^3 \subseteq J\mathfrak{m}$. In both case we have the following useful structure of the square of the maximal ideal of A.

Proposition 2.1 *Let $e = h + 3$. Then*
$$\mathfrak{m}^2 = \begin{cases} J\mathfrak{m} + (cf), & c, f \in \mathfrak{m},\ c, f \notin J \quad \textit{if } A \textit{ is stretched.} \\ J\mathfrak{m} + (cf, cg), & c, f, g \in \mathfrak{m},\ c, f, g \notin J \quad \textit{if } A \textit{ is short.} \end{cases}$$

Proof. If A is stretched, then we have

$$H_{A/J}(2) = 1 = dim(\mathfrak{m}^2/\mathfrak{m}^3 + (J \cap \mathfrak{m}^2)) = dim(\mathfrak{m}^2/\mathfrak{m}^3 + J\mathfrak{m}).$$

Hence

$$\mathfrak{m}^2 = \mathfrak{m}^3 + J\mathfrak{m} + (cf)$$

with $c, f \in \mathfrak{m}$, $c, f \notin J$. The conclusion follows by Nakayama Lemma.

If A is short, then $\mathfrak{m}^3 \subseteq J\mathfrak{m}$ so that $\mathfrak{m}^2/J\mathfrak{m}$ is a k-vector space of dimension 2 by (3). This implies

$$\mathfrak{m}^2 = J\mathfrak{m} + (xy, zt)$$

for suitable $x, y, z, t \in \mathfrak{m}$, $x, y, z, t \notin J$. Now, if $yz \notin J\mathfrak{m}$, then either $\mathfrak{m}^2 = J\mathfrak{m} + (xy, yz)$ or $\mathfrak{m}^2 = J\mathfrak{m} + (zt, yz)$ and the conclusion follows. If $xt \notin J\mathfrak{m}$, then either $\mathfrak{m}^2 = J\mathfrak{m} + (xy, xt)$ or $\mathfrak{m}^2 = J\mathfrak{m} + (zt, xt)$ and again we are through. Finally if $yz, xt \in J\mathfrak{m}$, then

$$\mathfrak{m}^2 = J\mathfrak{m} + (xy, zt) = J\mathfrak{m} + ((x + z)y, (x + z)t)$$

and the conclusion follows as well. $\square$

After the above result has been proved, we clearly need a deeper investigation of Cohen-Macaulay local rings $(A, \mathfrak{m})$ with the property that $\mathfrak{m}^2 = J\mathfrak{m} + cI$. In the next proposition we do not use the assumption $e = h + 3$.

Proposition 2.2 *Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring such that $\mathfrak{m}^2 = J\mathfrak{m}+cI$ for some $c \in \mathfrak{m}$, $c \notin J$ and some ideal $I \subseteq \mathfrak{m}$. Then we have:*

i) $\mathfrak{m} = I + (J : c)$.

ii) $\mathfrak{m}^{n+1} = J\mathfrak{m}^n + c^n I$ *for every* $n \geq 1$.

iii) $v_{n+1} \leq v_n$ *for every* $n \geq 0$.

iv) If $J : \mathfrak{m} \neq J : c$, *then* $v_2 < v_1$.

v) If $dim(A) = 1$, *then* $H_A(n) \leq H_A(n + 1)$ *for every* $n \geq 0$.

vi) If $I = (d)$, *then* $v_n \leq v_{n+1} + 1$ *for every* $n \geq 1$.

Proof. i) We have $c\mathfrak{m} \subseteq \mathfrak{m}^2 = J\mathfrak{m} + cI$ hence

$$\mathfrak{m} \subseteq I + (J\mathfrak{m} : c) \subseteq I + (J : c) \subseteq \mathfrak{m}.$$

ii) If $n = 1$ there is nothing to prove. We make induction on n. Let $n \geq 2$ and $\mathfrak{m}^n = J\mathfrak{m}^{n-1} + c^{n-1}I$. Then

$$\mathfrak{m}^{n+1} = J\mathfrak{m}^n + c^{n-1}I\mathfrak{m} \subseteq J\mathfrak{m}^n + c\mathfrak{m}^n \subseteq J\mathfrak{m}^n + c(J\mathfrak{m}^{n-1} + c^{n-1}I) =$$

$$= J\mathfrak{m}^n + c^n I \subseteq \mathfrak{m}^{n+1}.$$

iii) We have a canonical map

$$\mathfrak{m}^{n+1}/J\mathfrak{m}^n \to \mathfrak{m}^{n+2}/J\mathfrak{m}^{n+1}$$

given by the multiplication by c. By ii) this map is surjective and the conclusion follows.

