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Minimal free resolution of a finitely generated module over a regular local ring

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ABSTRACT

Numerical invariants of a minimal free resolution of a module M over a regular local ring $(R,\mathfrak{n})$ can be studied by taking advantage of the rich literature on the graded case. The key is to fix suitable $\mathfrak{n}$-stable filtrations $\mathbb{M}$ of M and to compare the Betti numbers of M with those of the associated graded module $gr_{\mathbb{M}}(M)$. This approach has the advantage that the same module M can be detected by using different filtrations on it. It provides interesting upper bounds for the Betti numbers and we study the modules for which the extremal values are attained. Among others, the Koszul modules have this behavior. As a consequence of the main result, we extend some results by Aramova, Conca, Herzog and Hibi on the rigidity of the resolution of standard graded algebras to the local setting.

Introduction

Consider a local ring $(R,\mathfrak{n})$ and let M be a finitely generated R-module. In the literature, starting from classical results by Northcott, Abhyankar, Matlis and Sally, several authors detected basic numerical characters of the module M by means of the Hilbert function of M arising from the standard $\mathfrak{n}$-adic filtration or, more in general, from $\mathfrak{n}$-stable filtrations (see [RV] for an extensive overview). Deeper information can be achieved from the numerical invariants of a minimal free resolution of M. It is a classical tool to equip M with a suitable filtration and to get information on M from the graded free resolution of the corresponding associated graded module. This approach has the advantage to benefit from the rich literature concerning the graded cases to return the information to the local

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ones. In particular, the main goal of the paper is to compare the numerical invariants of a local ring $(A, \mathfrak{m})$ with those of the associated graded ring with respect to the $\mathfrak{m}$-adic filtration on A, that is the standard graded algebra $gr_{\mathfrak{m}}(A) := \bigoplus_{t \geqslant 0} \mathfrak{m}^t/\mathfrak{m}^{t+1}$. This is a graded object which corresponds to a relevant geometric construction that encodes several information on A. In fact if A is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then the associated graded ring $gr_{\mathfrak{m}}(A)$ is the coordinate ring of the *tangent cone* of V. Often, we may assume that $A = R/I$ where R is a regular local ring. In this case $gr_{\mathfrak{m}}(A) \simeq P/I^*$ where I^* is a homogeneous ideal of a polynomial ring P generated by the initial forms (w.r.t. the $\mathfrak{n}$-adic filtration) of the elements of I. In this setting, the theory of the Gröbner bases and the leading term ideals offers interesting results on the extremal values of the Betti numbers (see for example [AHH,Bi,CHH,C]).

It is the reason why we are mainly interested in studying finitely generated modules M over a regular local ring $(R, \mathfrak{n})$. In this case $gr_{\mathfrak{n}}(R) = \bigoplus_{i \geqslant 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$ is a polynomial ring, say P, and the associated graded module with respect to the any $\mathfrak{n}$-stable filtration of M is a finitely generated P-module.

In general, the problem is how to lift the information from the associated graded module with respect to an $\mathfrak{n}$-stable filtration to the original module M. If the associated graded module with respect to the $\mathfrak{n}$-adic filtration $gr_{\mathfrak{n}}(M) = \bigoplus_{i \geqslant 0} \mathfrak{n}^i M/\mathfrak{n}^{i+1} M$ has a linear resolution (as a $gr_{\mathfrak{n}}(R)$-module), then it is easy to see that the Betti numbers of M and $gr_{\mathfrak{n}}(M)$ coincide. In this case, the module M is said to have a *linear resolution* in the terminology of Herzog, Simis and Vasconcelos (see [HSV]) or of Sega (see [Se]), equivalently M is *Koszul* in the terminology of Herzog and Iyengar (see [HI]). Koszul modules are important examples in our investigation. In this paper, we prefer to say that M is Koszul because to have linear resolution is misleading in the graded case. We recall that a local ring $(A, \mathfrak{m})$ is said Koszul if the residue field k is Koszul as an A-module, that is the graded k-algebra $gr_{\mathfrak{m}}(A) = \bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is Koszul in the classical meaning introduced by Priddy (see [HI, Remark 1.10]). By combining results by Herzog, Iyengar (see [HI, Proposition 1.5, Definition 1.7]) and Römer (see [R, Theorem 3.2.8]) or Martinez and Zacharia (see [MZ, Theorems 2.4, 2.5 and 3.4]), we can conclude that Koszul graded modules and componentwise linear modules coincide. Given these considerations, in this paper we will speak of Koszul modules in the case of modules over a local ring and of componentwise linear modules or, indifferently, of Koszul modules in the graded case.

Recently, many papers have been written extending classical results of the theory of the associated graded ring with respect to the $\mathfrak{n}$-adic filtration to the more general case of a stable (or good) filtration $\mathbb{M} = \{M_i\}_{i \geqslant 0}$ on a finitely generated module M over a local ring $(R, \mathfrak{n})$. The associated graded module $gr_{\mathbb{M}}(M) = \bigoplus_i M_i/M_{i+1}$ has a natural structure as a finitely generated G-module where $G = gr_{\mathfrak{n}}(R) = \bigoplus_{i \geqslant 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$. We are interested to compare the free resolution of M as an R-module and the free resolution of $gr_{\mathbb{M}}(M)$ as a G-module.

An important starting point of our investigation is a result due to Robbiano (see [Rob] and also [HRV,Se]) which says that from a minimal G-free resolution of $gr_{\mathbb{M}}(M)$ we can build up an R-free resolution on M which is not necessarily minimal. Hence, for the Betti numbers of M and $gr_{\mathbb{M}}(M)$ one has

$$\beta_i(M) \leqslant \beta_i\bigl(gr_{\mathbb{M}}(M)\bigr)$$

for every $i \geqslant 0$. In particular, the free resolution of M is minimal if and only if $\beta_i(gr_{\mathbb{M}}(M)) = \beta_i(M)$ for every $i \geqslant 0$.

According to [HRV], a finitely generated module M is called of *homogeneous type* with respect to an $\mathfrak{n}$-stable filtration $\mathbb{M}$, if $\beta_i(M) = \beta_i(gr_{\mathbb{M}}(M))$ for every $i \geqslant 0$. If M is of homogeneous type, then in particular $\operatorname{depth} M = \operatorname{depth} gr_{\mathbb{M}}(M)$. Notice that a Koszul module is a module of homogeneous type with respect to the $\mathfrak{n}$-adic filtration, conversely there are modules of homogeneous type which are not Koszul.

Under the assumption that R is a regular local ring, the main result of this paper (Theorem 3.1) says that if the minimal number of generators of M (as an R-module) and the minimal number of generators of $gr_{\mathbb{M}}(M)$ (as a G-module) coincide, then M is of homogeneous type with respect to $\mathbb{M}$, provided that $gr_{\mathbb{M}}(M)$ is a componentwise linear module or equivalently $gr_{\mathbb{M}}(M)$ is a Koszul module.

Moreover, Theorem 3.6 says that if $gr_{\mathbb{M}}(M)$ is Koszul for some $\mathfrak{n}$-stable filtration $\mathbb{M}$, then M is Koszul, provided the minimal number of generators of M and $gr_{\mathbb{M}}(M)$ coincide. It is an interesting question to ask whether the converse holds.

The first application deals with the classical case of the associated graded ring of a local ring $A = R/I$ ($(R, \mathfrak{n})$ is a regular local ring). In this case, we will apply Theorem 3.1 in the case $M = I$ equipped with the $\mathfrak{n}$-stable filtration $\mathbb{M} = \{I \cap \mathfrak{n}^i\}$. The point is that $I^* = gr_{\mathbb{M}}(I)$.

As a consequence we prove that if I is minimally generated by an $\mathfrak{n}$-standard base and I^* (the ideal generated by the initial forms of the elements of I) is componentwise linear, then the numerical invariants of a minimal free resolution of A and those of $gr_{\mathfrak{m}}(A)$ coincide (see Corollary 3.3). In particular, under these assumptions, $\operatorname{depth} A = \operatorname{depth} gr_{\mathfrak{m}}(A)$. We point out that, if this is the case, I is a Koszul module by Theorem 3.6.

It is worth remarking that we cannot delete the condition which I^* is componentwise linear, in fact in general A and $gr_{\mathfrak{m}}(A)$ do not have the same Betti numbers even if the minimal number of generators of I and I^* coincide, that is I is minimally generated by an $\mathfrak{n}$-standard base of I. For example this is the case if we consider the defining ideal I of the semigroup ring $A = k[[t^{19}, t^{26}, t^{34}, t^{40}]]$ (see [HRV, Example, Section 3]). Notice that both A and $gr_{\mathfrak{m}}(A)$ are Cohen–Macaulay.

On the analogy with the graded case and by taking advantage of it, we can rephrase the result in terms of suitable monomial ideals attached to the ideal I. If we define $Gin(I) := Gin(I^*)$ the generic initial ideal of I^* w.r.t. the reverlex order, then we may deduce from the homogeneous case (see [AHH,CHH,C]),

$$\beta_i(I) \leqslant \beta_i\big(Gin(I)\big)$$

for every $i \geqslant 0$ and, as a consequence of Theorem 3.1, the equality holds if and only if $\beta_0(I) = \beta_0(Gin(I))$ or equivalently I is minimally generated by an $\mathfrak{n}$-standard base and I^* is componentwise linear (see Corollary 3.3). A similar result can be also presented in terms of $Lex(I)$ which is the unique lex segment ideal in P such that R/I and $P/Lex(I)$ have the same Hilbert function. Taking advantage from the graded case, Corollary 3.4 is an extension to the local case of a result by Herzog and Hibi.

