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The Hilbert function of the Ratliff–Rush filtration ☆

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

The Ratliff–Rush filtration has been shown to be a very useful tool for studying numerical invariants of the associated graded ring $G := \bigoplus_{t \geqslant 0}(I^t/I^{t+1})$ of a local ring $(A, \mathfrak{m})$ with respect to the classical I-adic filtration. The advantage of this approach is that the associated graded ring $\widetilde{G}$ of A with respect to the Ratliff–Rush filtration has positive depth, but unfortunately $\widetilde{G}$ is not necessarily a standard graded algebra.

In this paper, we study some numerical invariants of $\widetilde{G}$ when I is an $\mathfrak{m}$-primary ideal of a local Cohen–Macaulay ring and, as consequence, we prove an upper bound on the first coefficient of the Hilbert polynomial of G which extends the already known bounds.

MSC: Primary: 13D40; secondary: 13P99

0. Introduction

The notion of Ratliff–Rush closure

$$\widetilde{I} := \bigcup_{n \geqslant 1} (I^{n+1} : I^n)$$

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of an ideal I in a Noetherian local ring A has been introduced in [12] where the authors show that, if I contains a regular element, then I is a reduction of $\widetilde{I}$ and, even more, $(\widetilde{I})^n = I^n$ for all large n, $\widetilde{I}$ being the largest ideal with this property. More generally it has also been proved in [12] that

$$\widetilde{I} \supseteq \widetilde{I^2} \supseteq \cdots \supseteq \widetilde{I^i} \supseteq \widetilde{I^{i+1}} \supseteq \cdots \supseteq \widetilde{I^n} = I^n$$

for all large n.

Since it is clear that $\widetilde{I^i}\widetilde{I^j} \subseteq \widetilde{I^{i+j}}$ for every i and j, the collection of ideals $\{\widetilde{I^n}\}_{n\in\mathbb{N}}$ is a filtration of A which is called the *Ratliff–Rush filtration* induced by I and which is a Noetherian filtration.

The Ratliff–Rush filtration has been shown to be a very useful tool for studying numerical invariants of the associated graded ring $G := \bigoplus_{t\geqslant 0}(I^t/I^{t+1})$ of A with respect to the classical I-adic filtration (see [3,4,7,8,10,11,13,15,16]).

For example, for all not negative integer n the degree n component of the zeroth local cohomology module of G with respect to the ideal $G_+ = \bigoplus_{t\geqslant 1}(I^t/I^{t+1})$ can be written as

$$[H^0_{G_+}(G)]_n = (\widetilde{I^{n+1}} \cap I^n)/I^{n+1}.$$

Hence G has positive depth if and only if $\widetilde{I^n} = I^n$ for all $n \geqslant 0$.

Since $\widetilde{I^p} \supseteq \widetilde{I^{p+1}}$, we can consider the abelian group

$$\widetilde{G} := \bigoplus_{p\geqslant 0}(\widetilde{I^p}/\widetilde{I^{p+1}})$$

which has a natural structure of graded algebra over its degree zero part, the local ring $\widetilde{G}_0 = A/\widetilde{I}$, with multiplication induced by the multiplication map $\widetilde{I^p} \times \widetilde{I^q} \to \widetilde{I^{p+q}}$.

The ring $\widetilde{G}$ is called the associated graded ring of A with respect to the Ratliff–Rush filtration induced by I. If I is $\mathfrak{m}$-primary, then $\widetilde{G}_0$ is an Artinian local ring and we can consider the Hilbert function of $\widetilde{G}$ which is by definition

$$\widetilde{H}_I(t) := \lambda_{A/\widetilde{I}}(\widetilde{G}_t) = \lambda(\widetilde{I^t}/\widetilde{I^{t+1}}),$$

where we simply write $\lambda(M)$ for the length of the A-module M. This function gives useful information on some numerical invariants related to the classical Hilbert function of I. The advantage is that $\widetilde{G}$ has positive depth, but unfortunately $\widetilde{G}$ is not a standard graded algebra because we do not necessarily have $\widetilde{G}_{i+1} = \widetilde{G}_1\widetilde{G}_i$.

Hence, the classical tools used for the computation of the Hilbert function in the standard case, are no more available here. However, if I is an $\mathfrak{m}$-primary ideal of a one-dimensional Cohen–Macaulay local ring $(A, \mathfrak{m})$, we can prove in Theorem 2.1 that the Hilbert function of $\widetilde{G}$ is strictly increasing up to reach the multiplicity e of I, the same behaviour which the Hilbert function of G has in the case G is Cohen–Macaulay. By using this result and as a particular case of a more precise bound, we prove in Corollary 2.3 that for every $t \geqslant 0$

$$\widetilde{H}_I(t) \geqslant \min(e, t + \lambda(A/\widetilde{I})).$$

This inequality should be compared with the inequality

$$H_R(t) \geqslant \min(e, t+1)$$

which holds for a given one-dimensional standard graded algebra R over an Artinian local ring R_0 and where the Hilbert function of R is defined as $H_R(t) := \lambda_{R_0}(R_t)$.

If R_0 is a field, this last result can be found in [9] or can be achieved as a consequence of the classical Macaulay's theorem, while, in the case R_0 is Artinian, it follows from an extension of Macaulay's theorem due to Blancafort (see [2, Corollary 2.11]).