iv) If $a \in J : c$, $a \notin J : \mathfrak{m}$, let $b \in \mathfrak{m}$ such that $ab \notin J$. Then $ac \in J \cap \mathfrak{m}^2 = J\mathfrak{m}$ so that $\overline{ab} \in \mathfrak{m}^2/J\mathfrak{m}$ is a non zero element of the kernel. The conclusion follows.

v) By iii) we have $v_{n+1} \le v_n$ for every $n \ge 0$. If $dim(A) = 1$, the conclusion follows by (7).

vi) We must prove that the kernel of the map as in iii) has length smaller or equal than one. But this kernel is the module

$$\mathfrak{m}^{n+1} \cap (J\mathfrak{m}^{n+1} : c)/J\mathfrak{m}^n.$$

Since for $t >> 0$ we have $\mathfrak{m}^t \subseteq J$, we get $\mathfrak{m}^{n+1+t} \subseteq J\mathfrak{m}^{n+1}$; on the other hand $\mathfrak{m}^{n+1} \not\subset J\mathfrak{m}^{n+1}$, hence there exists an integer $r > n$ such that

$$\mathfrak{m}^r \not\subset J\mathfrak{m}^{n+1}, \quad \mathfrak{m}^{r+1} \subseteq J\mathfrak{m}^{n+1}.$$

We claim that

$$\mathfrak{m}^{n+1} \cap (J\mathfrak{m}^{n+1} : c) = J\mathfrak{m}^n + \mathfrak{m}^r.$$

The claim gives the conclusion because $J\mathfrak{m}^n + \mathfrak{m}^r/J\mathfrak{m}^n$ is a k-vector space generated by $\overline{c^{r-1}d}$.

Let us prove the claim. If $r = n+1$, then $\mathfrak{m}^{n+2} \subseteq J\mathfrak{m}^{n+1}$ so that

$$\mathfrak{m}^{n+1} \subseteq J\mathfrak{m}^{n+1} : c$$

and we are done. Let $r \ge n+2$, and $a \in \mathfrak{m}^{n+1} \cap (J\mathfrak{m}^{n+1} : c)$. Then $a \in J\mathfrak{m}^n + \mathfrak{m}^{n+1}$ and if, by contradiction, $a \notin J\mathfrak{m}^n + \mathfrak{m}^r$, we could find an integer t such that $n+1 \le t \le r-1$ and

$$a \in J\mathfrak{m}^n + \mathfrak{m}^t, \quad a \notin J\mathfrak{m}^n + \mathfrak{m}^{t+1}.$$

This implies $a = bc^{t-1}d + w$ where $b \notin \mathfrak{m}$ and $w \in J\mathfrak{m}^n$. Hence $c^{t-1}d \in J\mathfrak{m}^{n+1} : c$ so that $\mathfrak{m}^r \subseteq \mathfrak{m}^{t+1} \subseteq J\mathfrak{m}^{n+1}$, a contradiction. □

A recent result found independently by Huckaba, Elias, Corso-Polini-Vaz Pinto, Rossi (see [7], [5], [4], [10],) says that the associated graded ring G of any Cohen-Macaulay local ring A has depth at least $d-1$ if $v_2 \le 1$. When $e = h+3$, this happens if the Cohen-Macaulay type of A, which will be denoted simply by τ, is not too big.

Theorem 2.3 *Let $e = h+3$. If $\tau < h$, then $v_2 \le 1$.*

Proof. By Proposition 2.1 we may write

$$\mathfrak{m}^2 = J\mathfrak{m} + cI$$

with $c \notin J$. Since by (3) $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = 2$, we get the conclusion by iv) in Proposition 2.2 if we can prove that $J : \mathfrak{m} \neq J : c$. We have by i) of the same proposition

$$\mathfrak{m} = I + (J : c),$$

hence, if by contradiction $J : \mathfrak{m} = J : c$, then

$$J \subseteq J : \mathfrak{m} \subseteq (J : \mathfrak{m}) + I = (J : c) + I = \mathfrak{m}.$$