In [CHH], Conca, Herzog and Hibi proved an upper bound for the Betti numbers of the local ring A in terms of the so-called generic annihilators of A; Corollary 3.7 shows that, under the assumption of the main result, the extremal Betti numbers are achieved.

Theorem 3.6 has another interesting consequence. Assume $I \subseteq \mathfrak{n}^2$ is a non-zero ideal of R. It is known that the symmetric algebra of the maximal ideal $\mathfrak{m}$ of $A = R/I$ is $S_A(\mathfrak{m}) \simeq \bigoplus_{i\geqslant 0} \mathfrak{n}^i/I\mathfrak{n}^{i-1}$. If we consider the $\mathfrak{n}$-adic filtration on the ideal I, then the associated graded module $gr_{\mathfrak{n}}(I) = \bigoplus_{i\geqslant 0} I\mathfrak{n}^i/I\mathfrak{n}^{i+1}$ with respect to the filtration $\mathbb{M} = \{I\mathfrak{n}^i\}$ sits inside $S_A(\mathfrak{m})$ as a graded submodule, via the canonical embedding $gr_{\mathfrak{n}}(I)(-2) \to S_A(\mathfrak{m})$. Herzog, Rossi and Valla in [HRV, Theorem 2.13] showed that we can deduce the homological properties of the symmetric algebra by taking advantage from this comparison. By using this result and the fact that the ideals under investigation are Koszul, Theorem 3.8 extends a recent result by Herzog, Restuccia and Rinaldo [HRR, Theorem 3.9] on the depth of the symmetric algebra.

1. Preliminaries on filtered modules

Throughout the paper $(R, \mathfrak{n})$ is a local ring and M is a finitely generated R-module. We say, according to the notation in [RV], that a filtration of submodules $\mathbb{M} = \{M_n\}_{n\geqslant 0}$ on M is an $\mathfrak{n}$-filtration if $\mathfrak{n}M_n \subseteq M_{n+1}$ for every $n \geqslant 0$, and a stable (or good) $\mathfrak{n}$-filtration if $\mathfrak{n}M_n = M_{n+1}$ for all sufficiently large n. In the following a *filtered module* M will be always an R-module equipped with a stable $\mathfrak{n}$-filtration $\mathbb{M}$.

If $\mathbb{M} = \{M_j\}$ is an $\mathfrak{n}$-filtration of M, define

$$gr_{\mathbb{M}}(M) = \bigoplus_{j\geqslant 0}(M_j/M_{j+1})$$

which is a graded $gr_{\mathfrak{n}}(R)$-module in a natural way. It is called the **associated graded module** to the filtration $\mathbb{M}$.

To avoid triviality, we assume that $gr_{\mathbb{M}}(M)$ is not zero or equivalently $M \neq 0$. If N is a submodule of M, by Artin–Rees lemma, the sequence $\{N \cap M_j \mid j \geqslant 0\}$ is a good $\mathfrak{n}$-filtration of N. Since

$$(N \cap M_j)/(N \cap M_{j+1}) \simeq (N \cap M_j + M_{j+1})/M_{j+1} \tag{1}$$

$gr_{\mathbb{M}}(N)$ is a graded submodule of $gr_{\mathbb{M}}(M)$ denoted by N^*.

If $m \in M \setminus \{0\}$, we denote by $\nu_{\mathbb{M}}(m)$ the largest integer p such that $m \in M_p$ (the so-called valuation of m with respect to $\mathbb{M}$) and we denote by m^* or $gr_{\mathbb{M}}(m)$ the residue class of m in M_p/M_{p+1} where $p = \nu_{\mathbb{M}}(m)$. If $m = 0$, we set $\nu_{\mathbb{M}}(m) = +\infty$.

Using (1), it is clear that $gr_{\mathbb{M}}(N)$ is generated by the elements x^* with $x \in N$, we write

$$gr_{\mathbb{M}}(N) = \langle x^*\colon x \in N \rangle.$$

On the other hand it is clear that $\{(N + M_j)/N \mid j \geqslant 0\}$ is a good $\mathfrak{n}$-filtration of M/N which we denote by $\mathbb{M}/N$. These graded modules are related by the graded isomorphism

$$gr_{\mathbb{M}/N}(M/N) \simeq gr_{\mathbb{M}}(M)/gr_{\mathbb{M}}(N).$$

For completeness we collect, in this section, a part of the well-known results concerning the homomorphisms of filtered modules.

Definition 1.1. If M and N are filtered R-modules and $f : M \to N$ is an R-homomorphism, f is said to be a *homomorphism of filtered modules* if $f(M_p) \subseteq N_p$ for every $p \geqslant 0$ and f is said *strict* if $f(M_p) = f(M) \cap N_p$ for every $p \geqslant 0$.

The morphism of filtered modules $f : M \to N$ clearly induces a morphism of graded $gr_{\mathfrak{n}}(R)$-modules

$$gr(f) : gr_{\mathbb{M}}(M) \to gr_{\mathbb{N}}(N).$$

It is clear that $gr(\cdot)$ is a functor from the category of the filtered R-modules into the category of the graded $gr_{\mathfrak{n}}(R)$-modules. Furthermore we have a canonical embedding $(\operatorname{Ker} f)^* \to \operatorname{Ker}(gr(f))$.

Proposition 1.2. *(See [RoV].) Let* $\mathbf{F} : M \xrightarrow{g} N \xrightarrow{f} Q$ *be a complex of filtered modules and*

$$gr(\mathbf{F}) : gr_{\mathbb{M}}(M) \xrightarrow{gr(g)} gr_{\mathbb{N}}(N) \xrightarrow{gr(f)} gr_{\mathbb{Q}}(Q)$$

be the induced complex of graded $gr_{\mathfrak{n}}(R)$*-modules. Then* $gr(\mathbf{F})$ *is exact if and only if* $\mathbf{F}$ *is exact and* f *and* g *are strict morphisms.*

As a consequence of the above result we get that a morphism $f : M \to N$ of filtered modules is strict if and only if the canonical embedding $(\operatorname{Ker} f)^* \to \operatorname{Ker}(gr(f))$ is an isomorphism.

Definition 1.3. Let $L = \bigoplus_{i=1}^{s} Re_i$ be a free R-module of rank s and $\nu_1, \ldots, \nu_s$ be integers. We define the filtration $\mathbb{L} = \{L_p\colon\ p \in \mathbf{Z}\}$ on L as follows

$$L_p := \bigoplus_{i=1}^{s} \mathfrak{n}^{p-\nu_i} e_i = \{(a_1, \ldots, a_s)\colon\ a_i \in \mathfrak{n}^{p-\nu_i}\}.$$

We denote the filtered free R-module L by $\bigoplus_{i=1}^{s} R(-\nu_i)$ and we call it *special filtration* on L.

So when we write $L = \bigoplus_{i=1}^{s} R(-\nu_i)$ it means that we consider the free module L of rank s with the special filtration defined above. It is clear that $\mathbb{L}$ is an $\mathfrak{n}$-stable filtration. A filtered free R-module $L = \bigoplus_{i=1}^{s} R(-\nu_i)$ is a free module with canonical basis $(e_1, \ldots, e_s)$ such that $\nu_{\mathbb{L}}(e_i) = \nu_i$. It is obvious that $gr_{\mathbb{L}}(L) = \bigoplus_p L_p/L_{p+1}$ is isomorphic as a $gr_{\mathfrak{n}}(R)$-module to $\bigoplus_{i=1}^{s} gr_{\mathfrak{n}}(R)(-\nu_i) = \bigoplus_{i=1}^{s} G(-\nu_i)$ where for short $G = gr_{\mathfrak{n}}(R)$.

The canonical basis $(gr_{\mathbb{L}}(e_1), \ldots, gr_{\mathbb{L}}(e_s))$ of $gr_{\mathbb{L}}(L)$ will be simply denoted by $(e_1, \ldots, e_s)$. Note that R with the $\mathfrak{n}$-adic filtration is the filtered module $R(0)$.

If $(\mathbf{F.}, \delta.)$ is a complex of finitely generated free R-modules, a special filtration on $\mathbf{F.}$ is a special filtration on each F_i that makes $(\mathbf{F.}, \delta.)$ a filtered complex (complex of filtered modules). Our goal is to consider special filtrations on an R-free resolution of a filtered module M. We recall that over local rings, each finitely generated module has a minimal free resolution, and this is unique (up to isomorphism). Thus, one may speak of the minimal free resolution of such a module. We introduce now the main objects of interest.

Let M be a finitely generated filtered R-module and $S = \{f_1, \ldots, f_s\}$ be a system of elements of M and let $\nu_{\mathbb{M}}(f_i)$ be the corresponding valuations. As in Definition 1.3, let $L = \bigoplus_{i=1}^{s} Re_i$ be a free R-module of rank s equipped with the filtration $\mathbb{L}$ where $\nu_i = \nu_{\mathbb{M}}(f_i)$. Then we denote the filtered free R-module L by $\bigoplus_{i=1}^{s} R(-\nu_{\mathbb{M}}(f_i))$, hence $\nu_{\mathbb{L}}(e_i) = \nu_{\mathbb{M}}(f_i)$.

