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# Depth of associated graded rings via Hilbert coefficients of ideals

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**Abstract**

Given a local Cohen–Macaulay ring $(R, \mathfrak{m})$, we study the interplay between the integral closedness —or even the normality—of an $\mathfrak{m}$-primary $R$-ideal $I$ and conditions on the Hilbert coefficients of $I$. We relate these properties to the depth of the associated graded ring of $I$.


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## 1. Introduction

Let $I$ be an $\mathfrak{m}$-primary ideal of a local Cohen–Macaulay ring $(R, \mathfrak{m})$ of dimension $d > 0$ and with infinite residue field. The *Hilbert–Samuel function* of $I$ is the numerical function that measures the growth of the length of $R/I^n$ for all $n \geqslant 1$. For $n \gg 0$ this function $\lambda(R/I^n)$

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is a polynomial in $n$ of degree $d$

$$\lambda(R/I^n) = e_0 \binom{n+d-1}{d} - e_1 \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d$$

and $e_0, e_1, \ldots, e_d$ are called the Hilbert coefficients of $I$.

It is well known that $e_0 = \lambda(R/J)$ for any minimal reduction $J$ of $I$ and that the integral closure $\bar{I}$ of $I$ can also be characterized as the largest ideal containing $I$ with the same *multiplicity* $e_0$ [21]. More generally, Ratliff and Rush introduced the ideal $\tilde{I}$, which turns out to be the largest ideal containing $I$ with the same Hilbert coefficients as $I$ [24]. In particular one has the inclusions $I \subseteq \tilde{I} \subseteq \bar{I}$, where equalities hold if $I$ is integrally closed. A very useful technique—that we also exploit—is to consider the generating functions of $\lambda(R/\widetilde{I^n})$ or $\lambda(R/\overline{I^n})$ instead of the one of $\lambda(R/I^n)$: They clearly coincide if $I$ is normal (that is, all powers of $I$ are integrally closed).

Little is known about the higher Hilbert coefficients of $I$, unless we are in presence of good depth properties of the *associated graded ring* $\mathrm{gr}_I(R) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$ of $I$. For example, if the depth of $\mathrm{gr}_I(R)$ is at least $d-1$ then all the Hilbert coefficients of $I$ are positive [18]. Conversely, numerical information on the $e_i$'s has been used to obtain information on the depth of $\mathrm{gr}_I(R)$. For instance, Northcott showed that the bound $\lambda(R/I) \geqslant e_0 - e_1$ always holds [20]. Later Huneke [14] and Ooishi [22] showed that the equality $\lambda(R/I) = e_0 - e_1$ holds if and only if $I^2 = JI$. In particular, $\mathrm{gr}_I(R)$ is Cohen–Macaulay.

Translating information from the Hilbert coefficients of $I$ into good depth properties of $\mathrm{gr}_I(R)$ has also been a constant theme in the work of Sally [28–34]. The most recent results along this line of investigation can be found in [3,5–7,10,13,17,23,25,27,38–40]. The general philosophy is that an 'extremal' behavior of some of the $e_i$'s controls the depth of the associated graded ring of $I$, or of some of its powers, and at the same time determines its Hilbert–Samuel function. We remark that these results are somewhat unexpected since the Hilbert coefficients give asymptotic information on the Hilbert–Samuel function.

It is clear that $e_0$ and $e_1$ are positive integers. As far as the higher Hilbert coefficients of $I$ are concerned, it is a famous result of Narita that $e_2 \geqslant 0$ [19]. In this case the minimal value for $e_2$ does not imply the Cohen–Macaulayness of $\mathrm{gr}_I(R)$. In the very same paper, he also showed that if $d = 2$, then $e_2 = 0$ if and only if $I^n$ has reduction number one for some $n \gg 0$. In particular, $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay. Examples show that the result cannot be extended to higher dimension. In [3] an elementary proof of the positivity of $e_2$ has been given by using the structure of the so-called Sally module $S_J(I)$.

Unfortunately, the well-behavior of the Hilbert coefficients stops with $e_2$. Indeed, in [19] Narita showed that it is possible for $e_3$ to be negative. However, a remarkable result of Itoh says that if $I$ is a normal ideal then $e_3 \geqslant 0$ [16]. A recent proof of this result was given by Huckaba and Huneke [12]. In general, it seems that the integral closedness (or the normality) of the ideal $I$ yields non-trivial consequences on the Hilbert coefficients of $I$ and, ultimately, on the depth of $\mathrm{gr}_I(R)$.

To be more specific, our goal is to characterize a sufficiently high depth of the associated graded ring of $I$ in terms of conditions on the first Hilbert coefficients, and in particular on $e_2$ and $e_3$. Our approach is to study the interplay between the integral closedness (or the normality) of the ideal $I$ and (upper or lower) bounds on the Hilbert coefficients of $I$ and relate it to the depth of the corresponding associated graded ring of $I$. Among our tools, we make systematic use of the standard technique of modding out a superficial sequence in order to decrease the dimension of the ring. This explains why some of our results are formulated for rings of small dimension.

We first establish in Theorem 3.1 a general upper bound on the second Hilbert coefficient which is reminiscent of a similar bound on the first Hilbert coefficient due to Huckaba and Marley [13] and Vaz Pinto [39]. Namely, we show that $e_2 \leqslant \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n)$, for any minimal reduction $J$ of $I$. Furthermore, the upper bound is attained if and only if depth $\mathrm{gr}_I(R) \geqslant d-1$. Next, we characterize in Theorem 3.3 the depth of the associated graded ring of ideals whose second Hilbert coefficient has value 'close enough' to the upper bound established in Theorem 3.1. That is, the condition $e_2 \geqslant \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 2$ implies that depth $\mathrm{gr}_I(R) \geqslant d-2$. Noteworthy is the fact that the same conclusion holds whenever $I$ is an integrally closed ideal satisfying the less restrictive bound $e_2 \geqslant \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 4$. Still in the case of integrally closed ideals we improve Narita's positivity result on $e_2$. Indeed, in Theorem 3.6 we show that for any integrally closed ideal $I$ one has $e_2 \geqslant \lambda(I^2/JI)$ for any minimal reduction $J$ of $I$. The interesting fact is that equality holds in the previous formula if and only if $I^3 = JI^2$ if and only if $\lambda(R/I) = e_0 - e_1 + \lambda(I^2/JI)$. In this case $\mathrm{gr}_I(R)$ is Cohen–Macaulay and the Hilbert–Samuel function is determined. This result fully generalizes the ones of Itoh [17], who handled the cases $e_2 \leqslant 2$.