Our approach also gives a bound on the regularity $\widetilde{s}$ of $\widetilde{G}$ in terms of the invariants of I. More precisely we prove in Theorem 2.4 and 2.5 that

$$\widetilde{s} \leqslant e - \max(v(I), v(\widetilde{I})) + 1,$$

where $v(J)$ denotes the minimal number of generators of J.

We remark that in most of the cases $v(\widetilde{I}) \geqslant v(I)$, but Example 3.6. in [14] shows that $v(I) - v(\widetilde{I})$ can be positive and large as you want even in a regular local ring.

In the last section, as a simple numerical consequence of the described properties of the Hilbert functions of $\widetilde{G}$, we recover and extend in Theorem 3.1 a remarkable result proved by Elias [5].

For a Cohen–Macaulay one-dimensional local ring $(A, \mathfrak{m})$ one has for every $n \geqslant 0$

$$\lambda(A/\mathfrak{m}^{n+1}) = \sum_{t=0}^{n} H_G(t) = e(n+1) - e_1,$$

where e is the multiplicity of A and e_1 is an integer which is called the first Hilbert coefficient of A.

In the quoted paper, by using deep methods related to the strict transform of the blowing up of A, Elias proved that

$$e_1 \leqslant \binom{e}{2} - \binom{v-1}{2}$$

where v is the embedding dimension of A. This bound is sharp and it can be used to give all the possible Hilbert–Samuel polynomials for the class of one-dimensional Cohen–Macaulay local rings with multiplicity e and embedding dimension v.

In Theorem 3.2, as a consequence of a more general result, we improve the upper bound for e_1 proved by Elias by showing that for an $\mathfrak{m}$-primary ideal of a Cohen–Macaulay local ring $(A, \mathfrak{m})$ one has

$$e_1 \leqslant \binom{e}{2} - \binom{v(I)-d}{2} - \lambda(A/I) + 1.$$

This result can be used to give strict constraints on the Hilbert function of an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay local ring, for example it says that the Hilbert series

$$P_{\mathfrak{m}}(z) = \frac{1+3z-z^2+z^3+z^4}{1-z}$$

is not admissible since $e = 5$, $v = 4$ and $e_1 = 8$.

The paper ends with a short proof (see Proposition 3.3) that, in the case I is the maximal ideal of A, if e_1 reaches its maximal value, then A has a specified Hilbert function, a result which was the main theorem in [6].

1. Preliminaries

Let $(A, \mathfrak{m})$ be a local ring of dimension d and I an $\mathfrak{m}$-primary ideal in A. Let us recall a construction due to Ratliff and Rush (see [12]). For every $n \geqslant 0$ we have a chain of ideals

$$I^n \subseteq I^{n+1}\colon I \subseteq I^{n+2}\colon I^2 \subseteq \cdots \subseteq I^{n+k}\colon I^k \subseteq \cdots .$$

This chain stabilizes at an ideal which we will denote by

$$\widetilde{I^n} := \bigcup_{k \geqslant 1} (I^{n+k}\colon I^k).$$

Hence there is a positive integer t, depending on n, such that $\widetilde{I^n} = I^{n+k}\colon I^k$ for every $k \geqslant t$.

It is clear that we have $\widetilde{I^0} = A$ and for every non-negative integers i and j

$$I^i \subseteq \widetilde{I^i}, \quad \widetilde{I^i}\,\widetilde{I^j} \subseteq \widetilde{I^{i+j}}, \quad \widetilde{I^{i+1}} \subseteq \widetilde{I^i}.$$

We will denote by $\widetilde{G} := \bigoplus_{i \geqslant 0} (\widetilde{I^i}/\widetilde{I^{i+1}})$ the associated graded ring of A with respect to the Ratliff–Rush filtration and by

$$\widetilde{H}_I(t) := \lambda_{A/\widetilde{I}}(\widetilde{G}_t) = \lambda(\widetilde{I^t}/\widetilde{I^{t+1}})$$

its Hilbert function. This is the Hilbert function we refer to in the title.

Superficial elements play an important role in this paper. We recall that an element x in I is called superficial for I if $d \geqslant 1$ and there exists an integer $c > 0$ such that

$$(I^n : x) \cap I^c = I^{n-1}$$

for every $n > c$.

It is well known that if the residue field is infinite, superficial elements always exist. Further, if A has positive depth, every superficial element for I is also a regular element in A.

If x is superficial for I and a non-zero divisor, it is an easy consequence of the Artin Rees lemma that for every integer $j \gg 0$ we have $I^j\colon x = I^{j-1}$. From this we easily get $I^i = \widetilde{I^i}$, for $i \gg 0$.

Finally, for every $n \geqslant 0$, we have

$$\widetilde{I^{n+1}}\colon x = \widetilde{I^n}. \tag{1}$$

which implies that $\widetilde{G}$ has positive depth.

If $G := \bigoplus_{i \geqslant 0} (I^i/I^{i+1})$ is the associated graded ring of A with respect to the I-adic filtration, we have $\widetilde{G}_i = G_i$ for $i \gg 0$. We recall that G is a standard graded algebra which has not necessarily positive depth, while $\widetilde{G}$ is not a standard graded algebra, but depth $\widetilde{G} > 0$ by (1).