If we can prove that

$$\lambda(J : \mathfrak{m} + I/J : \mathfrak{m}) \leq 2$$

then we get

$$h + 2 = \lambda(\mathfrak{m}/J) \leq \tau + 2,$$

a contradiction. Now if A is stretched, then by Proposition 2.1, $I = (t)$ and we have a filtration

$$J : \mathfrak{m} \subseteq (J : \mathfrak{m}) + (t^2) \subseteq (J : \mathfrak{m}) + (t) = (J : \mathfrak{m}) + I.$$

Since $t^2\mathfrak{m} \subseteq \mathfrak{m}^3 \subseteq J : \mathfrak{m}$ and $t\mathfrak{m} = t((t) + (J : c)) \subseteq (J : \mathfrak{m}) + (t^2)$,

$$(J : \mathfrak{m}) + (t^2)/(J : \mathfrak{m}) \text{ and } (J : \mathfrak{m}) + (t)/(J : \mathfrak{m}) + (t^2)$$

are k-vector spaces of dimension ≤ 1, hence

$$\lambda(J : \mathfrak{m} + I/J : \mathfrak{m}) \leq 2.$$

If A is short, then by Proposition 2.1 I is a two generated ideal; hence, since $I\mathfrak{m} \subseteq \mathfrak{m}^2 \subseteq J : \mathfrak{m}$, $J : \mathfrak{m} + I/J : \mathfrak{m}$ is a k-vector space of dimension

$$\lambda(J : \mathfrak{m} + I/J : \mathfrak{m}) \leq 2.$$

□

We come now to the main results of the paper concerning the Hilbert Function of a local Cohen-Macaulay ring with $e = h + 3$.

We want to prove the following theorems:

Theorem 2.4 *Let $e = h + 3$ and $d = 1$. Then*

a) $h_A(z) = 1 + hz + z^t + z^s$ with $2 \leq t \leq s \leq \tau + 2$.

b) If $v_2 = 0$ then G is Cohen-Macaulay and $s = t = 2$.

c) If $v_2 = 1$ then $t = 2$.

d) If A is stretched then $t < s$.

Theorem 2.5 *Let $e = h + 3$. Then*

a) If $v_2 = 0$ then G is Cohen-Macaulay and $h_A(z) = 1 + hz + 2z^2$.

b) If $v_2 = 1$ then $depth(G) \geq d - 1$ and $h_A(z) = 1 + hz + z^2 + z^s$ where

$$\begin{cases} 3 \leq s \leq \tau + 2 & \textit{if } A \textit{ is stretched.} \\ 2 \leq s \leq \tau + 2 & \textit{if } A \textit{ is short.} \end{cases}$$

We first claim that the proof of Theorem 2.5 is reduced to the proof of Theorem 2.4.

Namely, if $v_2 = 0$, then $\mathfrak{m}^3 = J\mathfrak{m}^2$, hence A is short and G is Cohen-Macaulay with $h_A(z) = 1 + hz + 2z^2$. This proves part a) of Theorem 2.5

If $v_2 = 1$, then we have already remarked that $depth(G) \geq d - 1$ so that, if we let $R = A/I$, where I is the ideal generated by the first $d-1$ elements in a maximal superficial sequence, then $h_A(z) = h_R(z)$. We thus have $e(R) = h + 3$, $dim(R) = 1$, $\tau(A) = \tau(R)$ and $\mathfrak{m}^{j+1} \cap I = I\mathfrak{m}^j$ for every $j \geq 0$. It follows that

$$v_2(A) = \lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = \lambda(\mathfrak{m}^3/J\mathfrak{m}^2 + \mathfrak{m}^3 \cap I) = \lambda\left((\mathfrak{m}/I)^3/(J/I)(\mathfrak{m}/I)^2\right) = v_2(R).$$

We can thus apply Theorem 2.4 to get part b) of Theorem 2.5. The claim is proved.

We want to prove now Theorem 2.4. Hence, in the rest of this section, A is a local Cohen-Macaulay ring of dimension one with $e = h + 3$.

By Proposition 2.2 iii), we have $v_j \leq v_{j-1}$ for every $j \geq 1$, so that by (9) $h_j \geq 0$ for every $j \geq 0$. This means that

$$h_A(z) = 1 + hz + z^t + z^s$$

with $2 \leq t \leq s$.

If $v_2 = 0$, then, as before, A is short and G is Cohen-Macaulay with

$$h_A(z) = 1 + hz + 2z^2;$$

hence $s = t = 2$ and b) is proved.