In [34] Sally proved that if $d = 2$ then $e_2 \geqslant e_1 - e_0 + \lambda(R/\tilde{I})$. Starting from Sally's inequality, Itoh proved that if $d \geqslant 1$ and $I$ is integrally closed then $e_2 \geqslant e_1 - e_0 + \lambda(R/I)$ [17]. It was not known which conditions are forced on the ideal $I$ if the equality $e_2 = e_1 - e_0 + \lambda(R/I)$ holds whenever $I$ is integrally closed. In the particular case of the maximal ideal $\mathfrak{m}$ of $R$, Valla had conjectured in [37] that the reduction number of $\mathfrak{m}$ is at most two, which in turn would imply the Cohen–Macaulayness of $\mathrm{gr}_{\mathfrak{m}}(R)$. A recent example of Wang shows, though, that Valla's conjecture is false in general. In Theorem 3.12 we prove Valla's conjecture for normal ideals. In particular, if $I$ is normal and $\lambda(R/I) = e_0 - e_1 + e_2$ then $e_2 = \lambda(I^2/JI)$ and $I^3 = JI^2$ for some minimal reduction $J$ of $I$. In particular, $\mathrm{gr}_I(R)$ is Cohen–Macaulay and the Hilbert function is known. The key to the result is a theorem of Itoh on the normalized Hilbert coefficients of ideals generated by a system of parameters.

As far as the third Hilbert coefficient of $I$ is concerned, our first result in Section 4 is a generalization of Itoh's result on the positivity of $e_3$ in case $d = 3$. The thrust of our calculation is to replace the normality assumption on $I$ with the weaker requirement of the integral closedness of $I^n$ for some large $n$ (see Theorem 4.1). The proof reduces to comparing the Hilbert coefficients of $I$ and those of a large power of $I$. Combining this result with Theorem 3.12 we are able to characterize when $e_3 = 0$ for asymptotically normal ideals. If this is the case, then for $n \gg 0$ we have that $I^n$ has reduction number at most two, which in turn yields that $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay. This result is reminiscent of Narita's characterization of $e_2 = 0$ when $d = 2$.

## 2. Preliminaries

Thus far we have described the Hilbert–Samuel function associated with the $I$-adic filtration $\mathscr{F} = \{I^n\}_{n\geqslant 0}$. It is important to observe that the theory also applies to other filtrations of ideals of $R$: The so-called Hilbert filtrations (see [13,8]). Let $(R, \mathfrak{m})$ be a local ring of dimension $d$. A filtration of $R$-ideals $\mathscr{F}=\{F_n\}_{n\geqslant 0}$ is called an *Hilbert filtration* if $F_1$ is an $\mathfrak{m}$-primary ideal and the Rees algebra $\mathscr{R}(\mathscr{F})$ is a finite $\mathscr{R}(F_1)$-module. As in the case of the $I$-adic filtration we can define the *Hilbert–Samuel function* of $\mathscr{F}$ to be $\lambda(R/F_n)$. For $n\gg 0$ one also has that $\lambda(R/F_n)$ is a polynomial in $n$ of degree $d$

$$\lambda(R/F_n)=\sum_{j=0}^{d}(-1)^j e_j(\mathscr{F})\binom{n+d-j-1}{d-j},$$

where the $e_j(\mathscr{F})$'s are called the *Hilbert coefficients* of $\mathscr{F}$.

Another related object is the *Hilbert series* of $\mathscr{F}$, which is defined as

$$P_{\mathscr{F}}(t)=\sum_{n\geqslant 1}\lambda(F_{n-1}/F_n)t^{n-1}.$$

The numerical function $\lambda(F_{n-1}/F_n)$ is called *Hilbert function* with respect to the filtration $\mathscr{F}$. It is well known that there exists a unique polynomial $f_{\mathscr{F}}(t)\in\mathbb{Z}[t]$, called the *h-polynomial* of $\mathscr{F}$, with degree $s(\mathscr{F})$, $f_{\mathscr{F}}(1)\neq 0$ and such that

$$P_{\mathscr{F}}(t)=\frac{f_{\mathscr{F}}(t)}{(1-t)^d}.$$

We recall that $e_j(\mathscr{F})=f^{(j)}_{\mathscr{F}}(1)/j!$, where $f^{(j)}_{\mathscr{F}}(t)$ denotes the $j$th formal derivative of $f_{\mathscr{F}}(t)$, and we also point out that it is useful to consider the Hilbert coefficients $e_j(\mathscr{F})$ even when $j>d$. Finally, we denote by $\mathrm{gr}_{\mathscr{F}}(R)=\bigoplus_{n\geqslant 0}F_n/F_{n+1}$ the associated graded ring with respect to $\mathscr{F}$.

If $I$ is an $\mathfrak{m}$-primary ideal and $\mathscr{F}=\{I^n\}_{n\geqslant 0}$ is the usual $I$-adic filtration we write $e_j(I)$ instead of $e_j(\mathscr{F})$ or simply $e_j$ if there is not confusion on the ideal under consideration and we denote by $\mathrm{gr}_I(R)$ the corresponding associated graded ring. Of particular interest is the filtration $\mathscr{F}=\{\overline{I^n}\}_{n\geqslant 0}$ given by the integral closure of the powers of an $\mathfrak{m}$-primary ideal $I$ of an analytically unramified local ring. It is customary to denote with $\bar{e}_0(I),\bar{e}_1(I),\ldots,\bar{e}_d(I)$ the Hilbert coefficients with respect to this filtration $\mathscr{F}$. We also recall that if $J$ is a reduction of $I$ (that is $J\subset I$ and $I^{n+1}=JI^n$ for some integer $n$), then $\overline{J^n}=\overline{I^n}$ for every $n$ because $J^n$ is a reduction of $I^n$. It follows that if $I$ is normal then $e_j(I)=\bar{e}_j(J)$. Another crucial example is the Ratliff–Rush filtration of the powers of an $\mathfrak{m}$-primary ideal $I$ of a local ring. We recall that [24]

$$\widetilde{I^n}=\bigcup_{k\geqslant 1}I^{k+n}:I^k.$$

If $I$ contains a non-zero divisor then $\widetilde{I^n}=I^n$ for $n\gg 0$ and hence $\mathscr{F}=\{\widetilde{I^n}\}_{n\geqslant 0}$ is an Hilbert filtration. In particular $e_j(\mathscr{F})=e_j(I)$ for $j=0,\ldots,d$.

The advantage of considering $\mathrm{gr}_{\mathscr{F}}(R)$ with the above filtrations rather than $\mathrm{gr}_I(R)$ is that they are graded rings with positive depth. Unfortunately, they are not standard algebras.

A classical technique for studying the Hilbert coefficients of any filtration $\mathscr{F}$ is to reduce the dimension of the ring by modding out a superficial sequence for $\mathscr{F}$. We recall that an element $x\in F_1$ is called a *superficial element* for $\mathscr{F}$ if there exists an integer $c$ such that $(F_n:x)\cap F_c=F_{n-1}$ for all $n>c$. A sequence $x_1,\ldots,x_k$ is then called a *superficial* *sequence* for $\mathscr{F}$ if $x_1$ is superficial for $\mathscr{F}$ and $x_i$ is superficial for the quotient filtration $\mathscr{F}/(x_1,\dots,x_{i-1})=\{F_n+(x_1,\dots,x_{i-1})/(x_1,\dots,x_{i-1})\}$ for $2\leqslant i\leqslant k$. Now, if grade $F_1\geqslant k$ and $x_1,\dots,x_k$ is a superficial sequence for $\mathscr{F}$ it can be showed that $e_j(\mathscr{F})=e_j(\mathfrak{F})$, for $0\leqslant j\leqslant d-k$, where $\mathfrak{F}=\mathscr{F}/(x_1,\dots,x_k)=\{F_n+(x_1,\dots,x_k)/(x_1,\dots,x_k)\}$ (see for instance [13]).