In this paper, we study some properties of $\widetilde{H}_I(t)$ and we show how these properties give information on the Hilbert function $H_I(t)$ of I which, as usual, is defined as

$$H_I(t) = H_G(t) = \lambda_{A/I}(I^t/I^{t+1}) = \lambda(I^t/I^{t+1}).$$

The generating function of the numerical function $H_I(t)$ is the power series

$$P_I(z) = \sum_{t \geqslant 0} H_I(t) z^t.$$

This series is called the Hilbert series of I. It is well known that this series is rational and that, even more, there exists a polynomial $h_I(z)$ with integers coefficients such that $h_I(1) \neq 0$ and

$$P_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

For every $i \geqslant 0$, the integers

$$e_i(I) := \frac{h_I^{(i)}(1)}{i!}$$

are called the *Hilbert coefficients* of I. The integer $e_0(I) = h_I(1)$ is the *multiplicity* of I and it is simply denoted by $e(I)$.

It is well known that the polynomial

$$p_I(X) := \sum_{i=0}^{d} (-1)^i e_i(I) \binom{X+d-i}{d-i}$$

has the property that for every $n \gg 0$

$$p_I(n) = \lambda(A/I^{n+1}) = \sum_{j=0}^{n} H_I(j).$$

Since we have $I^{n+1} = \widetilde{I^{n+1}}$ for every n big enough, we also get for every $n \gg 0$

$$p_I(n) = \lambda(A/\widetilde{I^{n+1}}) = \sum_{j=0}^{n} \widetilde{H}_I(j).$$

A well-known property we will use in the paper is the following: if $x_1, \ldots, x_r$ is a superficial sequence for I (which means x_1 is superficial for I and $\overline{x_i}$ is superficial for $I/(x_1, \ldots, x_{i-1})$ for every $2 \leqslant i \leqslant r$) and we put $\bar{I} := I/(x_1, \ldots, x_r)$, then, for $i=0, \ldots, d-r$, we have $e_i(I) = e_i(\bar{I})$. Hence, for example, if $d=1$ and x is a superficial element in I, then $e_0(I) = e_0(I/xA) = \lambda(A/xA)$.

When the ring A has dimension one, we have nice properties of the above-defined integers. Hence, from now on, we are assuming that $(A, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension $d = 1$ and we will simply write e and e_1 for the Hilbert coefficients $e_0(I)$ and $e_1(I)$, respectively.

Further, we let x be a superficial element of the $\mathfrak{m}$-primary ideal I and we recall that, since A is Cohen–Macaulay, x is regular on A and $\widetilde{G}$ as well.

We consider for every $i \geqslant 0$ the following diagram:

$$\begin{array}{ccccc} A & \supseteq & \widetilde{I^{i+1}} & \supseteq & I^{i+1} \\ \cup & & \cup & & \cup \\ xA & \supseteq & x\widetilde{I^i} & \supseteq & xI^i. \end{array}$$

Accordingly, we set

$$\rho_i := \lambda(\widetilde{I^{i+1}}/x\widetilde{I^i}), \quad v_i := \lambda(I^{i+1}/xI^i)$$

and then from the diagram we get

$$e = \lambda(A/xA) = H_I(i) + v_i = \widetilde{H}_I(i) + \rho_i. \tag{2}$$

Hence $H_I(i) = e$ if and only if $v_i = 0$, that is $I^{i+1} = xI^i$, and similarly $\widetilde{H}_I(i) = e$ if and only if $\rho_i = 0$, that is $\widetilde{I^{i+1}} = x\widetilde{I^i}$.

Let s be the integer defined by

$$v_j > 0 \quad \text{if } i \leqslant s-1,$$

$$v_i = 0 \quad \text{if } i \geqslant s \tag{3}$$

so that s is exactly the *reduction number* of I. It is well known that $s \leqslant e-1$ (see for example [17, Remark 6.16]).

We have $I^{i+1} = xI^i$ for every $i \geqslant s$, from which we easily get by induction on $t \geqslant 0$ and for every $p \geqslant s$,

$$I^{t+p} = x^t I^p. \tag{4}$$

Let j be an integer, $j \geqslant s$, and let t be a positive integer such that $\widetilde{I^j} = I^{j+t} : I^t$; we have

$$\widetilde{I^j} = I^{j+t} : I^t \subseteq I^{j+t} : x^t = x^{j+t-s} I^s : x^t = x^{j-s} I^s \subseteq I^j,$$

so that, for every $j \geqslant s$,

$$\widetilde{I^j} = I^j, \quad \widetilde{H}_I(j) = H_I(j) = e, \quad v_j = \rho_j = 0.$$

Since for $n \gg 0$

$$p_I(n) = e(n+1) - e_1 = \sum_{i=0}^{n} H_I(i) = \sum_{i=0}^{n} \widetilde{H}_I(i),$$

by (2) we get

$$e_1 = \sum_{i=0}^{n} v_i = \sum_{i=0}^{s-1} v_i$$

and, similarly,

$$e_1 = \sum_{i=0}^{s-1} \rho_i. \tag{5}$$

We want now to describe the components of the Ratliff–Rush filtration in the one-dimensional case.

Let $t \geqslant 0$ and j and p integers such that $0 \leqslant j \leqslant s \leqslant p$; if $ax^t \in I^{j+t}$, then, by (4),

$$ax^t I^{p-j} \subseteq I^{t+p} = x^t I^p$$

so that $a \in I^P : I^{p-j}$. This proves that

$$I^{j+t} : x^t \subseteq I^p : I^{p-j} \tag{6}$$

for every $t \geqslant 0$ and $0 \leqslant j \leqslant s \leqslant p$.