If $v_2 = 1$, then by (9) and (3)

$$h_2 = v_1 - v_2 = e - h - 1 - 1 = 1,$$

hence $t = 2$. This proves c).

Finally if A is stretched, then by Proposition 2.2 vi) we get $v_j \leq v_{j-1} \leq v_j + 1$ for every $j \geq 2$ so that $0 \leq h_j \leq 1$ for the same $j's$. This implies $t < s$ and proves d).

We are left to prove that, in any case, $s \leq \tau + 2$.

This is clear if $\tau \geq h$, because by (12)

$$s \leq e - 1 = h + 2 \leq \tau + 2.$$

Hence we may assume $\tau < h$ which, by Proposition 2.3, implies $v_2 \leq 1$. Since in the case $v_2 = 0$ we have $s = t = 2 \leq \tau + 2$, the proof of Theorem 2.4, and thus of Theorem 2.5, is reduced to the proof of the following result.

Theorem 2.6 *Let $e = h + 3$ and $d = 1$. If $v_2 = 1$ then $s \leq \tau + 2$.*

Proof. Since $v_2 = 1$, by (4) we have for every $n \geq 2$

$$\mathfrak{m}^{n+1} = x\mathfrak{m}^n + (a^{n-1}b) \tag{15}$$

where $a \in \mathfrak{m}$, $b \in \mathfrak{m}^2$. Further by (9) and (3) we have

$$h_2 = v_1 - v_2 = e - h - 1 - 1 = 1,$$

hence, by v) in Proposition 2.2,

$$h_A(z) = 1 + hz + z^2 + z^s$$

with $2 \leq s$.

Since $\tau \geq 1$, the conclusion is clear if $s \leq 3$; hence we may also assume $s \geq 4$.

We will prove the theorem in several steps.

Step 1. We may assume $a_2 = s - 3$, and

$$\rho_j = \begin{cases} s-1 & \text{if } j = 1. \\ 1 & \text{if } j = 2. \\ 0 & \text{if } j \geq 3. \end{cases}$$

Proof. By Proposition 1.4 we have

$$a_2 \geq H_A(1) - H(s-1) + s - 2$$

so that

$$a_2 \geq h + 1 - (h+2) + s - 2 = s - 3.$$

By (8), we get

$$\rho_1 = v_1 + a_2 \geq 2 + s - 3 = s - 1.$$

Further, if $\rho_2 = 0$, then $\widetilde{\mathfrak{m}^3} \subseteq x\widetilde{\mathfrak{m}^2}$ which implies $\widetilde{\mathfrak{m}^2} \subseteq x : \mathfrak{m}$. By using Proposition 1.5 we get $s \leq \tau + 1$.

Hence we may assume $\rho_2 \geq 1$. From the true definition of e_1 we have

$$e_1 = \sum_{j=0}^{s} jh_j = h + 2 + s;$$

on the other hand by (14) and (1) we have

$$e_1 = \sum_{j=0}^{s-1} \rho_j = h + 2 + \sum_{j=1}^{s-1} \rho_j.$$

This implies $\sum_{j=1}^{s-1} \rho_j = s$ hence $\rho_1 = s-1$, $\rho_2 = 1$, $\rho_j = 0$, for every $j \geq 3$ and $a_2 = s-3$. This proves step 1.

Now it is clear that we have a filtration of length $s-2$ connecting $\mathfrak{m}^2$ to $\widetilde{\mathfrak{m}^2}$, namely

$$\mathfrak{m}^2 \subseteq \mathfrak{m}^3 : x \subseteq \mathfrak{m}^4 : x^2 \subseteq \cdots \subseteq \mathfrak{m}^s : x^{s-2} = \widetilde{\mathfrak{m}^2},$$

where the last equality follows by Proposition 1.2.

Since $a_2 = s-3$ there exists an integer j such that $0 \leq j \leq s-3$ and

$$\mathfrak{m}^2 \subset \mathfrak{m}^3 : x \subset \mathfrak{m}^4 : x^2 \subset \cdots \subset \mathfrak{m}^{j+2} : x^j = \mathfrak{m}^{j+3} : x^{j+1}.$$

We remark that we have

$$H_{A/xA}(2) = \lambda(\mathfrak{m}^2 + x/\mathfrak{m}^3 + x) = \lambda(\mathfrak{m}^2/\mathfrak{m}^3 + \mathfrak{m}^2 \cap x) =$$

$$= H_A(2) - \lambda(\mathfrak{m}^3 + \mathfrak{m}^2 \cap x/\mathfrak{m}^3) = H_A(2) - \lambda(x(\mathfrak{m}^2 : x)/x(\mathfrak{m}^3 : x)) =$$

$$= H_A(2) - \lambda(\mathfrak{m}/\mathfrak{m}^3 : x) = H_A(2) - H_A(1) + \lambda(\mathfrak{m}^3 : x/\mathfrak{m}^2) = 1 + \lambda(\mathfrak{m}^3 : x/\mathfrak{m}^2).$$

From this we get that A is stretched if and only if $j = 0$.