## 3. Results on the second Hilbert coefficient

Our first result is an upper bound on $e_2$ which is reminiscent of the one on $e_1$ established by Huckaba and Marley [13, 4.7] and by Vaz Pinto [39, 1.1], which characterizes when the depth of the associated graded ring is at least $d-1$.

**Theorem 3.1.** *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension $d\geqslant 1$ and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then*

$$e_2\leqslant \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n)$$

*for any minimal reduction $J$ of $I$. Furthermore, equality holds for some minimal reduction $J$ of $I$ if and only if* depth $\mathrm{gr}_I(R)\geqslant d-1$.

**Proof.** If $d=1$ the result follows from [8, 1.9]. Let us assume then $d\geqslant 2$. Let $J$ be a minimal reduction of $I$ and let $x_1,\dots,x_{d-1}$ be a superficial sequence for $I$ contained in $J$. Let $H$ and $K$ denote the ideals $I/(x_1,\dots,x_{d-2})$ and $I/(x_1,\dots,x_{d-1})$, respectively. By Elias et al. [6, 1.2(a),(b)] and Guerrieri and Rossi [8, 1.9] we get $e_2(I)=e_2(H)\leqslant e_2(K)=\sum_{n\geqslant 1} n\lambda(K^{n+1}/JK^n)\leqslant\sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n)$, which establishes the desired inequality.

If depth $\mathrm{gr}_I(R)\geqslant d-1$ the equality follows from [8, 1.9]. Conversely, if equality holds one has that $e_2(H)=e_2(K)$, which in turn forces depth $\mathrm{gr}_H(R/(x_1,\dots,x_{d-2}))\geqslant 1$ by Elias et al. [6, 1.2(c)]. Hence by Huckaba and Marley [13, 2.2] we conclude that depth $\mathrm{gr}_I(R)\geqslant d-1$. $\square$

The following example, due to Huckaba and Huneke [12, 3.12], provides an instance in which the bound in Theorem 3.1 is attained. This example will also play a role in the next section.

**Example 3.2.** Let $k$ be a field of characteristic $\neq 3$ and set $R=k[\![X,Y,Z]\!]$, where $X,Y,Z$ are indeterminates. Let $N=(X^4, X(Y^3+Z^3), Y(Y^3+Z^3), Z(Y^3+Z^3))$ and set $I=N+\mathfrak{m}^5$, where $\mathfrak{m}$ is the maximal ideal of $R$. The ideal $I$ is a normal $\mathfrak{m}$-primary ideal whose associated graded ring $\mathrm{gr}_I(R)$ has depth $d-1$, where $d(=3)$ is the dimension of $R$. We checked that

$$P_I(t)=\frac{31+43t+t^2+t^3}{(1-t)^3},$$

thus yielding $e_2=4$. Moreover, we also checked that $\lambda(I^2/JI)=2$, $\lambda(I^3/JI^2)=1$ and $I^4=JI^3$, for any minimal reduction $J$ of $I$. Hence the bound in Theorem 3.1 is sharp.

Since $I$ is, in particular, integrally closed we have that $J \cap I^2 = JI$ by Huneke [14, 2.1] and Itoh [15, 1]. Thus depth $\mathrm{gr}_I(R) \geqslant 2$ also follows from the main result of [3,5,11,26].

Theorem 3.3 below deals with the depth property of the associated graded ring of ideals whose second Hilbert coefficient is close enough to the upper bound established in Theorem 3.1.

**Theorem 3.3.** *Let* $(R, \mathfrak{m})$ *be a local Cohen–Macaulay ring of dimension* $d \geqslant 1$. *Let* $I$ *be an* $\mathfrak{m}$*-primary ideal and let* $J$ *denote a minimal reduction of* $I$. *If any of the following conditions holds*:

(a) $e_2 \geqslant \sum_{n \geqslant 1} n\lambda(I^{n+1}/JI^n) - 2$;

(b) *$I$ is an integrally closed ideal and* $e_2 \geqslant \sum_{n \geqslant 1} n\lambda(I^{n+1}/JI^n) - 4$,

*then* depth $\mathrm{gr}_I(R) \geqslant d - 2$.

**Proof.** Throughout the proof we use the same notation as in the proof of Theorem 3.1. By Theorem 3.1 we may assume that depth $\mathrm{gr}_I(R) < d-1$, which implies that $e_2(H) < e_2(K)$ and

$$\sum_{n \geqslant 1} n\lambda(K^{n+1}/JK^n) < \sum_{n \geqslant 1} n\lambda(H^{n+1}/JH^n) \leqslant \sum_{n \geqslant 1} n\lambda(I^{n+1}/JI^n).$$

Indeed $e_2(H) = e_2(K)$ implies depth $\mathrm{gr}_H(R/(x_1, \ldots, x_{d-2})) > 0$ by Elias et al. [6, 1.2(c)] and hence depth $\mathrm{gr}_I(R) \geqslant d-1$ by Huckaba and Marley [13, 2.2]. If

$$\sum_{n \geqslant 1} n\lambda(K^{n+1}/JK^n) = \sum_{n \geqslant 1} n\lambda(H^{n+1}/JH^n),$$

then $\lambda(K^{n+1}/JK^n) = \lambda(H^{n+1}/JH^n)$ for every $n$ so that

$$e_1(H) = e_1(K) = \sum_{n \geqslant 0} \lambda(K^{n+1}/JK^n) = \sum_{n \geqslant 0} \lambda(H^{n+1}/JH^n),$$

hence again depth $\mathrm{gr}_H(R/(x_1, \ldots, x_{d-2})) > 0$ by Huckaba and Marley [13, 4.7(b)] and, as before, depth $\mathrm{gr}_I(R) \geqslant d-1$.

Let us assume that (a) holds. We have

$$\begin{aligned}
\sum_{n \geqslant 1} n\lambda(I^{n+1}/JI^n) - 2 \leqslant e_2(I) = e_2(H) & \\
&\leqslant e_2(K) - 1 \\
&= \sum_{n \geqslant 1} n\lambda(K^{n+1}/JK^n) - 1 \\
&\leqslant \sum_{n \geqslant 1} n\lambda(I^{n+1}/JI^n) - 2.
\end{aligned}$$

Hence we obtain that

$$\sum_{n\geqslant 1} n\lambda(K^{n+1}/JK^n) = \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 1,$$

which implies that $\lambda(K^2/JK) = \lambda(I^2/JI) - 1$ and $\lambda(K^{n+1}/JK^n) = \lambda(I^{n+1}/JI^n)$ for all $n \geqslant 2$. Hence

$$e_1(I) = e_1(K) = \sum_{n\geqslant 0} \lambda(K^{n+1}/JK^n) = \sum_{n\geqslant 0} \lambda(I^{n+1}/JI^n) - 1,$$

from which it follows that depth $\mathrm{gr}_I(R) \geqslant d-2$ by Polini [23] and Wang [40].