Let $\phi : L \to M$ be a morphism of filtered R-modules defined by

$$\phi(e_i) = f_i.$$

It is clear that ϕ is a morphism of filtered modules and $gr_{\mathbb{L}}(L)$ is isomorphic to the graded free G-module $\bigoplus_{i=1}^{s} G(-\nu_{\mathbb{M}}(f_i))$ with a basis $(e_1, \ldots, e_s)$ where $\deg(e_i) = \nu_{\mathbb{M}}(f_i)$. In particular ϕ induces a natural graded morphism (of degree zero) $gr(\phi) : gr_{\mathbb{L}}(L) \to gr_{\mathbb{M}}(M)$ sending e_i to $gr_{\mathbb{M}}(f_i) = f_i^*$.

Let $c = {}^t(c_1, \ldots, c_s)$ be an element of L. By the definition of the filtration $\mathbb{L}$ on L, we have

$$\nu_{\mathbb{L}}(c) = \min\bigl\{\nu_R(c_i) + \nu_{\mathbb{M}}(f_i):\ 1 \leqslant i \leqslant s\bigr\} \leqslant \nu_{\mathbb{M}}\bigl(\phi(c)\bigr).$$

Set $gr_{\mathbb{L}}(c) = {}^t(c'_1, \ldots, c'_s)$ and $\nu = \nu_{\mathbb{L}}(c)$, then

$$c'_i = \begin{cases} gr_{\mathfrak{n}}(c_i) & \text{if } \nu_R(c_i) + \nu_{\mathbb{M}}(f_i) = \nu, \\ 0 & \text{if } \nu_R(c_i) + \nu_{\mathbb{M}}(f_i) > \nu. \end{cases}$$

If we denote by $\operatorname{Syz}(S)$ the submodule of L generated by the first syzygies of $f_1, \ldots, f_s$, then $\operatorname{Syz}(S) = \operatorname{Ker}\phi$. Likewise let $\operatorname{Syz}(gr_{\mathbb{M}}(S))$ be the module generated by the first syzygies of $gr_{\mathbb{M}}(f_1), \ldots, gr_{\mathbb{M}}(f_s)$, then $\operatorname{Syz}(gr_{\mathbb{M}}(S)) = \operatorname{Ker}(gr(\phi))$.

Then we have the following fundamental diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Syz}(S) & \longrightarrow & \bigoplus_{i=1}^{s} R(-\nu_{\mathbb{M}}(f_i)) & \xrightarrow{\phi} & M \\ & & & & \downarrow{\scriptstyle gr_{\mathbb{L}}} & & \downarrow{\scriptstyle gr_{\mathbb{M}}} \\ 0 & \longrightarrow & \operatorname{Syz}(gr_{\mathbb{M}}(S)) & \longrightarrow & \bigoplus_{i=1}^{s} G(-\nu_{\mathbb{M}}(f_i)) & \xrightarrow{gr(\phi)} & gr_{\mathbb{M}}(M) \end{array}$$

Definition 1.4. Let M be a filtered module. An element $g \in M$ is a *lifting* of an element $h \in gr_{\mathbb{M}}(M)$ if

$$gr_{\mathbb{M}}(g) = h.$$

Remark 1.5. If $p \in \operatorname{Syz}(S)$, then we have $gr_{\mathbb{L}}(p) \in \operatorname{Syz}(gr_{\mathbb{M}}(S))$; in particular the map $gr_{\mathbb{L}}$ induces a map

$$gr_{\mathbb{L}}| : \operatorname{Syz}(S) \to \operatorname{Syz}\bigl(gr_{\mathbb{M}}(S)\bigr).$$

Following the setting in [Sh, Section 2] and [KR], we introduce the concept of standard bases of a module.

Definition 1.6. Let M be a filtered R-module. A subset $S = \{f_1, \ldots, f_s\}$ of M is called a *standard basis* of M if

$$gr_{\mathbb{M}}(M) = \langle f_1^*, \ldots, f_s^* \rangle.$$

If any proper subset of S is not a standard basis, we call S a *minimal standard basis*.

Let $f : M \to N$ be a morphism of filtered R-modules and $gr(f) : gr_{\mathbb{M}}(M) \to gr_{\mathbb{N}}(N)$ the induced homomorphism. If $gr(f)$ is surjective, then f is a strict surjective homomorphism. By using this fact and Proposition 1.2, we can prove next theorem which gives a criteria for standard bases. In the case of ideals, the result had been proved in [RoV]. We omit here the proof because it is essentially the same as Theorem 2.9 in [Sh].

Theorem 1.7. *Let M be a filtered R-module, $f_1, \ldots, f_s \in M$ and $S = \{f_1, \ldots, f_s\}$. The following facts are equivalent*:

1. $\{f_1, \ldots, f_s\}$ *is a standard basis of M.*
2. $\{f_1, \ldots, f_s\}$ *generates M and every element of* $\mathrm{Syz}(gr_{\mathbb{M}}(S))$ *can be lifted to an element in* $\mathrm{Syz}(S)$.
3. $\{f_1, \ldots, f_s\}$ *generates M and* $\mathrm{Syz}(gr_{\mathbb{M}}(S)) = gr_{\mathbb{L}}(\mathrm{Syz}(S))$.

Note that a similar result is well known for Gröbner bases (see for example [KR, Theorem 2.4.1(D1) and (B2)]).

With the notation of the previous fundamental diagram, the equivalent conditions of Theorem 1.7 are also equivalent to the following:

4. $gr(\phi)$ *is surjective.*
5. $\{f_1, \ldots, f_s\}$ *generates M and ϕ is strict.*

In this setting it comes natural the following result presented in [Rob] and also in [HRV, Theorem 3.1], which gives a comparison between an R-free resolution of M and a G-free resolution of $gr_{\mathbb{M}}(M)$. The result will be a central tool in our investigation and we present here a proof in terms of standard bases because this constructive approach will be fundamental in the following.

Theorem 1.8. *Let M be a filtered R-module and $(\mathbf{G}., d.)$ a G-free graded resolution of $gr_{\mathbb{M}}(M)$. Then we can build up an R-free resolution $(\mathbf{F}., \delta.)$ of M and a special filtration $\mathbb{F}$ on it such that $gr_{\mathbb{F}}(\mathbf{F}.) = \mathbf{G}.$.*

Proof. Let

$$\mathbf{G}. : \cdots \to \bigoplus_{i=1}^{\beta_l} G(-a_{li}) \xrightarrow{d_l} \bigoplus_{i=1}^{\beta_{l-1}} G(-a_{l-1i}) \xrightarrow{d_{l-1}} \cdots \xrightarrow{d_1} \bigoplus_{i=1}^{\beta_0} G(-a_{0i}) \xrightarrow{d_0} gr_{\mathbb{M}}(M) \to 0$$

be a G-free resolution (not necessarily minimal) of $gr_{\mathbb{M}}(M)$. We define now $(\mathbf{F}., \delta.)$ by inductive process focused on Theorem 1.7.

We put $g_i = d_0(e_{0i}) \in gr_{\mathbb{M}}(M)$ and let $f_i \in M$ be a lifting of g_i. Starting from the integers $a_{0i} = \nu_{\mathbb{M}}(f_i)$, define F_0 the R-free module of rank β_0 with the special filtration

$$F_0 = \bigoplus_{i=1}^{\beta_0} R(-a_{0i}) \quad \text{and} \quad \delta_0 : F_0 \to M$$

such that $\delta_0(e_{0i}) = f_i$. Since d_0 is surjective, then the f_i's generate a standard base of M hence, by Theorem 1.7, $\operatorname{Ker}(d_0) = gr_{\mathbb{F}_0}(\operatorname{Ker}(\delta_0))$ and $F_0 \xrightarrow{\delta_0} M \to 0$ is exact.

Suppose that we have defined filtered free modules $F_0, \ldots, F_j$ with $0 \leqslant j < l$ such that

$$F_j \xrightarrow{\delta_j} F_{j-1} \to \cdots \to F_0 \xrightarrow{\delta_0} M \to 0$$

is a part of an R-free resolution of M. In particular, for every $i \leqslant j$, $gr_{\mathbb{F}_i}(F_i) = G_i$, $gr_{\mathbb{F}_i}(\delta_i) = d_i$ and moreover

$$\operatorname{Ker}(d_j) = gr_{\mathbb{F}_j}\bigl(\operatorname{Ker}(\delta_j)\bigr). \tag{2}$$

Let $\operatorname{Ker}(d_j) = \langle g_{j1}, \ldots, g_{j\beta_{j+1}} \rangle$, then there exist $f_{ji} \in \operatorname{Ker}(\delta_j)$ which are lifting of g_{ji}. So $g_{ji} = gr_{\mathbb{F}_j}(f_{ji})$ and $a_{j+1i} = \nu_{\mathbb{F}_j}(f_{ji})$. Define now the filtered free R-module

$$F_{j+1} = \bigoplus_{i=1}^{\beta_{j+1}} R(-a_{j+1i}) \quad \text{and} \quad \delta_{j+1} : F_{j+1} \to F_j$$

such that $\delta_{j+1}(e_{j+1i}) = f_{ji}$. Then

$$F_{j+1} \xrightarrow{\delta_{j+1}} F_j \xrightarrow{\delta_j} F_{j-1}$$

is a complex such that $gr_{\mathbb{F}_{j+1}}(\delta_{j+1}) = d_{j+1}$. Because of (2), $f_{j1}, \ldots, f_{j\beta_{j+1}}$ is a standard basis of $\operatorname{Ker}(\delta_j)$ as a submodule of the filtered module F_j. So again by Theorem 1.7, we get $\operatorname{Ker}(d_{j+1}) = gr_{\mathbb{F}_{j+1}}(\operatorname{Ker}(\delta_{j+1}))$ and we can continue by inductive process. $\square$

It is worth remarking that if we start from a minimal free resolution of $gr_{\mathbb{M}}(M)$, then the R-free resolution of M, given in the proof of Theorem 1.8, is not necessarily minimal and it is minimal if and only if the corresponding Betti numbers coincide, i.e. $\beta_i(gr_{\mathbb{M}}(M)) = \beta_i(M)$ for every $i \geqslant 0$. In general the following inequalities hold:

- $\beta_i(gr_{\mathbb{M}}(M)) \geqslant \beta_i(M)$ for every $i \geqslant 0$.