Step 2. Let j be the integer defined above. Then

$$\mathfrak{m}^{j+2} \subseteq x^j A$$

and

$$\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = j.$$

Proof. If $j = 0$, everything is clear. Let $j \geq 1$; by the choice of j we have

$$\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) \geq j.$$

From the exact sequence

$$0 \to \mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^{j+1}/\mathfrak{m}^{j+2} \to \mathfrak{m}^{j+1}/x^j\mathfrak{m} + \mathfrak{m}^{j+2} \to 0,$$

where the map in the middle is multiplication by x^j, we get

$$\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = h + 1 - H_A(j+1) + \lambda(\mathfrak{m}^{j+1}/x^j\mathfrak{m} + \mathfrak{m}^{j+2}).$$

Since $2 \leq j+1 \leq s-2$, $H_A(j+1) = h+2$ so that

$$\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = \lambda(\mathfrak{m}^{j+1}/x^j\mathfrak{m} + \mathfrak{m}^{j+2}) - 1.$$

By using the filtration

$$\mathfrak{m}^{j+1} \supseteq x\mathfrak{m}^j \supseteq x^2\mathfrak{m}^{j-1} \supseteq \cdots \supseteq x^j\mathfrak{m},$$

we get

$$\lambda(\mathfrak{m}^{j+1}/x^j\mathfrak{m}+\mathfrak{m}^{j+2}) + \lambda(x^j\mathfrak{m}+\mathfrak{m}^{j+2}/x^j\mathfrak{m}) = \lambda(\mathfrak{m}^{j+1}/x^j\mathfrak{m}) =$$

$$= \sum_{t=0}^{j-1} \lambda(x^t\mathfrak{m}^{j+1-t}/x^{t+1}\mathfrak{m}^{j-t}) = \sum_{t=0}^{j-1} \lambda(\mathfrak{m}^{j+1-t}/x\mathfrak{m}^{j-t}) = \sum_{t=0}^{j-1} v_{j-t} = j-1+2 = j+1.$$

From this we get

$$j \leq \lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = j+1-\lambda(x^j\mathfrak{m}+\mathfrak{m}^{j+2}/x^j\mathfrak{m}) - 1 = j - \lambda(x^j\mathfrak{m}+\mathfrak{m}^{j+2}/x^j\mathfrak{m}).$$

Hence

$$\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = j$$

and

$$x^j\mathfrak{m}+\mathfrak{m}^{j+2} = x^j\mathfrak{m}$$

which implies

$$\mathfrak{m}^{j+2} \subseteq x^jA.$$

This proves step 2.

Since $\mathfrak{m}^{j+2} \subseteq x^jA$ we can write

$$a^jb = x^jz, \qquad z \in \mathfrak{m}^{j+2} : x^j \subseteq \widetilde{\mathfrak{m}^2}.$$

By (15) we get

$$\mathfrak{m}^{j+3} = x\mathfrak{m}^{j+2} + (a^{j+1}b) = x\mathfrak{m}^{j+2} + (ax^jz). \tag{16}$$

<u>Step</u> 3. With the above notation we have

$$\widetilde{\mathfrak{m}^3} = x\widetilde{\mathfrak{m}^2} + (az).$$

<u>Proof.</u> Since $z \in \widetilde{\mathfrak{m}^2}$, we have $az \in \widetilde{\mathfrak{m}^3}$. By Step 1 $\rho_2 = \lambda(\widetilde{\mathfrak{m}^3}/x\widetilde{\mathfrak{m}^2}) = 1$, hence the conclusion follows if we can prove that $az \notin x\widetilde{\mathfrak{m}^2}$.