Let us assume now that (b) holds. Since $I$ is integrally closed $I^2 \cap J = JI$ by Huneke [14, 4.7(b)] and Itoh [15, 1], hence $\lambda(H^2/JH) = \lambda(K^2/JK) = \lambda(I^2/JI)$. We claim that our assumption on $e_2$ forces $\lambda(H^{n+1}/JH^n) = \lambda(I^{n+1}/JI^n)$ for all $n \geqslant 0$ and $\sum_{n\geqslant 0} \lambda(K^{n+1}/JK^n) = \sum_{n\geqslant 0} \lambda(I^{n+1}/JI^n) - 1$. Thus we conclude, as before, that

$$e_1(I) = \sum_{n\geqslant 0} \lambda(I^{n+1}/JI^n) - 1,$$

which forces depth $\mathrm{gr}_I(R) \geqslant d-2$.

Notice that if $\lambda(H^{n+1}/JH^n) \neq \lambda(I^{n+1}/JI^n)$ for some $n(\geqslant 2)$ then

$$\begin{aligned}
\sum_{n\geqslant 1} n\lambda(H^{n+1}/JH^n) - 2 &\leqslant \sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 4 \\
&\leqslant e_2(I) = e_2(H) \\
&\leqslant e_2(K) - 1 \\
&= \sum_{n\geqslant 1} n\lambda(K^{n+1}/JK^n) - 1 \\
&= \lambda(H^2/JH) + \sum_{n\geqslant 2} n\lambda(K^{n+1}/JK^n) - 1 \\
&\leqslant \lambda(H^2/JH) + \sum_{n\geqslant 2} n\lambda(H^{n+1}/JH^n) - 3
\end{aligned}$$

which is impossible. Hence we may assume that $\lambda(I^{n+1}/JI^n) = \lambda(H^{n+1}/JH^n)$ for all $n$. Since $e_2(H) \leqslant e_2(K) - 1$, we have to consider two cases.

If $e_2(H) = e_2(K) - 1$, by Elias et al. [6, 1.2(b)] we have that $\sum_{n\geqslant 1} \lambda((H^{n+1} : x_{d-1})/H^n) = 1$. This implies (by using, for example, the exact sequence in the proof of 1.7 [25]) that $\sum_{n\geqslant 1} \lambda(H^{n+1}/JH^n) = \sum_{n\geqslant 1} \lambda(K^{n+1}/JK^n) + 1$. This proves our claim.

If $e_2(H) \leqslant e_2(K) - 2$, then we have

$$\begin{aligned}
e_2(H) &\leqslant e_2(K) - 2 \\
&= \sum_{n\geqslant 1} n\lambda(K^{n+1}/JK^n) - 2 \\
&\leqslant \sum_{n\geqslant 1} n\lambda(H^{n+1}/JH^n) - 4,
\end{aligned}$$

which implies $\sum_{n\geqslant 2} n\lambda(K^{n+1}/JK^n) = \sum_{n\geqslant 2} n\lambda(H^{n+1}/JH^n) - 2$. This proves again our claim. $\square$

**Remark 3.4.** From the proof of Theorem 3.3 we conclude that $e_2(I)$ can never be equal to $\sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 1$. Moreover, if $I$ is an integrally closed ideal we also conclude that it can be neither $\sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 1$ nor $\sum_{n\geqslant 1} n\lambda(I^{n+1}/JI^n) - 2$.

We illustrate Theorem 3.3 with the following example which has been slightly modified from one suggested to us by Wang.

**Example 3.5.** Let $R$ be the three-dimensional local Cohen–Macaulay ring

$$k[\![X, Y, Z, U, V, W]\!]/(Z^2, ZU, ZV, UV, YZ - U^3, XZ - V^3),$$

with $k$ a field and $X, Y, Z, U, V, W$ indeterminates. Let $x, y, z, u, v, w$ denote the corresponding images of $X, Y, Z, U, V, W$ in $R$. One has that the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ of $\mathfrak{m} = (x, y, z, u, v, w)$ has depth $d - 2$, where $d(=3)$ is the dimension of $R$. Indeed, we checked that

$$P_{\mathfrak{m}}(t) = \frac{1 + 3t + 3t^3 - t^4}{(1-t)^3},$$

so that $e_2=3$. Moreover $\lambda(\mathfrak{m}^2/J\mathfrak{m})=2$, $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2)=2$ and $\mathfrak{m}^4=J\mathfrak{m}^3$, where $J=(x, y, w)$. Thus Theorem 3.3 applies.

Next, we present an improvement of Narita's positivity result on $e_2$, which holds for any integrally closed ideal. We give a more concrete lower bound and we characterize the integrally closed ideals for which the minimal value of $e_2$ is attained.

**Theorem 3.6.** *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension $d \geqslant 1$. Let $I$ be an $\mathfrak{m}$-primary integrally closed ideal. Then*

$$e_2 \geqslant \lambda(I^2/JI),$$

*where $J$ is any minimal reduction of $I$.*
*In addition, the following conditions are equivalent*:

(a) $e_2 = \lambda(I^2/JI)$;
(b) $I^3 = JI^2$;
(c) $\lambda(R/I) = e_0 - e_1 + \lambda(I^2/JI)$.

*Moreover, if any of the previous equivalent conditions holds then $\mathrm{gr}_I(R)$ is Cohen–Macaulay and*

$$P_I(t) = \frac{\lambda(R/I) + (e_0 - \lambda(R/I) - \lambda(I^2/JI))t + \lambda(I^2/JI)t^2}{(1-t)^d}.$$

**Proof.** Let $J$ be a minimal reduction of $I$. By Huckaba and Marley [13, 4.7(a)] we have the inequality

$$e_1 \geqslant \sum_{n\geqslant 0} \lambda(I^{n+1}/J\cap I^n) = e_0 - \lambda(R/I) + \sum_{n\geqslant 1} \lambda(I^{n+1}/J\cap I^n).$$

Since $I$ is integrally closed, by Itoh [17, 12] we also have that $e_2 \geqslant e_1 - e_0 + \lambda(R/I)$. If we now take into account the above inequality on $e_1$ we conclude that

$$e_2 \geqslant \sum_{n\geqslant 1} \lambda(I^{n+1}/J\cap I^n).$$

On the other hand, $I$ being integrally closed implies that $J\cap I^2 = JI$ by Huneke [14, 4.7(b)] and Itoh [15, 1]. Hence we have that

$$e_2 \geqslant \lambda(I^2/JI) + \sum_{n\geqslant 2} \lambda(I^{n+1}/J\cap I^{n+1}) \geqslant \lambda(I^2/JI),$$

which is the desired inequality.