Let R be is a regular local ring and denote by $pd(\ )$ the projective dimension of a module:

- $pd(gr_{\mathbb{M}}(M)) \geqslant pd(M)$;
- $\operatorname{depth}(gr_{\mathbb{M}}(M)) \leqslant \operatorname{depth} M$.

We are interested in finding classes of finitely generated R-modules M for which the equalities hold. Accordingly with [HRV, Section 3] we give the following definition.

Definition 1.9. A filtered module M is said to be of homogeneous type with respect to the given filtration $\mathbb{M}$ if $\beta_i(gr_{\mathbb{M}}(M)) = \beta_i(M)$ for every $i \geqslant 0$.

When the R-module M is of homogeneous type *with respect to the* $\mathfrak{n}$*-adic filtration*, we simply say that *M is of homogeneous type*.

The $\mathfrak{n}$-adic filtration has a particular interest and it produces the first interesting class of modules of homogeneous type: the Koszul modules introduced by J. Herzog and S. Iyengar in [HI]. In fact, by [HI, Proposition 1.5], M is a Koszul R-module if and only if $gr_{\mathfrak{n}}(M)$ has a linear resolution as a $gr_{\mathfrak{n}}(R)$-module which implies in particular that M is of homogeneous type.

Remark 1.10. Consider $M = I$ an ideal of a regular local ring $(R, \mathfrak{n})$. In this paper we will focus our attention on the following filtrations of the ideal I.

1. $\mathbb{M} = \{\mathfrak{n}^p I\}$ (the $\mathfrak{n}$-adic filtration on I): in this case $gr_{\mathbb{M}}(I) = gr_{\mathfrak{n}}(I) = \bigoplus_{p\geqslant 0} I\mathfrak{n}^p / I\mathfrak{n}^{p+1}$. Accordingly with our setting, we say that I is of homogeneous type if $\beta_i(gr_{\mathfrak{n}}(I)) = \beta_i(I)$ for every $i \geqslant 0$.
2. $\mathbb{M} = \{\mathfrak{n}^p \cap I\}$: in this case $gr_{\mathbb{M}}(I) = I^* = \bigoplus_{p\geqslant 0} I \cap \mathfrak{n}^p / I \cap \mathfrak{n}^{p+1} = \bigoplus_{p\geqslant 0} I \cap \mathfrak{n}^p + \mathfrak{n}^{p+1}/\mathfrak{n}^{p+1}$ is the ideal of the polynomial ring $P = gr_{\mathfrak{n}}(R)$ generated by the initial forms of the elements of I. Hence if $A = R/I$ and $\mathfrak{m} = \mathfrak{n}/I$, the associated graded ring $gr_{\mathfrak{m}}(A) \simeq P/I^*$. According to our setting, we say that I is of homogeneous type with respect to $\mathbb{M}$ if $\beta_i(gr_{\mathbb{M}}(I)) = \beta_i(I)$ for every $i \geqslant 0$. This is equivalent to say that $\beta_i(A) = \beta_i(gr_{\mathfrak{m}}(A))$ for every $i \geqslant 0$, that is the local ring $(A, \mathfrak{m})$ is of homogeneous type.

The following examples show how it is difficult to find modules of homogeneous type.

Example 1.11. (1) Let $I = (x^3 - y^7, x^2y - xt^3 - z^6)$ be in $R = k[[x, y, z, t]]$. The ideal is a complete intersection and hence the resolution of I as an R-module is given by the Koszul complex. But

$$I^* = \left(x^3, x^2y, x^2t^3, xt^6, x^2z^6, xy^9 - xz^6t^3, xy^8t^3, y^7t^9\right) \subseteq P = k[x, y, z, t]$$

and hence $\beta_0(I^*) = 8 > \beta_0(I) = 2$. Using [CoCoA], it is possible to check that $\beta_1(I^*) = 12 > \beta_1(I) = 1$, $\beta_2(I^*) = 6 > \beta_2(I) = 0$, $\beta_3(I^*) = 1 > \beta_3(I) = 0$.

The following example shows that $\beta_0(I) = \beta_0(I^*)$ and $pd(I) = pd(I^*)$ do not force I to be of homogeneous type.

(2) Consider the local ring

$$A = k\left[\left[t^{19}, t^{26}, t^{34}, t^{40}\right]\right] = k[[x, y, z, t]]/I,$$

one can prove that I is minimally generated by an $\mathfrak{n}$-standard base, i.e. $\beta_0(I) = \beta_0(I^*) = 5$, I and I^* are perfect ideals (hence they have the same projective dimension), nevertheless I is not of homogeneous type with respect to $\mathbb{M} = \{\mathfrak{n}^p \cap I\}$, neither A is of homogeneous type (see [HRV, Example (3)]).

Nevertheless examples of local rings of homogeneous type (not necessarily Koszul) can be given.

Example 1.12. (1) Let I be an ideal of R generated by a super-regular sequence. This means that $I = (f_1, \ldots, f_r)$ where $f_1, \ldots, f_r$ is a regular sequence and an $\mathfrak{n}$-standard base of I, equivalently the initial forms $f_1^*, \ldots, f_r^*$ are a regular sequence in $P = gr_{\mathfrak{n}}(R)$ (see [VV]). Then both $A = R/I$ and I are of homogeneous type (see [HRV, Example 1, Theorem 3.6]).

(2) Let I be the ideal generated by the maximal minors of a generic $r \times s$ ($r \leqslant s$) matrix $X = (x_{ij})$ in $R = k[[x_{ij}]]$, then $gr_{\mathfrak{n}}(I) \simeq I(-r)$ has a linear resolution and it is easy to prove that I is of homogeneous type.

(3) Let I be an ideal of the regular ring $(R, \mathfrak{n})$ such that $A = R/I$ is Cohen–Macaulay of minimal multiplicity and let $\mathfrak{m} = \mathfrak{n}/I$, then J. Sally (see [Sa]) proved that $gr_{\mathfrak{m}}(A)$ is Cohen–Macaulay of minimal degree and I has a standard base of equimultiple elements of degree 2. From this, using [HRV, Lemma 3.3], one can prove that I is of homogeneous type.

(4) Let I be an ideal of the regular ring $(R, \mathfrak{n})$ generated by two elements, then I is of homogeneous type (see [HRV, Proposition 3.4]).

(5) Let I be the defining ideal of a monomial curve in $\mathcal{A}^3$ in the regular ring R such that $\nu(I) = \nu(I^*)$, then $A = R/I$ is of homogeneous type. Because I is a perfect ideal of codimension two, it is enough to recall that Robbiano and Valla in [RoV1] proved that, in this case, $gr_{\mathfrak{m}}(A)$ is Cohen–Macaulay.

We remark that Proposition 2.4 in [RS] gives us a criterion for producing more modules of homogeneous type.

2. Properties of componentwise linear modules

The componentwise linear modules over a polynomial ring had been introduced by Herzog and Hibi by enlarging the class of the graded modules with a d-linear resolution. Interesting results concerning their graded Betti numbers had been proved by Aramova, Conca, Herzog and Hibi (see [H,HH, AHH,CHH,C]). Later Römer (see [R]) studied more homological properties of the componentwise linear graded modules in the general setting of finitely generated modules over Koszul algebras (instead of polynomial rings), some of them partially overlap with those of Martinez and Zacharia in [MZ]. Thanks to the fact that componentwise linear modules and graded Koszul modules coincide, in the literature one can find two different approaches: the first coming from Herzog and Hibi's methods (see [HH,CHH]) dealing with graded ideals in the polynomial ring and a purely homological approach (see [MZ] and [HI]) on modules over Koszul algebras. However, in view of the applications, in this section we consider graded modules over a polynomial ring $P = k[x_1, \dots, x_n]$ even if most of the results hold in a more general setting.

Let N be a graded P-module. For $d \in \mathbf{Z}$ we write $N_{\langle d \rangle}$ for the submodule of N which is generated by all homogeneous elements of N with degree d. In the graded case we may also define the graded Betti numbers, i.e.

$$\beta_{i,j}(N) := \dim_k Tor_i^P(k, N)_j.$$

If the context is clear, we will simply write $\beta_{i,j}$.

Definition 2.1. Let N be a graded P-module.

(i) Let $d \in \mathbf{Z}$. Then N has a d-linear resolution if $\beta_{i,j} = 0$ for $j \neq d + i$.
(ii) N is componentwise linear if for all integers d the module $N_{\langle d \rangle}$ has a d-linear resolution.

For more information concerning the componentwise linear modules, see [HH,C,R,CHH]. We select here some good properties of their graded minimal free resolutions.