Let us assume, by contradiction, that $az = xp$; then

$$a^{j+1}b = a(a^jb) = x^jaz = x^{j+1}p$$

hence $p \in \mathfrak{m}^{j+3} : x^{j+1} = \mathfrak{m}^{j+2} : x^j$. This implies

$$a^{j+1}b = x(x^jp) \in x\mathfrak{m}^{j+2}$$

so that, by (15),

$$\mathfrak{m}^{j+3} = x\mathfrak{m}^{j+2}.$$

This means $v_{j+2} = 0$, which, since $j + 2 \leq s - 1$, is a contradiction to (10).

Step 4. If $j \geq 1$, then we have:

a) $\mathfrak{m}^{j+1} : x^{j-1} \subseteq x : \mathfrak{m}$.

b) $\mathfrak{m}^{j+2} : x^j \subseteq (z) + (x : \mathfrak{m})$.

Proof. If $cx^{j-1} \in \mathfrak{m}^{j+1}$, then, by Step 2,

$$cx^{j-1}\mathfrak{m} \subseteq \mathfrak{m}^{j+2} \subseteq x^j A.$$

This implies $c \in x : \mathfrak{m}$ which proves a).

If $cx^j \in \mathfrak{m}^{j+2} = x\mathfrak{m}^{j+1} + (a^j b) = x\mathfrak{m}^{j+1} + (x^j z)$, then $(c - pz)x^j \in x\mathfrak{m}^{j+1}$ for some $p \in A$. Hence, by a),

$$c - pz \in x\mathfrak{m}^{j+1} : x^j \subseteq \mathfrak{m}^{j+1} : x^{j-1} \subseteq x : \mathfrak{m}.$$

This proves b) so that the proof of step 4 is now complete.

In the following $v(I)$ denotes the minimal number of generators of the ideal I.

Step 5. There exists an ideal I such that $v(I) = s - 3$, $aI \subseteq x\widetilde{\mathfrak{m}^2}$ and

$$\widetilde{\mathfrak{m}^2} = (\mathfrak{m}^{j+2} : x^j) + I.$$

Proof. By Step 1 we have $s-3 = a_2 = \lambda(\widetilde{\mathfrak{m}^2}/\mathfrak{m}^2)$, while, by step 2, $\lambda(\mathfrak{m}^{j+2} : x^j/\mathfrak{m}^2) = j$. Hence $\widetilde{\mathfrak{m}^2}/\mathfrak{m}^{j+2} : x^j$ is a k-vector space of dimension $s - 3 - j$ and we can write

$$\widetilde{\mathfrak{m}^2} = (\mathfrak{m}^{j+2} : x^j) + (w_1, \ldots, w_{s-3-j}).$$

Since $w_i \in \widetilde{\mathfrak{m}^2}$, we have $aw_i \in \widetilde{\mathfrak{m}^3}$ and by Step 3 we can write

$$aw_i = r_i + l_i az$$

where $r_i \in x\widetilde{\mathfrak{m}^2}$. It follows that for every i

$$a(w_i - l_i z) \in x\widetilde{\mathfrak{m}^2}.$$

Since $z \in \mathfrak{m}^{j+2} : x^j$, we have

$$\widetilde{\mathfrak{m}^2} = (\mathfrak{m}^{j+2} : x^j) + (w_1 - l_1 z, \ldots, w_{s-3-j} - l_{s-3-j} z)$$

and the conclusion follows with $I = (w_1 - l_1 z, \ldots, w_{s-3-j} - l_{s-3-j} z)$.

Step 6. We have

$$I \subseteq x : \mathfrak{m}.$$

Proof.

If, by contradiction, there exist $w \in I$, $c \in \mathfrak{m}$ such that $cw \notin xA$, then, since $cw \in \widetilde{\mathfrak{m}^3}$ and $\rho_2 = 1$, we can write

$$\widetilde{\mathfrak{m}^3} = x\widetilde{\mathfrak{m}^2} + (cw).$$

By using Step 5, we get

$$\widetilde{\mathfrak{m}^3} = x\widetilde{\mathfrak{m}^2} + (cw) = xI + x(\mathfrak{m}^{j+2} : x^j) + (cw) \subseteq x(\mathfrak{m}^{j+2} : x^j) + I\mathfrak{m}.$$

Since $z \in \widetilde{\mathfrak{m}^2}$, we get

$$az \in x(\mathfrak{m}^{j+2} : x^j) + I\mathfrak{m}.$$

At this point we want to use Theorem 1.7 to get a contradiction. We will use Theorem 1.7 with the following setting:

$$K := \mathfrak{m}^{j+2} : x^j, \quad v := s - 3 - j, \quad c := x^j \quad i := j + 2.$$

Then we have

$$v + i + 1 = s - 3 - j + j + 2 + 1 = s$$

and we need to verify the four conditions of the theorem.