Proposition 1.1. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one and let I be an $\mathfrak{m}$-primary ideal in A with reduction number s. Let $p \geqslant s$ be an integer, then for every $j \geqslant 0$ we have*

$$\widetilde{I^j} = \begin{cases} I^p : x^{p-j} = I^p : I^{p-j} & if\ \ j \leqslant s, \\ I^j & if\ \ j \geqslant s. \end{cases}$$

Proof. We have already seen that $\widetilde{I^j} = I^j$ if $j \geqslant s$.

Now, let t be a positive integer such that

$$\widetilde{I^j} = I^{j+t} : I^t.$$

If $j \leqslant s$ we can use (6) and (4) to get

$$\begin{aligned}\widetilde{I^j} &= I^{j+t} : I^t \subseteq I^{j+t} : x^t \subseteq I^p : I^{p-j} \subseteq I^p : x^{p-j} \subseteq I^{p+s} : x^{p-j} I^s \\ &= I^{p+s} : I^{p+s-j} \subseteq \widetilde{I^j}.\end{aligned}$$

The conclusion follows. □

2. The Hilbert function of $\widetilde{G}$

In this section $(A, \mathfrak{m})$ is a local Cohen–Macaulay ring of dimension one, I an ideal which is primary for $\mathfrak{m}$, x a superficial element in I and s the reduction number of I.

We will simply write $H(t)$ and $\widetilde{H}(t)$ instead of $H_I(t)$ and $\widetilde{H}_I(t)$ for the Hilbert function of G and $\widetilde{G}$, respectively, and e for the multiplicity $e(I)$ of I.

Since by (1) we have $\widetilde{I^{t+1}} : x = \widetilde{I^t}$, for every $t \geqslant 0$ the multiplication by x gives an injective map

$$0 \to \widetilde{G}_t \xrightarrow{x} \widetilde{G}_{t+1}$$

whose cokernel is

$$\widetilde{G}_{t+1}/x\widetilde{G}_t = \widetilde{I^{t+1}}/(x\widetilde{I^t} + \widetilde{I^{t+2}}).$$

Since we have

$$x\widetilde{I^t} + \widetilde{I^{t+2}} \subseteq I\widetilde{I^t} + \widetilde{I^{t+2}} \subseteq \widetilde{I^{t+1}},$$

if we let

$$b_t := \lambda(I\widetilde{I^t} + \widetilde{I^{t+2}}/x\widetilde{I^t} + \widetilde{I^{t+2}})$$

and

$$c_t := \lambda(\widetilde{I^{t+1}}/I\widetilde{I^t} + \widetilde{I^{t+2}})$$

for every $t \geqslant 0$ we get

$$\widetilde{H}(t+1) = \widetilde{H}(t) + c_t + b_t. \tag{7}$$

Further, since for every $t \geqslant s$ we have $\widetilde{H}(t) = e$, it is clear that $c_t = b_t = 0$ for every $t \geqslant s$.

The next result is the main theorem of this section. We recall that if R is a one-dimensional Cohen–Macaulay standard graded algebra over a field, its Hilbert function is strictly increasing until it reaches the multiplicity at which it stabilizes. We prove that the same property holds for the Cohen–Macaulay graded algebra $\widetilde{G}$, even if $\widetilde{G}$ is an algebra over an Artinian local ring and it is not standard.

Theorem 2.1. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one, let I be an $\mathfrak{m}$-primary ideal in A and let $t \geqslant 0$ be an integer. The following conditions are equivalent*:

(a) $\widetilde{H}(t+1) = \widetilde{H}(t)$.
(b) $b_t = 0$.
(c) $\widetilde{H}(t) = e$.
(d) $\widetilde{H}(n) = e$ *for every* $n \geqslant t$.

Proof. It is clear by (7) that (a) implies (b). Let us prove that (b) implies (c). If $t \geqslant s$, then $\widetilde{H}(t) = H(t) = e$. So let $t + 1 \leqslant s$. By assumption we have

$$I\widetilde{I^t} \subseteq x\widetilde{I^t} + \widetilde{I^{t+2}}$$

and we claim that

$$I^s = x^{s-t}\widetilde{I^t}.$$

We have

$$x^{s-t}\widetilde{I^t} \subseteq \widetilde{I^s} = I^s,$$

on the other hand

$$\begin{aligned} I^s &= I^{s-t-1}I^{t+1} \subseteq I^{s-t-1}I\widetilde{I^t} \subseteq I^{s-t-1}(x\widetilde{I^t} + \widetilde{I^{t+2}}) \subseteq xI^{s-t-1}\widetilde{I^t} + \widetilde{I^{s+1}} \\ &= xI^{s-t-1}\widetilde{I^t} + I^{s+1}. \end{aligned}$$

If $s = t + 1$ we are done by Nakayama. Otherwise $s > t + 1$ and we have

$$\begin{aligned} xI^{s-t-1}\widetilde{I^t} + I^{s+1} &= xI^{s-t-2}I\widetilde{I^t} + I^{s+1} \subseteq xI^{s-t-2}(x\widetilde{I^t} + \widetilde{I^{t+2}}) + I^{s+1} \\ &\subseteq x^2 I^{s-t-2}\widetilde{I^t} + I^{s+1} \subseteq \cdots \subseteq x^{s-t}\widetilde{I^t} + I^{s+1}. \end{aligned}$$

The claim follows again by Nakayama.