Set $indeg(N) = \min\{d \in \mathbf{Z}\colon\ N_d \neq 0\}$. If N is componentwise linear, it is known that $N/N_{\langle indeg(N) \rangle}$ is componentwise linear too (see [R, Lemma 3.2.2]). Let $(\mathbf{G.}, d.)$ be the minimal graded free resolution of N and define the subcomplex $(\widetilde{\mathbf{G}}., \widetilde{d}.)$ of $(\mathbf{G.}, d.)$ by

$$\widetilde{G}_i = P\bigl(-\bigl(i + indeg(N)\bigr)\bigr)^{\beta_{i,i+indeg(N)}} \subseteq G_i \quad \text{and} \quad \widetilde{d}. = d.|_{\widetilde{\mathbf{G}}.}.$$

Römer proved that $\widetilde{\mathbf{G}}.$ is the *minimal* (linear) graded free resolution of $N_{\langle indeg(N) \rangle}$ and $\mathbf{G.}/\widetilde{\mathbf{G}}.$ is the minimal graded free resolution of $N/N_{\langle indeg(N) \rangle}$ (see [R, Lemma 3.2.4]).

As a consequence of these properties we easily get the following information that have an intrinsic interest in the theory of componentwise linear modules.

Proposition 2.2. *Let N be a graded P-module minimally generated in degrees $i_1, \dots, i_m$. Assume N is componentwise linear and let $(\mathbf{G.}, d.)$ be the minimal graded free resolution of N. Then for every $1 \leqslant s \leqslant pd(N)$ we have*

$$Tor_s^P(k, N)_j = 0 \quad \textit{for } j \neq i_1 + s, \dots, i_m + s.$$

In particular, if for some $1 \leqslant s \leqslant pd(N)$ and $1 \leqslant r \leqslant m$, $Tor_s^P(k, N)_{i_r+s} = 0$, then $Tor_{s+1}^P(k, N)_{i_r+s+1} = 0$.

Proof. Let $N_1 = N_{\langle indeg(N)\rangle}$. Since N is componentwise linear, N_1 has $i_1 = indeg(N)$-linear resolution and $\overline{N} = N/N_1$ is a componentwise linear module minimally generated in degrees $i_2, \dots, i_m$. Thus, from the short exact sequence

$$0 \to N_1 \to N \to \overline{N} \to 0$$

we obtain $Tor_i^P(k, N)_j = Tor_i^P(k, N_1)_j \oplus Tor_i^P(k, \overline{N})_j$. Repeating this procedure, we can find the sequence of graded modules $N_1, \dots, N_m$, such that N_i has an i_r-linear resolution and

$$Tor_i^P(k, N)_j = \bigoplus_{r=1}^{m} Tor_i^P(k, N_r)_j.$$

Hence, the conclusion follows. $\square$

The following remark will clarify the shape of the matrices associated to the differential maps of the resolutions of componentwise linear modules.

Remark 2.3. Let N be a graded P-module generated in degrees $i_1, \dots, i_m$. Assume N is componentwise linear and let $(\mathbf{G.}, d.)$ be the minimal graded free resolution of N. Then by the above proposition $G_s = \bigoplus_{j=1}^{m} P^{\beta_{s,i_j+s}}(-(i_j + s))$ for every $1 \leqslant s \leqslant pd(N)$. We want to describe the shape of the matrix $\mathcal{M}_s$ associated to $G_s \xrightarrow{d_s} G_{s-1}$ with respect to the canonical homogeneous bases of G_s and G_{s-1} of degrees respectively $i_1 + s, \dots, i_m + s$ and $i_1 + s - 1, \dots, i_m + s - 1$.

By using Proposition 2.2, without loss of generality, we may assume $\mathcal{M}_s$ of the following shape:

$$
\begin{array}{rc}
 & \begin{array}{cccc} \overbrace{\qquad\quad}^{i_1+s} & \overbrace{\qquad\quad}^{i_2+s} & \qquad\quad & \overbrace{\qquad\quad}^{i_m+s} \end{array} \\
\begin{array}{r} i_1+s-1\,\{ \\ i_2+s-1\,\{ \\ \\ i_m+s-1\,\{ \end{array} &
\begin{pmatrix}
B_{i_1i_1s} & B_{i_1i_2s} & \cdots & B_{i_1i_ms} \\
0 & B_{i_2i_2s} & \cdots & B_{i_2i_ms} \\
0 & 0 & \ddots & \vdots \\
0 & \cdots & 0 & B_{i_mi_ms}
\end{pmatrix}
\end{array}
$$

where all the non-zero entries of $B_{i_1i_1s}, B_{i_2i_2s}, B_{i_mi_ms}$ (diagonal blocks) are linear forms and the non-zero entries of $B_{i_pi_qs}$ with $p < q$ (up-diagonal blocks) are forms of degree at least two.

Remark 2.4. Let N be a componentwise linear graded P-module. Set $N_1 := N$ and for every $j = 2, \dots, m$ we define

$$N_j := N_{j-1}/(N_{j-1})_{\langle indeg(N_{j-1})\rangle} = N_{j-1}/(N_{j-1})_{\langle i_{j-1}\rangle}.$$

By [R, Lemmas 3.2.2 and 3.2.4], it is easy to show that, for every $1 \leqslant s \leqslant pd(N)$, the matrices $B_{i_j i_j s}$ (on the diagonal) in $\mathcal{M}_s$ have the following properties:

- in each column of $B_{i_j i_j s}$ at least one entry is different from zero;
- the columns of $B_{i_j i_j s}$ minimally generate the s-th syzygy module of $(N_j)_{\langle i_j\rangle}$.

Indeed it is enough to remark that the matrices $B_{i_j i_j s}$ can be considered as the matrices associated to the differential maps of the minimal free resolution of $(N_j)_{\langle i_j\rangle}$ which has linear resolution.

We present the following example in order to help the reader to visualize better the resolutions of componentwise linear modules.

Example 2.5. Let $P = k[x_1, x_2, x_3, x_4]$ and $I = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3^2, x_1x_4^3, x_3^4)$. The ideal I is Borel-fixed, so I is componentwise linear and the minimal free resolution of I is:

$$\begin{aligned} &0 \to P(-7) \xrightarrow{d_3} P(-4) \oplus P(-5) \oplus P^4(-6) \xrightarrow{d_2} P^4(-3) \oplus P^2(-4) \oplus P^5(-5) \\ &\xrightarrow{d_1} P^4(-2) \oplus P(-3) \oplus P^2(-4) \to I \to 0. \end{aligned}$$

According to Remark 2.3, we have

$$\mathcal{M}_1 = \left(\begin{array}{cccc|cc|ccccc} x_2 & x_1 & 0 & 0 & 0 & 0 & x_4^3 & 0 & 0 & 0 & x_3^3 \\ 0 & 0 & x_1 & 0 & x_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_3 & 0 & -x_2 & x_1 & 0 & x_3^2 & 0 & x_4^3 & 0 & 0 & 0 \\ 0 & -x_3 & 0 & -x_2 & 0 & 0 & 0 & 0 & x_4^3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -x_2 & -x_1 & 0 & 0 & 0 & x_3^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_2 & -x_1 & 0 & 0 \end{array}\right)$$

One can find a similar shape for $\mathcal{M}_2$ and $\mathcal{M}_3$.

3. Extremal Betti numbers

In this section, we present the main result of the paper and the application to the minimal free resolutions of a local ring. We denote by $\mu(\ )$ the minimal number of generators of a module over a local ring (or the minimal number of generators of a graded module over the polynomial ring).

Theorem 3.1. *Let M be a finitely generated filtered module over a regular local ring $(R, \mathfrak{n})$. Assume that*

1. $\mu(M) = \mu(gr_{\mathbb{M}}(M))$;
2. $gr_{\mathbb{M}}(M)$ *is a componentwise linear P-module.*

Then M is of homogeneous type with respect to $\mathbb{M}$.

Proof. For short we denote $gr_{\mathbb{M}}(M)$ by M^*. By Corollary 2.2, its minimal graded free resolution $(\mathbf{G}_{\boldsymbol{\cdot}}, d_{\boldsymbol{\cdot}})$ as a $P = gr_{\mathfrak{n}}(R)$-module has the following shape:

$$0 \to \bigoplus_{j=1}^{m} P^{\beta_{h,i_j+h}}\bigl(-(i_j+h)\bigr) \xrightarrow{d_h} \cdots \xrightarrow{d_1} \bigoplus_{j=1}^{m} P^{\beta_{0,i_j}}(-i_j) \to M^* \to 0$$

where $h = pd(M^*)$ and $0 < i_1 < i_2 < \cdots < i_m$ are the degrees of a minimal system of generators of M^*. Denote by $\beta_t(M^*)$ the total Betti numbers of M^*. From $(\mathbf{G}_{\boldsymbol{\cdot}}, d_{\boldsymbol{\cdot}})$ we can build up a free resolution $(\mathbf{F}_{\boldsymbol{\cdot}}, \delta_{\boldsymbol{\cdot}})$ of M by the inductive process described in Theorem 1.8:

$$0 \to \bigoplus_{j=1}^{m} R^{\beta_{h,i_j+h}}\bigl(-(i_j+h)\bigr) \xrightarrow{\delta_h} \cdots \xrightarrow{\delta_1} \bigoplus_{j=1}^{m} R^{\beta_{0,i_j}}(-i_j) \to M \to 0.$$