Condition 1. becomes $\mathfrak{m}^{j+3} = x\mathfrak{m}^{j+2} + (ax^j z)$ which follows from (16).

Condition 2 becomes $az \in x(\mathfrak{m}^{j+2} : x^j) + I\mathfrak{m}$ which has been verified before.

Condition 3 becomes $aI \subseteq x(I + \mathfrak{m}^{j+2} : x^j) = x\widetilde{\mathfrak{m}^2}$ which follows from Step 5.

Condition 4 becomes $x^j(\mathfrak{m}^{j+2} : x^j) \subseteq \mathfrak{m}^{j+2}$ which is trivial.

This proves Step 6.

Step 7. Conclusion.

Proof. By Step 6 we have $I \subseteq x : \mathfrak{m}$, hence, by using Proposition 1.5 and Step 5, we get

$$s \leq 1 + \tau + \lambda(x : \mathfrak{m} + \widetilde{\mathfrak{m}^2}/x : \mathfrak{m}) = 1 + \tau + \lambda\left((x : \mathfrak{m}) + (\mathfrak{m}^{j+2} : x^j)/x : \mathfrak{m}\right).$$

Now, if $j = 0$, then A is stretched and, according to Proposition 2.1, $\mathfrak{m}^2 = x\mathfrak{m} + (cf)$. Since $\mathfrak{m}^4 \subseteq x\mathfrak{m}$, it is clear that $x : \mathfrak{m} + \mathfrak{m}^2/x : \mathfrak{m} = (x : \mathfrak{m}) + (cf)/x : \mathfrak{m}$ is a k-vector space of dimension ≤ 1. The conclusion follows.

If $j \geq 1$, then by Step 4 we have $\mathfrak{m}^{j+2} : x^j \subseteq (z) + x : \mathfrak{m}$, hence

$$\lambda\left((x : \mathfrak{m}) + (\mathfrak{m}^{j+2} : x^j)/x : \mathfrak{m}\right) \leq \lambda\left(x : \mathfrak{m} + (z)/x : \mathfrak{m}\right).$$

Since $\rho_3 = 0$, we have

$$z\mathfrak{m}^2 \subseteq \widetilde{\mathfrak{m}^4} = x\widetilde{\mathfrak{m}^3}$$

so that $x : \mathfrak{m} + (z)/x : \mathfrak{m}$ is k-vector space of dimension ≤ 1 and the conclusion follows also in this case.

The proof of the theorem is now complete.

$\square$

We want to remark that if A is stretched then, as noticed in Step 1 of the theorem, $j = 0$ so that the proof of this main result is strongly simplified. Namely, one does not need to prove Step 2 , Step 4 and Step 5.

3 Examples

In this last section we discuss several examples which put some light on the results we have proved before and on some possible extensions.

• First we show that in Theorem 2.3 we cannot delete the assumption $\tau < h$.

Let

$$A = k\left[\left[t^7, t^8, t^{13}, t^{19}, t^{25}\right]\right].$$

Then $h = 4$, $e = 7 = h + 3$ and $x = t^7$ is a superficial element of A. We have $\tau = 4$,

$$P_{A/xA}(z) = 1 + 4z + z^2 + z^3$$

and

$$P_A(z) = \frac{1 + 4z + z^3 + z^5}{1 - z}.$$

Hence A is stretched and $v_2 = 2$.

Let

$$A = k\left[\left[t^6, t^7, t^{15}, t^{23}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3$,

$$P_{A/xA}(z) = 1 + 3z + 2z^2$$

and

$$P_A(z) = \frac{1 + 3z + z^3 + z^5}{1 - z}.$$

Hence A is short and $v_2 = 2$.

• We remark that, as a consequence of Theorem 2.4, if $e = h + 3$ and A is Gorenstein then G is Cohen-Macaulay, a result which has been proved by Sally in [14] and which was the starting point of our investigation. The following examples show that the result is no more true if we make the weaker assumption $\tau = 2$.