From the claim we get

$$I^{s+1} = xI^s \subseteq x^{s-t}\widetilde{I^{t+1}} \subseteq \widetilde{I^{s+1}} = I^{s+1},$$

hence $I^{s+1} = x^{s-t}\widetilde{I^{t+1}}$, and we finally get

$$e = \lambda(I^s/I^{s+1}) = \lambda(x^{s-t}\widetilde{I^t}/x^{s-t}\widetilde{I^{t+1}}) = \lambda(\widetilde{I^t}/\widetilde{I^{t+1}}) = \widetilde{H}(t).$$

Let us finally prove that (c) implies (d). If $n \geqslant t$, we have

$$e \geqslant e - \rho_n = \widetilde{H}(n) \geqslant \widetilde{H}(t) = e$$

and the conclusion follows. □

As an easy consequence of this result, we have the following crucial corollary.

Corollary 2.2. *Let j be a non-negative integer; then for every $n \geqslant j$ we have*

$$\widetilde{H}(n) \geqslant \min\left(e, \widetilde{H}(j) + n - j + \sum_{i=0}^{n-1} c_i\right).$$

Proof. If $j = n$ there is nothing to prove. So let $n > j$ and consider the sequence

$$\widetilde{H}(j) \leqslant \widetilde{H}(j+1) \leqslant \cdots \leqslant \widetilde{H}(n).$$

If for some $j \leqslant i \leqslant n-1$ we have $\widetilde{H}(i) = \widetilde{H}(i+1)$, then $e = \widetilde{H}(i) \leqslant \widetilde{H}(n)$ and the conclusion follows. Otherwise $b_j, \ldots, b_{n-1} > 0$ and we have

$$\widetilde{H}(n) = \widetilde{H}(j) + \sum_{i=j}^{n-1}(c_i + b_i) \geqslant \widetilde{H}(j) + \sum_{i=j}^{n-1} c_i + n - j,$$

as wanted. □

We can get free of the nasty term involving the $c_i's$ in the above inequality by proving the following corollary. We will use throughout the notation

$$\lambda := \lambda(A/\widetilde{I}) = \widetilde{H}(0).$$

Corollary 2.3. *For every $n \geqslant 0$ we have*

$$\widetilde{H}(n) \geqslant \min(e, n + \lambda).$$

Proof. We have $\widetilde{H}(0) = \lambda$ so that by the above corollary we get

$$\widetilde{H}(n) \geqslant \min\left(e, \widetilde{H}(0) + n + \sum_{i=0}^{n-1} c_i\right) \geqslant \min(e, n + \lambda). \qquad \square$$

The next result of this main section gives an upper bound for the reduction number of the Ratliff–Rush filtration.

In the rest of the paper we let

$$\sigma := \lambda(I + \widetilde{I^2}/\widetilde{I^2})$$

so that

$$c_0 + \sigma = \lambda(\widetilde{I}/I + \widetilde{I^2}) + \lambda(I + \widetilde{I^2}/\widetilde{I^2}) = \lambda(\widetilde{I}/\widetilde{I^2}) = \widetilde{H}(1).$$

We also denote by g the integer

$$g := \sum_{i \geqslant 0} c_i + \sigma = \lambda(\widetilde{I}/\widetilde{I^2}) + \sum_{i \geqslant 1} c_i.$$

Theorem 2.4. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one and let I be an $\mathfrak{m}$-primary ideal in A. We have $e - g + 1 \geqslant 1$ and*

$$\widetilde{H}(e - g + 1) = e.$$

Proof. Since $c_j = 0$ for $j \gg 0$, we can consider the least integer $t \geqslant 0$ such that $c_j = 0$ for every $j \geqslant t$.

If $t = 0$, then $g = \sigma$ and, in this case, $e \geqslant \widetilde{H}(1) = c_0 + \sigma = \sigma$, so that $e - \sigma \geqslant 0$ and $e - g + 1 = e - \sigma + 1 \geqslant 1$. By Corollary 2.2 we get

$$\widetilde{H}(e - g + 1) = \widetilde{H}(e - \sigma + 1) \geqslant \min\left(e, \widetilde{H}(1) + e - \sigma + 1 - 1 + \sum_{i=1}^{e-g} c_i\right) = e.$$

If instead $t \geqslant 1$, then $g = \sum_{i=0}^{t-1} c_i + \sigma$ with $c_{t-1}, b_{t-1} > 0$. Since $b_{t-1} > 0$, we have $\widetilde{H}(t - 1) < e$, hence, if $t \geqslant 2$, we can apply Corollary 2.2 with $j = 1$, $n = t - 1$ to get

$$\begin{aligned}\widetilde{H}(t) &= \widetilde{H}(t-1) + b_{t-1} + c_{t-1} \geqslant \widetilde{H}(1) + t - 1 - 1 + \sum_{i=1}^{t-2} c_i + b_{t-1} + c_{t-1} \\ &= c_0 + \sigma + t - 2 + \sum_{i=1}^{t-2} c_i + b_{t-1} + c_{t-1} \geqslant g + t - 1.\end{aligned}$$

The inequality $\widetilde{H}(t) \geqslant g + t - 1$ holds true also if $t = 1$ because, in that case, $g = c_0 + \sigma = \widetilde{H}(1)$. Hence if $t \geqslant 1$, we have

$$e \geqslant \widetilde{H}(t) \geqslant g + t - 1$$

so that $e - g + 1 \geqslant t \geqslant 1$ and finally by Corollary 2.2 we get

$$\begin{aligned}\widetilde{H}(e-g+1) &\geqslant \min\left(e, \widetilde{H}(t) + e - g + 1 - t + \sum_{i=t}^{e-g} c_i\right) \\ &\geqslant \min\left(e, g + t - 1 + e - g + 1 - t + \sum_{i=t}^{e-g} c_i\right) = e,\end{aligned}$$

as wanted. □

We have seen that the integer g plays a central role in the above theorem. Unfortunately, it looks like a mysterious invariant of the ideal I involving unaccessible integers. Nevertheless, the next theorem proves that it is bounded below by nice numerical invariants of the ideal I.