We have to prove that $(\mathbf{F}_{\boldsymbol{\cdot}}, \delta_{\boldsymbol{\cdot}})$ is minimal. For every $t = 0, \dots, h$ denote by $\mathcal{M}_t^*$ (resp. $\mathcal{M}_t$) the matrix of the differential map d_t (resp. δ_t). We prove that the columns of $\mathcal{M}_t$ for $t = 0, \dots, h$ *minimally* generate the t-th syzygy module of M, that is $\operatorname{Ker}(\delta_{t-1})$. We proceed by inductive process. The first step $(t = 0)$ follows easily from the assumption $\mu(M) = \mu(M^*)$, which says that a minimal system of generators of M^* (say $g_{-1,i}$) build up a minimal system of generators of M (say $f_{-1,i}$). Suppose now that for each $0 \leqslant t \leqslant s < h$ we have proved that

$$F_s = \bigoplus_{j=1}^{m} R^{\beta_{s,i_j+s}}\bigl(-(i_j+s)\bigr) \xrightarrow{\delta_s} \cdots \xrightarrow{\delta_1} F_0 = \bigoplus_{j=1}^{m} R^{\beta_{0,i_j}}(-i_j) \xrightarrow{\delta_0} M \to 0$$

is part of a *minimal* free resolution of M. This means in particular that, if $d_s(e_{s,r})) = g_{s-1,r}$ and $f_{s-1,r}$ is the corresponding lifting in the filtered module F_{s-1} $(F_{-1} = M)$, then $\{f_{s-1,1}, \dots, f_{s-1,\beta_s}\}$ is a minimal generating set for the s-th syzygy module. By following the proof of Theorem 1.8, we have to prove now that

$$F_{s+1} = \bigoplus_{i=1}^{\beta_m} R^{\beta_{s+1,i_j+s+1}}\bigl(-(i_j+s+1)\bigr) \xrightarrow{\delta_{s+1}} F_s \to F_{s-1}$$

is part of a minimal free resolution of M. Because $\{f_{s-1,1}, \dots, f_{s-1,\beta_s}\}$ is a minimal generating set for the module their generated, we conclude that all entries of $\mathcal{M}_{s+1}$ belong to $\mathfrak{n}$. The goal is to prove that the columns of the matrix $\mathcal{M}_{s+1}$ *minimally* generate the $s+1$-st syzygy module of M.

Since M^* is componentwise linear, accordingly with Remark 2.3, we may assume that $\mathcal{M}_{s+1}^*$ has the following shape:

$$\begin{array}{c|c}
 & \begin{array}{cccc} \overbrace{i_1+s+1} & \overbrace{i_2+s+1} & & \overbrace{i_m+s+1} \end{array} \\
\begin{array}{r} i_1+s\{ \\ i_2+s\{ \\ \\ i_m+s\{ \end{array} &
\begin{pmatrix}
M^*_{i_1 i_1 s+1} & M^*_{i_1 i_2 s+1} & \cdots & M^*_{i_1 i_m s+1} \\
0 & M^*_{i_2 i_2 s+1} & \cdots & M^*_{i_2 i_m s+1} \\
0 & 0 & \ddots & \vdots \\
0 & \cdots & 0 & M^*_{i_m i_m s+1}
\end{pmatrix}
\end{array}$$

The columns of $\mathcal{M}^*_{s+1}$ are the initial forms of the columns of $\mathcal{M}_{s+1}$ with respect to the special filtration $(F_s)_p = \bigoplus_{j=1}^m \bigoplus_{k=1}^{\beta_{s,i_j+s}} \mathfrak{n}^{p-i_j-s} e^s_{k,i_j}$ on F_s. In particular by looking the degree matrix of $\mathcal{M}^*_{s+1}$, the elements of $M^*_{i_p i_q s+1}$ have degree $i_q - i_p + 1$ and the matrices $M^*_{i_j i_j s+1}$ (on the diagonal) have the good properties described in Remark 2.4.

Denote now by $M_{i_p i_q s+1}$ the blocks corresponding to the rows labeled by $i_p + s$ and the columns labeled by $i_q + s + 1$ in $\mathcal{M}_{s+1}$. Remark that in $\mathcal{M}_{s+1}$ the non-zero entries of the blocks $M_{i_p i_q s+1}$ have valuation at least $i_q - i_p + 1$. Hence if $p < q$ the non-zero elements have valuation at least 2. Notice that the non-zero entries of the blocks with $p > q$ (under the diagonal blocks) have valuation $\geqslant 1$ because all the entries belong to $\mathfrak{n}$.

Denote by $C_{i_k j}$ with $k = 1, \ldots, m$ and $j = 1, \ldots, \beta_{s+1, i_k+s+1}$ the columns of $\mathcal{M}_{s+1}$ (respectively $C^*_{i_k j}$ those of $\mathcal{M}^*_{s+1}$). By Remark 2.4 in each column $C_{i_k j}$ there is at least one element of valuation 1 and it belongs to the block $M_{i_k i_k s+1}$. Suppose now that there exists (k, j) such that

$$C_{i_k j} = \sum_{r=1}^{m} \sum_{p=1}^{\beta_{s+1, i_r+s+1}} \lambda_{(i_r, p)} C_{i_r p}$$

with $(i_r, p) \neq (i_k, j)$. Because in $C_{i_k j}$ there is at least one element of valuation 1 and the entries of $M_{i_k i_r s+1}$ with $r > k$ have valuation at least 2, necessarily there exists an integer u with $1 \leqslant u \leqslant k$ such that $\lambda_{(i_u, p)} \notin \mathfrak{n}$ for some p. Assume u the least integer with such property. This leads to prove that the columns of $M^*_{i_u i_u s+1}$ are not linearly independent against Remark 2.4. Assume $u = k$ and let $C_{i_k p_1}, \ldots, C_{i_k p_t}$ be columns corresponding to invertible coefficients $\lambda_{(i_k, p_1)}, \ldots, \lambda_{(i_k, p_t)}$ (in the band $i_k + s + 1$). By using again that the entries of $M_{i_k i_r s+1}$ with $r > k$ have valuation at least 2, one can easily prove that the columns of $M^*_{i_k i_k s+1}$ corresponding to $(i_k, j), (i_k, p_1), \ldots, (i_k, p_t)$ are not linearly independent. In the same way if $u < k$ we repeat the same argument on $M^*_{i_u i_u s+1}$ and we get the conclusion. $\square$

One of the main goal of the paper is the application of the above result to an ideal I of the regular local ring $(R, \mathfrak{n})$. In particular, we are interested in comparing the numerical invariants of the minimal R-free resolution of $A = R/I$ and those of the minimal graded P-free resolution of $gr_{\mathfrak{m}}(A) = P/I^*$ where $\mathfrak{m} = \mathfrak{n}/I$ and I^* is the graded ideal generated by the initial forms of I. We recall that if we apply the general theory on filtered modules to $M = I$ and $\mathbb{M} = \{\mathfrak{n}^p \cap I\}$, as we described in Remark 1.10, we obtain $gr_{\mathbb{M}}(M) = I^*$.

As we have already said

$$\beta_i(R/I) \leqslant \beta_i\big(P/I^*\big). \tag{3}$$

Theorem 3.1 says that if I is minimally generated by an $\mathfrak{n}$-standard base and I^* is componentwise linear, then the equality holds. Notice that in Theorem 3.1 the assumption which I is minimally generated by an $\mathfrak{n}$-standard base is necessary and it is not a consequence of the assumption that I^* is componentwise linear. For example, if I is the defining ideal of $A = k[\![t^{10}, t^{19}, t^{21}, t^{53}]\!]$, then one can check that I^* is componentwise linear, but $\mu(I) = 5$ and $\mu(I^*) = 7$.

The inequality (3) and Theorem 3.1 suggest us new upper bounds coming from the homogeneous context. From now on assume the residue field k of characteristic 0. We have two monomial ideals canonically attached to I^*: the generic initial ideal with respect to the revlex order and the lex-segment ideal of I^* characterized by Macaulay's theorem. They play a fundamental role in the investigation of many algebraic, homological, combinatorial and geometric properties of the ideal I itself. We denote

$$Gin(I) := Gin\big(I^*\big) \quad \text{and} \quad Lex(I) := Lex\big(I^*\big).$$

For the first equality, notice that in [B] it is proved that if $R = k[\![x_1, \ldots, x_n]\!]$, one can define an anti-degree-compatible ordering on the terms of R such that the leading term ideal of I, after performing a 'generic change' of coordinates, is a monomial ideal which coincides with $Gin(I^*)$. The second equality is clear from Macaulay's theorem because the Hilbert function of the local ring $A = R/I$ is the Hilbert function of $gr_{\mathfrak{m}}(A) = P/I^*$. All the involved monomial ideals have the same Hilbert function, indeed

$$HF_A(n) = HF_{gr_{\mathfrak{m}}(A)}(n) = HF_{P/I^*}(n) = HF_{P/Lex(I)}(n) = HF_{P/Gin(I)}(n),$$

nevertheless, since $\beta_i(R/I^*) \leqslant \beta_i(P/Gin(I^*)) \leqslant \beta_i(P/Lex(I^*))$, we have

$$\beta_i(R/I) \leqslant \beta_i\big(P/Gin(I)\big) \leqslant \beta_i\big(P/Lex(I)\big) \tag{4}$$

for every $i \geqslant 0$. The first inequality follows by standard deformation argument, the second was proved by A. Bigatti [B] and H.A. Hulett [Hu] in characteristic zero and extended later by K. Pardue [P] to positive characteristic.

Componentwise linear ideals have been characterized by A. Aramova, J. Herzog and T. Hibi in [AHH] as those ideals having the same Betti numbers as their generic initial ideal.

Theorem 3.2. *(See [AHH, Theorem 1.1].) Let J be a homogeneous ideal of P. The following facts are equivalent:*

(i) $\mu(J) = \mu(Gin(J))$.
(ii) $\beta_i(J) = \beta_i(Gin(J))$ *for every* $i \geqslant 0$.
(iii) $\beta_{ij}(J) = \beta_{ij}(Gin(J))$ *for every* $i, j \geqslant 0$.
(iv) J *is componentwise linear.*

Generalization of this result have been proved in [CHH,C,P].