Let us prove the equivalences. If $e_2 = \lambda(I^2/JI)$, then for every $n\geqslant 2$ we have that $\lambda(I^{n+1}/J\cap I^{n+1}) = 0$ and $e_1 = \sum_{n\geqslant 0} \lambda(I^{n+1}/J\cap I^n) = e_0 - \lambda(R/I) + \lambda(I^2/JI)$. This proves (c). If (c) holds, by Huckaba and Marley [13, 4.7(a)] we have that $\mathrm{gr}_I(R)$ is Cohen–Macaulay. In particular, we obtain that $J\cap I^{n+1} = JI^n$ for every $n$. Hence $I^{n+1} = J\cap I^{n+1} = JI^n$ for $n\geqslant 2$. This yields (b). Suppose now that (b) holds. Then we have that $\sum_{n\geqslant 2} n\lambda(I^{n+1}/JI^n) = 0$. Now Theorem 3.1 also gives us the upper bound $e_2 \leqslant \lambda(I^2/JI)$, so that (a) follows.

As far as the Hilbert series is concerned, since $\mathrm{gr}_I(R)$ is Cohen–Macaulay it follows that $P_I(t) = P_{I/J}(t)/(1-t)^d$. In particular, $P_{I/J}(t)$ is a polynomial of degree 2 because $I^3 \subseteq J$. If we write $P_{I/J}(t) = h_0 + h_1 t + h_2 t^2$, then we necessarily conclude that $h_0 = \lambda(R/I)$ and $h_2 = e_2 = \lambda(I^2/JI)$. □

**Remark 3.7.** We observe that the lower bound on $e_2$ given in Theorem 3.6 is well defined, as $\lambda(I^2/JI)$ is always independent of the minimal reduction $J$ of $I$ [36]. Also, Theorem 3.6 recovers previous results by Itoh, who treated the cases in which $e_2 = 0, 1, 2$: In such instances $\mathrm{gr}_I(R)$ always turns out to be Cohen–Macaulay [17, 5,6,7]. In addition to fully treating the general case, we also describe the Hilbert series of $I$.

We point out that if $e_2 = 3$ then $\mathrm{gr}_I(R)$ is not necessarily Cohen–Macaulay even if $I$ is the maximal ideal of a local Cohen–Macaulay ring (see Example 3.10).

Finally, we observe that in Theorem 3.6 the assumption on the ideal $I$ being 'integrally closed' cannot be weakened. The following example shows that $e_2 = 0$ does not imply the Cohen–Macaulayness of $\mathrm{gr}_I(R)$.

**Example 3.8.** Let $R$ be the three-dimensional regular local ring $k[\![X, Y, Z]\!]$, with $k$ a field and $X, Y, Z$ indeterminates. The ideal $I = (X^2 - Y^2, Y^2 - Z^2, XY, XZ, YZ)$ is not integrally closed and

$$P_I(t) = \frac{5 + 6t^2 - 4t^3 + t^4}{(1-t)^3}.$$

In particular, $e_2 = 0$ and $\mathrm{gr}_I(R)$ has depth zero. In fact, computing $P_{I/(XY)}(t)$ we can see that $XY$ is a superficial element for $I$ whose initial form is a zero-divisor in $\mathrm{gr}_I(R)$.

By using the techniques of this paper, we can also give here a short proof of a result of Narita who characterized $e_2 = 0$ for any $\mathfrak{m}$-primary ideal of a two-dimensional local Cohen–Macaulay ring.

**Proposition 3.9.** *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension two and let $I$ be an $\mathfrak{m}$-primary ideal. Then $e_2 = 0$ if and only if $I^n$ has reduction number one for some positive integer $n$. Under these circumstances then $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay.*

**Proof.** We first recall that $e_2 = e_2(I^m)$ for every positive integer $m$. Assume $e_2 = 0$ and let $n$ be an integer such that $\widetilde{I^n} = I^n$. By Sally [34, 2.5], $0 = e_2(I^n) \geqslant e_1(I^n) - e_0(I^n) + \lambda(R/\widetilde{I^n}) = e_1(I^n) - e_0(I^n) + \lambda(R/I^n)$. Hence $e_1(I^n) - e_0(I^n) + \lambda(R/I^n) = 0$ because it cannot be negative by Northcott's inequality. The result follows now by Huneke [14, 2.1] and Ooishi [22, 3.3]. For the converse, if $I^n$ has reduction number one for some $n$, then $e_2(I^n) = 0$ for example by Guerrieri and Rossi [8, 2.4]. In particular $e_2(I) = e_2(I^n) = 0$. It is clear that if $I^n$ has reduction number one, then $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay (see [35]). $\square$

We remark that Narita's result cannot be extended to a local Cohen–Macaulay ring of dimension $> 2$. The ideal $I$ described in Example 3.8 satisfies $e_2 = 0$, however $I^m$ has not reduction number one for every $m$. In fact, it is enough to remark that $I$ has not reduction number one ($\mathrm{gr}_I(R)$ is not Cohen–Macaulay) and $I^m = (X, Y, Z)^{2m}$ for $m > 1$ which has reduction number two.

In [17, 12] Itoh showed that if $I$ is an integrally closed ideal then $e_2 \geqslant e_1 - e_0 + \lambda(R/I)$. Later, it has been conjectured by Valla [37, 6.20] that if the equality $e_2 = e_1 - e_0 + \lambda(R/I)$ holds in the case in which $I$ is the maximal ideal $\mathfrak{m}$ of $R$ then the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay. Unfortunately, the following example given by Wang shows that the conjecture is, in general, false.

**Example 3.10.** Let $R$ be the two-dimensional local Cohen–Macaulay ring

$$k[\![X, Y, Z, U, V]\!]/(Z^2, ZU, ZV, UV, YZ - U^3, XZ - V^3),$$

with $k$ a field and $X, Y, Z, U, V$ indeterminates. Let $I$ be the maximal ideal $\mathfrak{m}$ of $R$. One has that the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ has depth zero and

$$P_I(t) = \frac{1 + 3t + 3t^3 - t^4}{(1 - t)^2}.$$

In particular, one has $e_2 = e_1 - e_0 + 1$, that is, $e_2$ is minimal according to Itoh's bound. However, $e_2$ is not minimal with respect to the bound given in Theorem 3.6.

Thus, the associated graded ring of the maximal ideal of the ring $R$ can have depth zero even if $e_2 = e_1 - e_0 + 1$. More generally, a condition such as $\lambda(R/I) = e_0 - e_1 + e_2$ is not sufficient to guarantee that $\mathrm{gr}_I(R)$ is Cohen–Macaulay even for an integrally closed ideal $I$. Motivated by this failure, we observe that the right setting is the one of normal ideals.