Let

$$A = k\left[\left[t^6, t^7, t^{11}, t^{15}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 2$,

$$P_{A/xA}(z) = 1 + 3z + 2z^2$$

and

$$P_A(z) = \frac{1 + 3z + z^2 + z^4}{1 - z}.$$

Hence A is short and G is not Cohen-Macaulay.

Let

$$I = (UY - X^2Y, UZ - X^2Z, U^2 - X^2U, YZ - XU, Y^3 - Z^2 - XZ - XY^2)$$

and $A = k[[X, Y, Z, U]]/I$. Then $h = 3$, $e = 6 = h + 3$ and $x = \overline{X}$ is a superficial element of A. We have $\tau = 2$,

$$P_{A/xA}(z) = 1 + 3z + z^2 + z^3$$

and

$$P_A(z) = \frac{1 + 3z + z^2 + z^4}{1 - z}.$$

Hence A is stretched and G is not Cohen-Macaulay.

The last example is a local ring which is not a domain. Since we could not find a domain with those properties, we ask the following question:

Problem 3.1 *If A is a Cohen-Macaulay stretched domain with multiplicity $e = h+3$ and Cohen-Macaulay type $\tau = 2$, is G Cohen-Macaulay?*

On the other hand the following example shows that there exists Cohen-Macaulay domains with multiplicity $e = h + 3$ and Cohen-Macaulay type $\tau = 3$ such that G is not Cohen-Macaulay.

Let

$$A = k\left[\left[t^6, t^7, t^{11}, t^{16}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3$,

$$P_{A/xA}(z) = 1 + 3z + z^2 + z^3$$

and

$$P_A(z) = \frac{1 + 3z + z^2 + z^4}{1 - z}.$$

Hence A is a Cohen-Macaulay stretched domain and G is not Cohen-Macaulay.

• In Theorem 2.4 d) we have proved that if A is stretched then $t < s$. The following example shows that this is not the case if A is short. It is enough to consider a short Cohen-Macaulay local ring such that G is Cohen-Macaulay.

Let

$$A = k\left[\left[t^6, t^8, t^{10}, t^{13}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have

$$P_{A/xA}(z) = 1 + 3z + 2z^2$$

and

$$P_A(z) = \frac{1 + 3z + 2z^2}{1 - z}.$$

Hence A is short and G is Cohen-Macaulay.

• The Hilbert series of the rings $k\left[[t^6, t^7, t^{11}, t^{15}]\right]$ and $k[[X, Y, Z, U]]/I$ where

$$I = (UY - X^2Y, UZ - X^2Z, U^2 - X^2U, YZ - XU, Y^3 - Z^2 - XZ - XY^2)$$

show that the bound $s \leq \tau + 2$ in Theorem 2.4 is sharp. This is clear if A is Gorenstein, but in the above examples we have $\tau = 2$.

• Finally we want to remark that if $\tau \geq h$ then we have no control on the Hilbert function of A in the sense that the degree of the h-polynomial can be the minimum, which is 2, and the maximum which is $e - 1$.

Let

$$A = k\left[\left[t^6, t^7, t^{15}, t^{23}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3 = h$,

$$P_{A/xA}(z) = 1 + 3z + 2z^2$$

and

$$P_A(z) = \frac{1 + 3z + z^3 + z^5}{1 - z}.$$

Hence A is short and $s = 5 = e - 1$.

Let

$$A = k\left[\left[t^6, t^8, t^{13}, t^{15}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3 = h$,

$$P_{A/xA}(z) = 1 + 3z + 2z^2$$

and

$$P_A(z) = \frac{1 + 3z + 2z^2}{1 - z}.$$

Hence A is short, G is Cohen-Macaulay and $s = 2$.

Let

$$A = k\left[\left[t^6, t^7, t^{16}, t^{17}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3 = h$,

$$P_{A/xA}(z) = 1 + 3z + z^2 + z^3$$

and

$$P_A(z) = \frac{1 + 3z + z^2 + z^5}{1 - z}.$$

Hence A is stretched and $s = 5 = e - 1$.

Let

$$A = k\left[\left[t^6, t^{11}, t^{19}, t^{20}\right]\right].$$

Then $h = 3$, $e = 6 = h + 3$ and $x = t^6$ is a superficial element of A. We have $\tau = 3 = h$,

$$P_{A/xA}(z) = 1 + 3z + z^2 + z^3$$

and

$$P_A(z) = \frac{1 + 3z + z^2 + z^3}{1 - z}.$$

Hence A is stretched, G is Cohen-Macaulay and $s = 2$.

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