Let now I be an ideal in the local ring R and we present the similar result in the local setting. Since $\mu(I) \leqslant \mu(I^*) \leqslant \mu(Gin(I))$, as a corollary of Theorem 3.1 and the above result, we deduce the following characterization.

Corollary 3.3. *Let I be an ideal of the regular local ring $(R, \mathfrak{n})$. The following facts are equivalent:*

(i) $\mu(I) = \mu(Gin(I))$.
(ii) $\beta_i(I) = \beta_i(I^*) = \beta_i(Gin(I))$ *for every* $i \geqslant 0$.
(iii) $\mu(I) = \mu(I^*)$ *and* I^* *is componentwise linear.*

J. Herzog and T. Hibi [HH, Corollary 1.4] proved the corresponding result of Theorem 3.2 for the lex-segment ideal associated to a homogeneous ideal of P. The result is very interesting because in general it is easier to determine $Lex(I)$ (it is uniquely determined by the Hilbert function) than $Gin(I)$.

Similarly, starting from the inequalities $\mu(I) \leqslant \mu(I^*) \leqslant \mu(Gin(I)) \leqslant \mu(Lex(I))$, it is easy to deduce:

Corollary 3.4. *Let I be an ideal of the regular local ring $(R, \mathfrak{n})$. The following facts are equivalent*:

(i) $\mu(I) = \mu(Lex(I))$.
(ii) $\beta_i(I) = \beta_i(I^*) = \beta_i(Lex(I))$ *for every* $i \geqslant 0$.
(iii) $\mu(I) = \mu(I^*)$ *and* I^* *is a Gotzmann ideal.*

Example 3.5. (1) Let $(A, \mathfrak{m}, k)$ be a stretched Cohen–Macaulay ring of embedding codimension h. Sally in [Sa] defined the stretched Cohen–Macaulay ring as the local rings which admit an Artinian reduction B with Hilbert function $H_B(i) \leqslant 1$ for $i \geqslant 2$. For example this is the case if A is Cohen–Macaulay of multiplicity $\leqslant h+2$.

If A has maximal Cohen–Macaulay type (i.e. h) and $gr_{\mathfrak{m}}(A)$ is Cohen–Macaulay, then A is of homogeneous type. We may assume $A = R/I$ where R is a regular local ring. By reducing the problem to the Artinian reduction B, it is known (see [Sa] and [EV]) that $\mu(I) = \binom{h+1}{2}$. In particular $\mu(I) = \mu(Gin(I))$, then B (hence A) is of homogeneous type by Corollary 3.3.

(2) The local ring $A = k[[t^9, t^{17}, t^{19}, t^{39}]]$ is of homogeneous type. In this case the defining ideal

$$I = \left(x_2x_3 - x_1^4, x_2^5 - x_1x_3^4, x_2x_4 - x_1^2x_3^2, x_3^2x_4 - x_1x_2^4, x_3^3 - x_1^2x_4, x_4^2 - x_1^3x_2^3\right)$$

is minimally generated by a standard base. Moreover $Gin(I) = (x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_2x_3^2, x_3^5)$, hence $\mu(I) = \mu(I^*) = \mu(Gin(I) = 6$ and we may apply Corollary 3.3. Notice that A is not stretched because $H_B(2) = 3$.

A. Conca, J. Herzog and T. Hibi proved that the Betti numbers of an ideal I of a regular local ring R of dimension n can be related to another sequence of numbers, $\alpha_1(I), \alpha_2(I), \ldots$ called the *generic annihilator numbers* of $A = R/I$. Assuming that the residue class field is infinite, regular system of parameters $y_1, \ldots, y_n$ can be chosen such that for every $p = 1, \ldots, n$

$$A_p := (y_1, \ldots, y_{p-1})A :_A y_p/(y_1, \ldots, y_{p-1})A$$

is of finite length. Denoting by

$$\alpha_p(A) := \text{length}\, A_p,$$

in [CHH, Corollary 1.2] it was proved that

$$\beta_i(A) \leqslant \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j(A).$$

If A is a graded standard algebra, in [CHH] it is proved that the equality holds provided the homogeneous ideal I is componentwise linear. In the local case we loose this characterization, but by [CHH, Theorem 1.5 and Remark 1.6], the equality holds if I is Koszul or equivalently $gr_{\mathfrak{n}}(I)$ has a linear resolution. By [HI, Proposition 1.5], for proving that a module is Koszul, it will be useful to introduce the *linearity defect* denoted by ld.

As usual let $(\mathbf{F.}, \delta.)$ be a minimal R-free resolution of a module M. For all integer i we have

$$gr_{\mathfrak{n}}(F_i)(-i) = \bigoplus_{j \geqslant i} \mathfrak{n}^{j-i} F_i / \mathfrak{n}^{j+1-i} F_i \simeq gr_{\mathfrak{n}}(R)^{\beta_i(M)}(-i).$$

Following this construction due to D. Eisenbud, G. Floystad and F.O. Schreyer in [EFS], the differential maps δ_i induces a bihomogeneous map:

$$\delta_{i+1}^{lin} : gr_{\mathfrak{n}}(F_{i+1})(-i-1) \to gr_{\mathfrak{n}}(F_i)(-i)$$

which can be described by matrices of *linear forms*. Precisely the matrices, say $\mathcal{M}_{i+1}^{lin}$, are obtained by replacing in $\mathcal{M}_{i+1}$ all entries of valuation > 1 by 0 and by replacing all the entries of valuation one by their initial forms with respect to the $\mathfrak{n}$-adic filtration. The minimality of $(\mathbf{F.}, \delta.)$ ensures that the maps $\{\delta_i^{lin}\}$ are well defined and form a complex homomorphism denoted by $lin^R(\mathbf{F.})$ which is not necessarily exact. It is called the *linear part of the resolution*. For the construction of this complex and related results see [EFS], as well [HI,R]. T. Römer introduced a measure for the lack of the exactness and he defined

$$ld(M) := \inf\bigl\{j\colon\ H_i\bigl(lin^R(\mathbf{F.})\bigr) = 0 \text{ for } i \geqslant j+1\bigr\}.$$

In particular $lpd(M) = 0$ if and only if $lin^R(\mathbf{F.})$ is exact. Following [HI], M is Koszul if and only if $ld(M) = 0$. Römer proved in [R, Theorem 3.2.8] that, for graded modules, having $ld(M) = 0$ is equivalent to be componentwise linear and hence to be Koszul. Herzog and Iyengar proved in [HI, Proposition 1.5] that to be *Koszul* is equivalent to the fact that $lin^R(\mathbf{F.})$ is the minimal free resolution of $gr_{\mathfrak{n}}(M) = \bigoplus_j \mathfrak{n}^j M / \mathfrak{n}^{j+1} M$. In particular this is the case if and only if $gr_{\mathfrak{n}}(M)$ has a linear resolution as a $gr_{\mathfrak{n}}(R)$-module.

The following theorem says that under the assumptions of Theorem 3.1, the module M is Koszul, hence M is of homogeneous type (w.r.t. the $\mathfrak{n}$-adic filtration).

Theorem 3.6. *Let M be a finitely generated filtered module over a regular local ring $(R, \mathfrak{n})$ such that $\mu(M) = \mu(gr_{\mathbb{M}}(M))$.*

If $gr_{\mathbb{M}}(M)$ is a componentwise linear P-module (equivalently a Koszul graded module), then M is Koszul.

Proof. We prove $ld(M) = 0$. Let $M^* = gr_{\mathbb{M}}(M)$ and $(\mathbf{G.}, d.)$ be a minimal P-free resolution of M^* (where $P = gr_{\mathfrak{n}}(R)$). Denote by $0 < i_1 < \cdots < i_m$ the degrees of the elements of a minimal set of generators of M^*.

From $\mathbf{G.}$ we can build $lin^P(\mathbf{G.})$ as defined before. Since M^* is a componentwise linear module, $lin^P(\mathbf{G.})$ is exact ($ld(M^*) = 0$). Moreover, we can split $lin^R(\mathbf{G.})$ as $\bigoplus_{r=1}^m lin^P(\mathbf{G_{i_r}.})$ where $lin^P(\mathbf{G_{i_r}.})$ is the linear part of the resolution of the submodule of M^* generated by the minimal set of generators of M^* of degree i_r.

In fact, by the construction, the matrices $lin(\mathcal{M}_j^*)$ associated to

$$d_j^{lin} : P^{\beta_j(M^*)}(-j) \to P^{\beta_{j-1}(M^*)}(-j+1)$$

are obtained from $\mathcal{M}_j^*$ by replacing all the entries of degree > 1 by 0. Then, by Remark 2.3, the matrices $lin(\mathcal{M}_j^*)$, $1 \leqslant j \leqslant pd(M)$ will present the following shape:

$$
\begin{array}{rc}
 & \begin{array}{ccc} \overbrace{\qquad\quad}^{i_1+j} & \qquad & \overbrace{\qquad\quad}^{i_m+j} \end{array} \\
\begin{array}{r} i_1+j-1\,\{ \\ \\ i_m+j-1\,\{ \end{array} & \left(\begin{array}{c:c:c} M^*_{i_1 i_1 j} & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & M^*_{i_m i_m j} \end{array}\right)
\end{array}
$$

We remark that all the blocks on the diagonal are the same as in $\mathcal{M}_j^*$ (they have linear entries) and the upper diagonal blocks are replaced by 0 because the corresponding entries have degree at least two.