The following result is essentially contained in [16], we present it for completeness with a simpler proof. As a piece of notation, we denote by $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_d$ the Hilbert coefficients with respect to the filtration $\mathscr{F} = \{\overline{I^n}\}_{n \geqslant 0}$ given by the integral closure of the powers of $I$.

**Theorem 3.11.** *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension $d \geqslant 1$. Let $I$ be an ideal generated by a system of parameters. If $\lambda(R/\bar{I}) = \bar{e}_0 - \bar{e}_1 + \bar{e}_2$ then $\overline{I^{n+2}} = I^n \overline{I^2}$ for all $n \geqslant 0$.*

**Proof.** If $d = 1$, the result follows from [13, 4.6]. Let assume then $d \geqslant 2$. Let $S$ be the ring obtained from $R$ by a purely transcendental residue field extension and by factoring out $d-2$ generic elements $a_1, \ldots, a_{d-2}$ of $I$. Notice that $S$ is a two-dimensional local Cohen–Macaulay ring and the ideal $IS$ is generated by a system of parameters. By Itoh [16, 1(2) and (3)]

$$\bar{I}S = \overline{IS}, \tag{1}$$
$$\overline{I^n}S = \overline{(IS)^n} \quad \text{for } n \geqslant 0.$$

This last fact coupled with the genericity of $a_1, \ldots, a_{d-2}$ yields that $\bar{e}_i = \bar{e}_i(I) = \bar{e}_i(IS)$ for $i = 0, 1, 2$. Also by (1) we have that $\lambda(R/\bar{I}) = \lambda(S/\overline{IS})$. This yields $\lambda(S/\overline{IS}) = \bar{e}_0(IS) - \bar{e}_1(IS) + \bar{e}_2(IS)$. Call $\mathfrak{F} = \{\overline{(IS)^n}\}$. As depth $\mathrm{gr}(\mathfrak{F}) \geqslant 1 = \dim S - 1$, by Guerrieri and Rossi [8, 1.11(4)] the degree of the $h$-polynomial $s(\mathfrak{F}) \leqslant 2$. Hence $\bar{e}_3(IS) = 0$. Now by Guerrieri and Rossi [8, 1.9] we obtain that $\overline{(IS)^{n+1}} = IS\overline{(IS)^n}$ for $n \geqslant 2$, and, in particular, $\overline{(IS)^{n+2}} = (IS)^n\overline{(IS)^2}$ for $n \geqslant 0$. Finally, using [16, 17], we have that $\overline{I^{n+2}} = I^n\overline{I^2}$ for $n \geqslant 0$. $\square$

**Theorem 3.12.** *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension $d \geqslant 1$. Let $I$ be an $\mathfrak{m}$-primary normal ideal. The following conditions are equivalent*:

(a) $\lambda(R/I) = e_0 - e_1 + e_2$;
(b) $I^3 = JI^2$ *for some minimal reduction $J$ of $I$*;
(c) $e_2 = \lambda(I^2/JI)$ *for some minimal reduction $J$ of $I$.*

*Moreover, if any of the previous equivalent conditions holds then $\mathrm{gr}_I(R)$ is Cohen–Macaulay and*

$$P_I(t) = \frac{\lambda(R/I) + (e_0 - \lambda(R/I) - \lambda(I^2/JI))t + \lambda(I^2/JI)t^2}{(1-t)^d}.$$

**Proof.** Assume that condition (a) holds and let $J = (x_1, \ldots, x_d)$ be a minimal reduction of $I$. Observe that $\mathscr{F} = \{\overline{J^n}\} = \{I^n\}$, as $J^n$ is still a reduction of $I^n$ (not minimal) hence $\bar{e}_i(J) = e_i$. By Theorem 3.11, applied to $J$, we have $\overline{J^{n+2}} = J^n\overline{J^2}$ for $n \geqslant 0$, hence $I^{n+2} = J^n I^2$ for $n \geqslant 0$. This yields $I^3 = JI^2$. Now (b) implies (c) and (c) forces (a) by Theorem 3.6 and the Cohen–Macaulayness of $\mathrm{gr}_I(R)$ and the Hilbert series follow as well from the same theorem. $\square$

## 4. Results on the higher Hilbert coefficients

Itoh showed in [16, 3(1)] that $e_3 \geqslant 0$ for an $\mathfrak{m}$-primary normal ideal. Earlier, Narita had given in [19] an example of a three-dimensional Cohen–Macaulay local ring and an ideal $I$ with $e_3 < 0$. The ring in Narita's example contains nilpotents, thus Marley subsequently gave in [18, 4.2] an example of an ideal in a polynomial ring in three variables with $e_3 < 0$. Finally, the previously mentioned example by Wang provides an example of a Cohen–Macaulay local ring $R$ in which the maximal ideal has $e_3 < 0$.

Let $n(I)$ denote the so-called postulation number of $I$, that is the smallest integer $n$ such that $\lambda(R/I^n)$ is a polynomial.

The following result improves the already known result of Itoh under a weaker assumption:

**Theorem 4.1.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension three and with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R such that $I^q$ is integrally closed for some $q \geqslant n(I)$. Then $e_3 \geqslant 0$.*

**Proof.** For $n \gg 0$ the Hilbert–Samuel function of $I$ can be written as

$$\lambda(R/I^n) = e_0 \binom{n+2}{3} - e_1 \binom{n+1}{2} + e_2 \binom{n}{1} - e_3. \tag{2}$$

Let $q \geqslant n(I)$ be an integer for which $\overline{I^q} = I^q$. Consider the Hilbert–Samuel function of $I^q$. For $n \gg 0$ one has that

$$\lambda(R/(I^q)^n) = \varepsilon_0 \binom{n+2}{3} - \varepsilon_1 \binom{n+1}{2} + \varepsilon_2 \binom{n}{1} - \varepsilon_3. \tag{3}$$

As $\lambda(R/(I^q)^n) = \lambda(R/I^{nq})$, an easy comparison between (2), with $nq$ in place of $n$, and (3) yields

$$\begin{aligned} &e_0 \binom{nq+2}{3} - e_1 \binom{nq+1}{2} + e_2 \binom{nq}{1} - e_3 \\ &\quad = \varepsilon_0 \binom{n+2}{3} - \varepsilon_1 \binom{n+1}{2} + \varepsilon_2 \binom{n}{1} - \varepsilon_3. \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\frac{1}{6} e_0(n^3q^3 + 3n^2q^2 + 2nq) - \frac{1}{2} e_1(n^2q^2 + nq) + e_2 nq - e_3 \\ &\quad = \frac{1}{6} \varepsilon_0(n^3 + 3n^2 + 2n) - \frac{1}{2} \varepsilon_1(n^2q + n) + \varepsilon_2 n - \varepsilon_3. \end{aligned}$$