We will denote by $v(J)$ the minimal number of generators of an ideal J of a local ring A.

Theorem 2.5. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one and let I be an $\mathfrak{m}$-primary ideal in A. Then*

$$g \geqslant \max(v(\widetilde{I}), v(I)).$$

Proof. Recall that

$$g = \lambda(\widetilde{I}/\widetilde{I^2}) + \sum_{i\geqslant 1} c_i = \lambda(\widetilde{I}/\widetilde{I^2}) + \sum_{i\geqslant 1} \lambda(\widetilde{I^{i+1}}/I\widetilde{I^i} + \widetilde{I^{i+2}}).$$

We remark that for every $i \geqslant 1$, $I\widetilde{I^i} \subseteq I\mathfrak{m} \cap \widetilde{I^{i+1}}$, hence we have

$$\lambda(\widetilde{I^{i+1}}/I\widetilde{I^i} + \widetilde{I^{i+2}}) \geqslant \lambda(\widetilde{I^{i+1}}/I\mathfrak{m} \cap \widetilde{I^{i+1}} + \widetilde{I^{i+2}}) = \lambda(\widetilde{I^{i+1}} + I\mathfrak{m}/\widetilde{I^{i+2}} + I\mathfrak{m}).$$

We know that there exists an integer N such that for every $j > N$ we have $\widetilde{I^j} = I^j$. Hence for $i \gg 0$ we have $\widetilde{I^{i+2}} = I^{i+2} = I\widetilde{I^{i+1}} \subseteq I\mathfrak{m}$ and so it is easy to see that

$$g \geqslant \lambda(\widetilde{I}/\widetilde{I^2}) + \lambda(\widetilde{I^2} + I\mathfrak{m}/I\mathfrak{m}).$$

Now

$$\lambda(\widetilde{I}/\widetilde{I^2}) + \lambda(\widetilde{I^2} + I\mathfrak{m}/I\mathfrak{m}) \geqslant \lambda(\widetilde{I}/\widetilde{I^2} + I\mathfrak{m}) + \lambda(\widetilde{I^2} + I\mathfrak{m}/I\mathfrak{m}) \geqslant v(\widetilde{I}).$$

On the other hand $\lambda(\widetilde{I}/\widetilde{I^2}) \geqslant \lambda(I + \widetilde{I^2}/\widetilde{I^2}) \geqslant \lambda(I + \widetilde{I^2}/\widetilde{I^2} + I\mathfrak{m})$, hence

$$g \geqslant \lambda(I + \widetilde{I^2}/I\mathfrak{m}) \geqslant v(I)$$

as desired. □

By analogy with the classical case, let $\widetilde{s}$ be the least integer t such that $\widetilde{H}(t) = e$. Since $e = \widetilde{H}_I(j) + \rho_j$, it is clear that $\widetilde{s}$ is also the least integer t such that $\rho_t = 0$. Since $H(s) = \widetilde{H}(s) = e$ we have $\widetilde{s} \leqslant s$.

We will denote by

$$v := \max(v(\widetilde{I}), v(I)).$$

By the above theorems we have

$$\widetilde{s} \leqslant e - g + 1 \leqslant e - v + 1.$$

We end this section by proving a far reaching property of the powers of an $\mathfrak{m}$-primary ideal I in a one-dimensional Cohen–Macaulay local ring.

Proposition 2.6. *With the above notation, we have* $I^s \subseteq x^s A : x^{e-g+1}$. *As a consequence*

$$I^{e-1} \subseteq x^{g-2} A \subseteq x^{v-2} A.$$

Proof. We know that $\widetilde{H}(e-g+1) = e$. This implies $\rho_j = 0$ for every $j \geqslant e - g + 1$ so that $\widetilde{I^{s+e-g+1}} = x^s \widetilde{I^{e-g+1}}$. Hence, using (4), we get

$$x^{e-g+1} I^s = I^{s+e-g+1} = I\widetilde{^{s+e-g+1}} = x^s \widetilde{I^{e-g+1}}.$$

The first assertion follows.

As for the second, we have $I^{e-1} = x^{e-1-s} I^s \subseteq x^{e-1-s}(x^s A : x^{e-g+1})$.

Now, if $s \geqslant e - g + 1$, then we get $I^{e-1} \subseteq x^{e-1-s} x^{s-e+g-1} A = x^{g-2} A$. If $s \leqslant e - g + 1$, then $e - 1 - s \geqslant g - 2$ so that $x^{e-1-s} A \subseteq x^{g-2} A$. This proves the second assertion. $\square$

3. The bound for e_1

In this section, we use the result on the Hilbert function of $\widetilde{G}$ to get an upper bound for the Hilbert coefficient e_1 of I.