Also, by Theorem 1.8, from $\mathbf{G.}$ we can build the minimal R-free resolution $(\mathbf{F.}, \delta.)$ of M. By Theorem 3.1, $\beta_j(M) = \beta_j(M^*)$. In its turn, from $(\mathbf{F.}, \delta.)$ we can build $lin^R(\mathbf{F.})$. We have to prove that

$$H_i\big(lin^P(\mathbf{G.})\big) = 0 \quad \Longrightarrow \quad H_i\big(lin^R(\mathbf{F.})\big) = 0 \quad \text{for every } i \geqslant 1.$$

We remark that $G_j^{lin} = F_j^{lin} \simeq P^{\beta_j(M)}(-j)$ where $\beta_j(M) = \beta_j(M^*) = \sum_{r=1}^m \beta_{j,i_r+j}$ and, without loss of generality, we denote by e_{s,i_r}^j for $r = 1, \dots, m$, $s = 1, \dots, \beta_{j,i_r+j}$ a basis of both G_j^{lin} and F_j^{lin}. The matrices $lin(\mathcal{M}_j)$ associated to $\delta_j^{lin} : P^{\beta_j(M)}(-j) \to P^{\beta_{j-1}(M)}(-j+1)$ with respect to these bases are obtained from $\mathcal{M}_j$ and they have the following shape:

$$
\begin{array}{rc}
 & \begin{array}{ccc} \overbrace{\qquad\quad}^{i_1+j} & \qquad & \overbrace{\qquad\quad}^{i_m+j} \end{array} \\
\begin{array}{r} i_1+j-1\,\{ \\ \\ i_m+j-1\,\{ \end{array} & \left(\begin{array}{c:c:c} M^*_{i_1 i_1 j} & 0 & 0 \\ \hdashline * & \ddots & 0 \\ \hdashline * & * & M^*_{i_m i_m j} \end{array}\right)
\end{array}
$$

where the "diagonal blocks" coincide with those of $lin(\mathcal{M}_j^*)$ (the non-zero entries have valuation 1) and $*$ denotes 0 or linear forms. Since we always have $\operatorname{Ker}\delta_j^{lin} \supseteq \operatorname{Im}\delta_{j+1}^{lin}$, we prove $\operatorname{Ker}\delta_j^{lin} \subseteq \operatorname{Im}\delta_{j+1}^{lin}$ for every $j = 1, \dots, pd(M)$.

Fixed an integer $r \in \{1, \dots, m\}$, denote by $N_{i_r}^j$ the submodule of G_j^{lin} generated by $e_{1,i_r}^j, \dots, e_{\beta_{j,i_r+j},i_r}^j$. Since $lin^P(\mathbf{G}_{\mathbf{i_r}})$ is exact, then $\operatorname{Ker}(d_j^{lin}|N_{i_r}^j) = \operatorname{Im}(d_{j+1}^{lin}|N_{i_r}^{j+1})$. Let

$$x = \sum_{r=1}^{m} \sum_{s=1}^{\beta_{j,i_r+j}} \lambda_{sr} e^{j}_{s,i_r} \in \operatorname{Ker} \delta^{lin}_{j}$$

with $\lambda_{sr} \in P$. Because $\sum_{s=1}^{\beta_{j,i_1+j}} \lambda_{s1} e^{j}_{s,i_1} \in \operatorname{Ker} d^{lin}_{j} \cap N^{j}_{i_1} = \operatorname{Im}(d^{lin}_{j+1} | N^{j+1}_{i_1})$, we get $\sum_{s=1}^{\beta_{j,i_1+j}} \lambda_{s1} e^{j}_{s,i_1} = d^{lin}_{j+1}(\alpha_1)$ with $\alpha_1 \in N^{j+1}_{i_1}$. Set

$$x_1 := x - \delta^{lin}_{j+1}(\alpha_1).$$

Notice that $x_1 \in \operatorname{Ker} \delta^{lin}_{j}$ because $x \in \operatorname{Ker} \delta^{lin}_{j}$ and $\delta^{lin}_{j} \circ \delta^{lin}_{j+1} = 0$. It is easy to see that $x_1 \in \langle e^{j}_{s,i_r} \rangle$ with $r \geqslant 2$. In fact $\delta_{j+1}(\alpha_1) - d_{j+1}(\alpha_1) \in P^{\beta_j(M)}(-j)/N^{j}_{i_1}$ because the blocks on the diagonal of $lin^R(\mathbf{F.})$ and $lin^P(\mathbf{G.})$ coincide.

Hence $x_1 = \sum_{r=2}^{m} \sum_{s=1}^{\beta_{j,i_r+j}} \lambda'_{sr} e^{j}_{s,i_r} \in \operatorname{Ker} \delta^{lin}_{j}$ with $\lambda'_{sr} \in P$ and $\sum_{s=1}^{\beta_{j,i_2+j}} \lambda'_{sr} e^{j}_{s,i_r} \in \operatorname{Ker} d^{lin}_{j} \cap N^{j}_{i_2} = \operatorname{Im}(d^{lin}_{j+1} | N^{j+1}_{i_2})$. We can repeat the same procedure and finally we find $\alpha_r \in N^{j+1}_{i_r}$, $r = 1, \ldots, m-1$, such that

$$x_m := x - \delta^{lin}_{j+1}\left(\sum_{i=1}^{m-1} \alpha_i\right) \in \operatorname{Ker} d^{lin}_{j} \cap N^{j}_{i_m} = \operatorname{Im}\bigl(d^{lin}_{j+1} \big| N^{j+1}_{i_m}\bigr) = \operatorname{Im}\bigl(\delta^{lin}_{j+1} \big| N^{j+1}_{i_m}\bigr).$$

Because $x_m \in \operatorname{Im}(\delta^{lin}_{j+1})$, it follows that $x \in \operatorname{Im}(\delta^{lin}_{j+1})$, as required. $\square$

As a consequence of Theorem 3.6 and of [CHH, Theorem 1.5 and Remark 1.6], we can prove the following result.

Corollary 3.7. *Let I be an ideal of a regular local ring R of dimension n which satisfies one of the equivalent conditions of Corollary* 3.3 *and let $A = R/I$. Then*

$$\beta_i(A) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j(A).$$

We remark that, under the assumption of the above result, we also have

$$\beta_i(A) = \sum_{j=1}^{n-i+1} \binom{n-j}{i-1} \alpha_j\bigl(gr_{\mathfrak{m}}(A)\bigr).$$

We present now an unexpected consequence on the theory of blowing-up algebras. If $I \subseteq \mathfrak{n}^2$ is a non-zero ideal of a regular local ring $(R, \mathfrak{n})$, we let $A = R/I$ the local ring with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. We denote by $S_A(\mathfrak{m})$ the symmetric algebra of $\mathfrak{m}$ over A. In this case it is known (see [HRV, Corollary 2.2]) that

$$S_A(\mathfrak{m}) = \bigoplus_{p \geqslant 0} \mathfrak{n}^p / I\mathfrak{n}^{p-1}.$$

Even if A is Cohen–Macaulay, the symmetric algebra has a strong tendency not to be Cohen–Macaulay. M.E. Rossi (see [Ro, Theorems 2.4 and 3.3]) proved that $S_A(\mathfrak{m})$ is Cohen–Macaulay if and only if A is an abstract hypersurface ring. One reason is that the Krull dimension of $S_A(\mathfrak{m})$, compared with the one of A, can be very higher. C. Huneke and M.E. Rossi (see [HR]) gave an explicit formula for the dimension of the symmetric algebra. Applied to the above setting, we get

$$\dim S_A(\mathfrak{m}) = \dim R. \tag{5}$$

It follows from a result of the same paper [HR] that

$$\operatorname{depth} S_A(\mathfrak{m}) \leqslant \dim A + 1.$$

In [HRV, Theorem 2.13], J. Herzog, M.E. Rossi and G. Valla proved that depth $S_A(\mathfrak{m})$ (with respect to homogeneous irrelevant maximal ideal) is strictly related to the depth of $gr_{\mathfrak{n}}(I)$. As a consequence, by using this connection and the results of this paper, we prove the following theorem.

Theorem 3.8. *Let $I \subseteq \mathfrak{n}^2$ be an ideal of a regular local ring $(R, \mathfrak{n})$, we let $A = R/I$ the local ring with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Assume that I satisfies one of the equivalent conditions of Corollary* 3.3. *If* depth $A > 0$*, then*

$$\operatorname{depth} S_A(\mathfrak{m}) \geqslant \operatorname{depth} A + 1.$$

If A is Cohen–Macaulay, then depth $S_A(\mathfrak{m}) = \dim A + 1$.

Proof. By Theorem 3.6 we know that I is of homogeneous type. It follows that depth $gr_{\mathfrak{n}}(I) =$ depth $I =$ depth $A + 1$ and the result follows by [HRV, Theorem 2.13(c) and Corollary 4.14]. $\square$

We remark that, if depth $A = 0$, then depth $S_A(\mathfrak{m}) = 0$ from [HRV, Proposition 2.3(d)].

Notice that Theorem 3.8 extends and reproves [HRR, Theorem 3.9] which was showed in the homogeneous context. Moreover, Theorem 3.8 and equality (5) show that if A is Cohen–Macaulay and the equivalent conditions of Corollary 3.3 hold, then $S_A(\mathfrak{m})$ is Cohen–Macaulay if and only if $\dim A = \dim R - 1$, which means A is a hypersurface ring. The result recovers, in a particular case, a more general result proved by M.E. Rossi in [Ro, Theorem 3.3].

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