Hence one concludes that

$$\begin{aligned} &\varepsilon_0 = e_0 q^3, \quad \varepsilon_1 = e_0 q^2(q-1) + e_1 q^2, \\ &\varepsilon_2 = e_0 \binom{q}{3} + e_1 \binom{q}{2} + e_2 \binom{q}{1}, \quad \varepsilon_3 = e_3. \end{aligned}$$

By Itoh [17, 12], the Hilbert coefficients of the ideal $I^q$ satisfy the inequality

$$\varepsilon_2 - \varepsilon_1 + \varepsilon_0 - \lambda(R/I^q) \geqslant 0,$$

as $q$ was chosen so that $\overline{I^q} = I^q$. After substituting the $\varepsilon_i$'s with the corresponding expressions in terms of the $e_i$'s we conclude that

$$\begin{aligned}
\varepsilon_2 - \varepsilon_1 + \varepsilon_0 - \lambda(R/I^q) &= \left[e_0\binom{q}{3} + e_1\binom{q}{2} + e_2\binom{q}{1}\right] - [e_0q^2(q-1) + e_1q^2] \\
&\quad + e_0q^3 - \lambda(R/I^q) \\
&= e_0\binom{q+2}{3} - e_1\binom{q+1}{2} + e_2\binom{q}{1} - \lambda(R/I^q) \\
&= e_3,
\end{aligned}$$

and therefore $e_3 \geqslant 0$, as claimed. $\square$

**Remark 4.2.** We note that in dimension three for all $n \geqslant 2$, $e_2(I^n)$ is always strictly positive. In particular, the reduction number of $I^n$ is at least two.

An ideal $I$ is said to be asymptotically normal if there exists an integer $N \geqslant 1$ such $I^n$ is integrally closed for all $n \geqslant N$. An interesting family of examples of asymptotically normal ideals that are not normal are described in the next remark.

**Remark 4.3.** If $I$ is an asymptotically normal ideal such that $e_3 = 0$ then $\operatorname{gr}_I(R)$ is not necessarily forced to be Cohen–Macaulay. For example, the ideal $I$ in Example 3.8 is such that $I^n$ is integrally closed for every $n \geqslant 2$, $e_3 = 0$ but $\operatorname{gr}_I(R)$ has depth zero.

More generally, let $I$ be any $\mathfrak{m}$-primary Gorenstein ideal of minimal multiplicity in $R = k[\![X, Y, Z]\!]$, where $k$ is a field and $X, Y, Z$ are indeterminates. Since $I : \mathfrak{m} = \mathfrak{m}^2$, by Corso et al. [2, 3.6(a)] we have that $\mathfrak{m}^2 = I : \mathfrak{m} \subseteq \bar{I}$, which forces $\bar{I} = \mathfrak{m}^2 \neq I$.

Furthermore by Herzog [9], $I/I^2$ has a Cohen–Macaulay deformation which is generically a complete intersection. Thus by Bruns and Herzog [1, 4.7.11, 4.7.17(a)] it follows that $\lambda(I/I^2) = \lambda(R/I)\operatorname{ht}(I) = 15$. Now $\lambda(R/I^2) = \lambda(R/\mathfrak{m}^4)$ yields $I^2 = \mathfrak{m}^4$. By Corso et al. [2, 3.6(b)] we also have that $\mathfrak{m}I = \mathfrak{m}^3$, which implies, in addition, that $I^3 = I^2I = \mathfrak{m}^4I = \mathfrak{m}^3\mathfrak{m}I = \mathfrak{m}^6$. Hence we conclude that $I^n = \mathfrak{m}^{2n}$ for all $n \geqslant 2$. Thus $I$ is asymptotically normal and its Hilbert series is given by

$$P_I(t) = \frac{5 + 6t^2 - 4t^3 + t^4}{(1-t)^3}.$$

In particular $e_3 = 0$. On the other hand, $\operatorname{gr}_I(R)$ has depth zero because for any superficial element $a \in I$ one has $I^2 : a = \mathfrak{m}^2 \neq I$.

In the above remark, $I^2$ is a normal ideal in $k[\![X, Y, Z]\!]$ with $e_3(I^2) = 0$, $\operatorname{gr}_{I^2}(R)$ Cohen–Macaulay and the reduction number is two. It is natural to ask the following question:

**Question 4.4.** *Let $I$ be a normal $\mathfrak{m}$-primary ideal of a local Gorenstein ring $R$. Does $e_3 = 0$ imply $\operatorname{gr}_I(R)$ Cohen–Macaulay? Does $e_3 = 0$ imply that the reduction number of $I$ is two?*

In [17, 3] Itoh gave a positive answer to Question 4.4 when $I$ is the maximal ideal of a Gorenstein ring. Notice that if $I$ is asymptotically normal, but not normal, the answer is negative by Remark 4.3.

In Corollary 4.5 below we show that, in a local Cohen–Macaulay ring of dimension three, the normality of $I$ implies the Cohen–Macaulayness of $\mathrm{gr}_{I^n}(R)$ for all large $n$ whenever $e_3 = 0$.

**Corollary 4.5.** *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension three and with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R such that I is asymptotically normal. Then $e_3 = 0$ if and only if there exists some n such that the reduction number of $I^n$ is at most two. Under these circumstances then $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay.*

**Proof.** Let $q \geqslant n(I)$ be an integer large enough so that $I^q$ is a normal ideal. From the proof of Theorem 4.1 we have that $0 = e_3 = \varepsilon_2 - \varepsilon_1 + \varepsilon_0 - \lambda(R/I^q)$, where the $\varepsilon_i$'s are the Hilbert coefficients of $I^q$. The statement now follows from Theorem 3.12. $\square$

**Remark 4.6.** A different proof of the above result can be obtained in the following way. Let $q$ be the integer such that $I^q$ is normal. By Huckaba and Huneke [12, 3.1], there exists an integer $N$ such that depth $\mathrm{gr}_{I^N}(R) \geqslant 2$ and by Huckaba and Marley [13, 4.6] we get

$$e_3(I) = e_3(I^N) = \sum_{n \geqslant 3} \binom{n-1}{2} \lambda(I^{nN}/JI^{nN-N}),$$

for any minimal reduction $J$ of $I^N$. Hence $e_3(I) = 0$ if and only if $I^{3N} = JI^{2N}$ and the result follows.

The latter proof suggests the following result in dimension four. By Huckaba and Marley [13, 4.5, 4.1], it is easy to obtain

$$e_4(I) = e_4(I^N) \leqslant \sum_{n \geqslant 4} \binom{n-1}{3} \lambda(I^{nN}/JI^{nN-N}),$$

for any minimal reduction $J$ of $I^N$.