We recall that we have defined for every $t \geqslant 0$ the integers

$$c_t := \lambda(\widetilde{I^{t+1}} / I\widetilde{I^t} + \widetilde{I^{t+2}}).$$

Further we set

$$\sigma := \lambda(I + \widetilde{I^2} / \widetilde{I^2}), \quad g := \sum_{i \geqslant 0} c_i + \sigma, \quad \lambda := \lambda(A/\widetilde{I}) = \widetilde{H}(0)$$

and $\widetilde{s}$ the least integer t such that $\widetilde{H}(t) = e$.

At the end of the last section, in Theorem 2.5, we proved that $g \geqslant v$. We remark now that the integer $\widetilde{s}$ can be zero, but, if this is the case, then $\lambda(A/I) \geqslant \lambda(A/\widetilde{I}) = e$, hence $H(t) = e$ for every $t \geqslant 0$ and $e_1 = 0$. Thus, we will tacitly assume in the rest of this section that $\widetilde{s} \geqslant 1$.

A final remark on the integer g is needed. Namely we claim that $g \geqslant 2$, unless $I = \mathfrak{m}$ and A is regular. In fact $g \geqslant c_0 + \sigma = \widetilde{H}(1) = \lambda + c_0 + b_0$, hence $g \geqslant 1$ and if $g = 1$ then $\lambda = 1$ and $b_0 = 0$. This implies $1 = \lambda = e$ so that $H(0) \geqslant \lambda = e = 1$, which implies $H(0) = 1$. Hence $I = \mathfrak{m}$ and A is regular.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension one and let I be an $\mathfrak{m}$-primary ideal in A. Then*

$$e_1 \leqslant \binom{e}{2} - \binom{g-1}{2} - \widetilde{s}(\lambda - 1).$$

Proof. We have by (5) and (2)

$$e_1 = \sum_{j=0}^{s-1} \rho_j = \sum_{j=0}^{\widetilde{s}-1} \rho_j = e\widetilde{s} - \sum_{j=0}^{\widetilde{s}-1} \widetilde{H}(j).$$

Since $0 \leqslant j \leqslant \widetilde{s} - 1$, we have $\widetilde{H}(j) < e$ so that, by Corollary 2.3, we get

$$\widetilde{H}(j) \geqslant j + \lambda.$$

Hence

$$e_1 = e\widetilde{s} - \sum_{j=0}^{\widetilde{s}-1} \widetilde{H}(j) \leqslant e\widetilde{s} - (1 + 2 + \cdots + \widetilde{s} - 1) - \widetilde{s}\lambda$$
$$= e\widetilde{s} - \binom{\widetilde{s}+1}{2} - \widetilde{s}(\lambda - 1).$$

By Theorem 2.4 we have $\widetilde{s} \leqslant e - g + 1$. Because $g \geqslant 2$ we also have

$$\widetilde{s} \leqslant e + g - 2.$$

An easy computation shows that

$$\binom{e}{2} - \binom{g-1}{2} - e\widetilde{s} + \binom{\widetilde{s}+1}{2} = \frac{(e - \widetilde{s} - g + 1)(e - \widetilde{s} + g - 2)}{2} \geqslant 0,$$

hence

$$e_1 \leqslant e\widetilde{s} - \binom{\widetilde{s}+1}{2} - \widetilde{s}(\lambda - 1) \leqslant \binom{e}{2} - \binom{g-1}{2} - \widetilde{s}(\lambda - 1). \qquad \square$$

Since by Theorem 2.5 we have $g \geqslant v$, we can give a weaker bound for e_1 which, however, uses more accessible invariants.

For every primary ideal I of the one-dimensional Cohen–Macaulay local ring A, we have

$$e_1 \leqslant \binom{e}{2} - \binom{v-1}{2} - \widetilde{s}(\lambda - 1) \leqslant \binom{e}{2} - \binom{v-1}{2} - \lambda + 1.$$

We would like to extend the inequality

$$e_1 \leqslant \binom{e}{2} - \binom{v-1}{2} - \lambda + 1$$

to the higher-dimensional case. Unfortunately, the integer $v(\widetilde{I})$ does not behave well under reduction modulo a superficial sequence. For this reason we will extend to higher dimension the inequality

$$e_1 \leqslant \binom{e}{2} - \binom{v(I)-1}{2} - \lambda + 1 \tag{8}$$

which holds in the one-dimensional case by the above remark.

Theorem 3.2. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal in A. Then*

$$e_1 \leqslant \binom{e}{2} - \binom{v(I)-d}{2} - \lambda(A/I) + 1.$$

Proof. In the one-dimensional case the result follows by (8). Let $d \geqslant 2$, let $x_1, \dots, x_{d-1}$ be a superficial sequence in I and denote $J := I/(x_1, \dots, x_{d-1})$ in the one-dimensional Cohen–Macaulay local ring $A/(x_1, \dots, x_{d-1})$. The multiplicity and e_1 do not change from I to J so that, by induction, we have

$$e_1 \leqslant \binom{e}{2} - \binom{v(J)-1}{2} - \lambda(A/I) + 1.$$

The conclusion follows because we clearly have

$$v(J) \geqslant v(I) - (d-1). \qquad \square$$

Coming back to the one-dimensional case we remark that, if $I = \mathfrak{m}$, then (8) becomes

$$e_1 \leqslant \binom{e}{2} - \binom{v(\mathfrak{m})-1}{2}.$$

In [6] we proved that equality holds if and only if

$$P_A(z) = \frac{1 + (v(\mathfrak{m})-1)z + \sum_{j=v(\mathfrak{m})}^{e-1} z^j}{(1-z)}.$$

The proof there was very hard and long. We end this paper by giving a shorter proof using the methods developed in the previous sections.