**Remark 4.7.** As we already remarked, a connection between the normality of $I$ and the depth of $\mathrm{gr}_{I^n}(R)$ has been observed in [12]. Indeed Huckaba and Huneke show that if $I$ is normal then $\mathrm{gr}_{I^n}(R)$ has depth at least 2 for $n \gg 0$. This result provides a two dimensional version of the Grauert–Riemenschneider vanishing theorem. More precisely, this is a generalization (in dimension two) of the following formulation of Grauert–Riemenschneider due to Sancho de Salas: If $R$ is a reduced Cohen–Macaulay local ring, essentially of finite type over an algebraically closed field of characteristic zero, and $I$ is an ideal of $R$ such that $\mathrm{Proj}(\mathscr{R})$ is regular, then $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay for some $n \gg 0$. While in dimension two the regularity of $\mathrm{Proj}(\mathscr{R})$ is not necessary (as shown in [12]), in dimension three the Grauert–Riemenschneider theorem fails if the assumption on $\mathrm{Proj}(\mathscr{R})$ being regular is dropped [4]. In [12] Huckaba and Huneke give another example of this failure.

**Example 4.8.** The same ideal $I$ considered in Example 3.2 also shows that Corollary 4.5 is sharp, that is the condition on $e_3$ cannot be relaxed. In fact, we checked that the ideal $I$ is such that

$$P_I(t) = \frac{31 + 43t + t^2 + t^3}{(1-t)^3}.$$

Thus one has $e_3 = 1$. On the other hand, Huckaba and Huneke show—in [12, 3.11]—that $I$ is a height 3 normal $R$-ideal for which $\mathrm{gr}_{I^n}(R)$ is not Cohen–Macaulay for any $n \geqslant 1$. In addition, one also has that $e_2 = 4$ while $\lambda(I^2/JI) = 2$, for any minimal reduction $J$ of $I$. Hence the bound in Theorem 3.6 is strict in this setting.

## References

[1] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge University Press, Cambridge, 1993.
[2] A. Corso, C. Huneke, W.V. Vasconcelos, On the integral closure of ideals, Manuscripta Math. 95 (1998) 331–347.
[3] A. Corso, C. Polini, M. Vaz Pinto, Sally modules and associated graded rings, Comm. Algebra 26 (1998) 2689–2708.
[4] S.D. Cutkosky, A new characterization of rational surface singularities, Invent. Math. 102 (1990) 157–177.
[5] J. Elias, On the depth of the tangent cone and the growth of the Hilbert function, Trans. Amer. Math. Soc. 351 (1999) 4027–4042.
[6] J. Elias, M.E. Rossi, G. Valla, On the coefficients of the Hilbert polynomial, J. Pure Appl. Algebra 108 (1996) 35–60.
[7] J. Elias, G. Valla, Rigid Hilbert functions, J. Pure Appl. Algebra 71 (1991) 19–41.
[8] A. Guerrieri, M.E. Rossi, Hilbert coefficients of Hilbert filtrations, J. Algebra 199 (1998) 40–61.
[9] J. Herzog, Deformationen von Cohen–Macaulay Algebren, J. Reine Angew. Math. 318 (1980) 83–105.
[10] S. Huckaba, A $d$-dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients, Proc. Amer. Math. Soc. 124 (1996) 1393–1401.
[11] S. Huckaba, On associated graded rings having almost maximal depth, Comm. Algebra 26 (1998) 967–976.
[12] S. Huckaba, C. Huneke, Normal ideals in regular rings, J. Reine Angew. Math. 510 (1999) 63–82.
[13] S. Huckaba, T. Marley, Hilbert coefficients and the depths of associated graded rings, J. London Math. Soc. 56 (1997) 64–76.
[14] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. 34 (1987) 293–318.
[15] S. Itoh, Integral closures of ideals generated by regular sequences, J. Algebra 117 (1988) 390–401.
[16] S. Itoh, Coefficients of normal Hilbert polynomials, J. Algebra 150 (1992) 101–117.
[17] S. Itoh, Hilbert coefficients of integrally closed ideals, J. Algebra 176 (1995) 638–652.
[18] T. Marley, The coefficients of the Hilbert polynomial and the reduction number of an ideal, J. London Math. Soc. 40 (1989) 1–8.
[19] M. Narita, A note on the coefficients of the Hilbert characteristic functions in semi-regular local rings, Proc. Cambridge Philos. Soc. 59 (1963) 269–275.
[20] D.G. Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc. 35 (1960) 209–214.
[21] D.G. Northcott, D. Rees, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc. 50 (1954) 145–158.
[22] A. Ooishi, $\Delta$-genera and sectional genera of commutative rings, Hiroshima Math. J. 17 (1987) 361–372.
[23] C. Polini, A filtration of the Sally module and the associated graded ring of an ideal, Comm. Algebra 28 (2000) 1335–1341.
[24] L.J. Ratliff, D.E. Rush, Two notes on reductions of ideals, Indiana Univ. Math. J. 27 (1978) 929–934.
[25] M.E. Rossi, A bound on the reduction number of a primary ideal, Proc. Amer. Math. Soc. 128 (2000) 1325–1332.

[26] M.E. Rossi, Primary ideals with good associated graded ring, J. Pure Appl. Algebra 145 (2000) 75–90.
[27] M.E. Rossi, G. Valla, A conjecture of J. Sally, Comm. Algebra 24 (1996) 4249–4261.
[28] J.D. Sally, On the associated graded ring of a Cohen–Macaulay local ring, J. Math. Kyoto Univ. 17 (1977) 19–21.
[29] J.D. Sally, Cohen–Macaulay local rings of maximal embedding dimension, J. Algebra 56 (1979) 168–183.
[30] J.D. Sally, Tangent cones at Gorenstein singularities, Compositio Math. 40 (1980) 167–175.
[31] J.D. Sally, Good embedding dimensions for Gorenstein singularities, Math. Ann. 249 (1980) 95–106.
[32] J.D. Sally, Cohen–Macaulay local rings with embedding dimension $e + d - 2$, J. Algebra 83 (1983) 325–333.
[33] J.D. Sally, Hilbert coefficients and reduction number 2, J. Algebraic Geom. 1 (1992) 325–333.
[34] J.D. Sally, Ideals whose Hilbert function and Hilbert polynomial agree at $n = 1$, J. Algebra 157 (1993) 534–547.
[35] P. Valabrega, G. Valla, Form rings and regular sequences, Nagoya Math. J. 72 (1978) 91–101.
[36] G. Valla, On form rings which are Cohen–Macaulay, J. Algebra 58 (1979) 247–250.
[37] G. Valla, Hilbert functions of graded algebras, in: J. Elias, J.M. Giral, R.M. Miró-Roig, S. Zarzuela (Eds.), Six Lectures on Commutative Algebra, Progress in Mathematics, vol. 166, Birkhäuser, Basel, 1998, pp. 293–344.
[38] W.V. Vasconcelos, Hilbert functions, analytic spread and Koszul homology, Contemp. Math. 159 (1994) 410–422.
[39] M. Vaz Pinto, Hilbert functions and Sally modules, J. Algebra 192 (1997) 504–523.
[40] H.-J. Wang, Hilbert coefficients and the associated graded rings, Proc. Amer. Math. Soc. 128 (2000) 963–973.