Proposition 3.3. *Let $(A, \mathfrak{m})$ be a one-dimensional Cohen–Macaulay local ring of embedding dimension v. If*

$$e_1 = \binom{e}{2} - \binom{v-1}{2}$$

then

$$P_A(z) = \frac{1 + (v-1)z + \sum_{j=v}^{e-1} z^j}{(1-z)}.$$

Proof. It is well known (see [1]) that we always have $e \geqslant v$ and if $e = v$ then $e_1 = 0$. Hence we have $e \geqslant v + 1$. By looking at the proof of Theorem 3.1 we see immediately that

$$e_1 = \binom{e}{2} - \binom{v-1}{2}$$

implies $v = g$ and

$$\widetilde{H}(t) = \begin{cases} t+1 & \text{if } t \leqslant e - v, \\ e & \text{if } t \geqslant e - v + 1. \end{cases}$$

This implies that $b_i = 1$ and $c_i = 0$ for every $i = 0, \ldots, e - v - 1$. Hence we have

$$\widetilde{\mathfrak{m}^{i+1}} = \mathfrak{m}\widetilde{\mathfrak{m}^i} + \widetilde{\mathfrak{m}^{i+2}} \tag{9}$$

for every $i = 0, \ldots, e - v - 1$.

Since $\widetilde{H}(1) = 2$, we can find an element $y \in \mathfrak{m}$ such that $\mathfrak{m} = (x, y) + \widetilde{\mathfrak{m}^2}$. Using this and (9) we easily get

$$\mathfrak{m} = (x, y) + \widetilde{\mathfrak{m}^{e-v+1}}.$$

By induction on j one gets for every $j \geqslant 1$

$$\mathfrak{m}^j = (x, y)^j + x^{j-1}\widetilde{\mathfrak{m}^{e-v+1}}.$$

We claim that this implies $\mathfrak{m}^j = (x, y)^j$ for every $j \geqslant v$. Since $j \geqslant v$, we have $e - v + j \geqslant e \geqslant s + 1$, hence we may apply Proposition 1.1 to get

$$\widetilde{\mathfrak{m}^{e-v+1}} = \begin{cases} \mathfrak{m}^{e-v+j} : x^{e-v+j-(e-v+1)} = \mathfrak{m}^{e-v+j} : x^{j-1} & \text{if } e - v + 1 \leqslant s, \\ \mathfrak{m}^{e-v+1} & \text{if } e - v + 1 \geqslant s. \end{cases}$$

It follows that

$$\mathfrak{m}^j = (x, y)^j + x^{j-1}\widetilde{\mathfrak{m}^{e-v+1}} \subseteq (x, y)^j + \widetilde{\mathfrak{m}^{e-v+j}}.$$

By Nakayama we get the claim.

Since $H(j) \geqslant \min(e, j + 1)$, this implies

$$H(j) = \begin{cases} j+1 & \text{if } v \leqslant j \leqslant e - 1, \\ e & \text{if } j \geqslant e. \end{cases}$$

On the other hand for every $j \geqslant 1$ we have

$$\mathfrak{m}^{j+1}/x\mathfrak{m}^j = \frac{(x, y)^{j+1} + x^j\widetilde{\mathfrak{m}^{e-v+1}}}{x(x, y)^j + x^j\widetilde{\mathfrak{m}^{e-v+1}}},$$

hence $\mathfrak{m}^{j+1}/x\mathfrak{m}^j$ is a cyclic module generated by $\overline{y^{j+1}}$ so that

$$\mathfrak{m}^{j+1}/x\mathfrak{m}^j \simeq A/(x\mathfrak{m}^j : y^{j+1}).$$

For every $t \geqslant e - v + 1$, $\widetilde{H}(t) = e$ so that $\rho_t = 0$; this implies

$$\widetilde{\mathfrak{m}^t} = x^{t-e+v-1}\widetilde{\mathfrak{m}^{e-v+1}}$$

for every $t \geqslant e - v + 1$. Hence for every $j \geqslant 1$ we have

$$y^{j+1}\widetilde{\mathfrak{m}^{e-v}} \subseteq \widetilde{\mathfrak{m}^{e-v+j+1}} = x^j\widetilde{\mathfrak{m}^{e-v+1}} \subseteq x\mathfrak{m}^j.$$

Thus we get a chain

$$xA \subseteq xA + \widetilde{\mathfrak{m}^{e-v}} \subseteq x\mathfrak{m}^j : y^{j+1} \subseteq A.$$

This implies for every $j \geqslant 1$

$$\begin{aligned} v_j &\leqslant e - \lambda(xA + \widetilde{\mathfrak{m}^{e-v}}/xA) = e - \lambda(\widetilde{\mathfrak{m}^{e-v}}/x\widetilde{\mathfrak{m}^{e-v-1}}) = e - \rho_{e-v-1} \\ &= \widetilde{H}(e - v - 1) = e - v. \end{aligned}$$

Finally we get

$$\begin{aligned} e_1 &= \sum_{i=0}^{e-2} v_i = \sum_{i=0}^{v-1} v_i + \sum_{i=v}^{e-2} v_i \leqslant e - 1 + (v-1)(e-v) + \sum_{i=v}^{e-2}(e - i - 1) \\ &= \binom{e}{2} - \binom{v-1}{2}. \end{aligned}$$

From this it follows that $v_i = e - v$ for $i = 1, \ldots, v - 1$ and this gives the conclusion. □

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