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Artinian level algebras of socle degree 4 ☆

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A B S T R A C T

In this paper we study the O-sequences of local (or graded) K-algebras of socle degree 4. More precisely, we prove that an O-sequence $h = (1, 3, h_2, h_3, h_4)$, where $h_4 \geq 2$, is the h-vector of a local level K-algebra if and only if $h_3 \leq 3h_4$. A characterization is also presented for Gorenstein O-sequences. In each of these cases we give an effective method to construct a local level K-algebra with a given h-vector. Moreover we refine a result of Elias and Rossi by showing that if $h = (1, h_1, h_2, h_3, 1)$ is a unimodal Gorenstein O-sequence, then h forces the corresponding Gorenstein K-algebra to be canonically graded if and only if $h_1 = h_3$ and $h_2 = \binom{h_1+1}{2}$, that is the h-vector is maximal. We discuss analogue problems for higher socle degrees.

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1. Introduction

Let $(A, \mathfrak{m})$ be an Artinian local or graded K-algebra where K is any arbitrary field unless otherwise specified. Let $\operatorname{Soc}(A) = (0 : \mathfrak{m})$ be the socle of A. We denote by s the *socle degree* of A, that is the maximum integer j such that $\mathfrak{m}^j \neq 0$. The *type* of A is $\tau := \dim_K \operatorname{Soc}(A)$. Recall that A is said to be *level* of type τ if $\operatorname{Soc}(A) = \mathfrak{m}^s$ and $\dim_K \mathfrak{m}^s = \tau$. If A has type 1, equivalently $\dim_K \operatorname{Soc}(A) = 1$, then A is *Gorenstein*. In the literature local rings with low socle degree, also called *short local rings*, have emerged as a testing ground for properties of infinite free resolutions (see [1], [2], [10], [20], [32], [35]). They have been also extensively studied in problems related to the irreducibility and the smoothness of the punctual Hilbert scheme $\operatorname{Hilb}_d(\mathbb{P}^n_K)$ parameterizing zero-dimensional subschemes in $\mathbb{P}^n_K$ of degree d, see among others [7], [8], [16], [31]. In this paper we study the structure of level K-algebras of socle degree 4, hence $\mathfrak{m}^5 = 0$. One of the most significant information on the structure is given by the Hilbert function.

By definition, the Hilbert function of A,

$$h_i = h_i(A) := \dim_K \mathfrak{m}^i/\mathfrak{m}^{i+1},$$

is the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(A) := \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. We also say that $h = (h_0, h_1, \ldots, h_s)$ is the h-vector of A. In [29] Macaulay characterized the possible sequences of positive integers h_i that can occur as the Hilbert function of A. Since then there has been a great interest in commutative algebra in determining the h-vectors that can occur as the Hilbert function of A with additional properties (for example, complete intersection, Gorenstein, level, etc). A sequence of positive integers $h = (h_0, h_1, \ldots, h_s)$ satisfying Macaulay's criterion, that is $h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for $i = 1, \ldots, s-1$, is called an O-sequence. A sequence $h = (1, h_1, \ldots, h_s)$ is said to be a *level (resp. Gorenstein) O-sequence* if h is the Hilbert function of some Artinian level (resp. Gorenstein) K-algebra A. Remark that h_1 is the embedding dimension and, if A is level, h_s is the type of A. Notice that a level O-sequence is not necessarily the Hilbert function of an Artinian level graded K-algebra. This is because the Hilbert function of the level ring $(A, \mathfrak{m})$ is the Hilbert function of $gr_{\mathfrak{m}}(A)$ which is not necessarily level. From now on we say that h is a graded level (resp. Gorenstein) O-sequence if h is the Hilbert function of a level (resp. Gorenstein) graded standard K-algebra. For instance it is well known that the h-vector of a Gorenstein graded K-algebra is symmetric, but this is no longer true for a Gorenstein local ring. Characterizing level O-sequences is a wide open problem in commutative algebra. The problem is difficult and very few results are known even in the graded case as evidenced by [18]. In the following table we give a summary of known results:

Characterization of	**Graded**	**Local**
Gorenstein O-sequences with $h_1 = 2$	[27,28]	[27,28][1]
level O-sequences with $h_1 = 2$	[27][2]	[27][3]
Gorenstein O-sequences with $h_1 = 3$	[6] (see also [36, Theorem 4.2], [19])	Open
level O-sequences with $h_1 = 3$ and $\tau \geq 2$	Open. In [18] authors gave a complete list with $s \leq 5$ or $s = 6$ and $\tau = 2$	Open
level O-sequences with $s \leq 3$	Open (see [18] for $h_1 = 3$ and [12] for discussion)	[12, Theorem 4.3]

In this paper we fill the above table by characterizing the Gorenstein and level O-sequences with a particular attention to socle degree 4 and embedding dimension $h_1 = 3$. In our setting we can assume that $A = R/I$ where $R = K[\![x_1, \dots, x_r]\!]$ is the formal power series ring or $R = K[x_1, \dots, x_r]$ the polynomial ring with standard grading and I an ideal of R. We say that A is graded when it can be presented as R/I where I is a homogeneous ideal in $R = K[x_1, \dots, x_r]$. Without loss of generality we assume that $h_1 = \dim_K \mathfrak{m}/\mathfrak{m}^2 = r$.

Recall that the *socle type* of $A = R/I$ is the sequence $E = (0, e_1, \dots, e_s)$, where

$$e_i := \dim_K((0 : \mathfrak{m}) \cap \mathfrak{m}^i/(0 : \mathfrak{m}) \cap \mathfrak{m}^{i+1}).$$

It is known that for all $i \geq 0$,

$$h_i \leq \min\{\dim_K R_i, e_i \dim_K R_0 + e_{i+1} \dim_K R_1 + \cdots + e_s \dim_K R_{s-i}\} \tag{1.1}$$

(see [21]). Hence a necessary condition for h to be a level O-sequence is that $h_{s-1} \leq h_1 h_s$ where $e_s = h_s$ and $e_i = 0$ otherwise. In the following theorem we prove that this condition is also sufficient for $h = (1, 3, h_2, h_3, h_4)$ to be a level O-sequence, provided $h_4 \geq 2$. However, if $h_4 = 1$, we need an additional assumption for h to be a Gorenstein O-sequence. We remark that the result for $h_4 = 2$ can not be extended to $h_1 > 3$ (see Example 3.9).

Theorem 1. *Let $h = (1, 3, h_2, h_3, h_4)$ be an O-sequence.*

[1] The characterization of local Gorenstein sequences was also obtained by Briançon in [5] and a later self-contained proof using symmetric decomposition was presented by Iarrobino in [22, Chapter 2].
[2] We refer the reader to [21, Theorem 4.6A] and [23] for more readable writing.
[3] See also [21, Theorem 4.6B] and [4, Theorem 2.6].

(a) *Let* $h_4 = 1$. *Then* h *is a Gorenstein O-sequence if and only if* $h_3 \leq 3$ *and* $h_2 \leq \binom{h_3+1}{2} + (3 - h_3)$.
(b) *Let* $h_4 \geq 2$. *Then* h *is a level O-sequence if and only if* $h_3 \leq 3h_4$.

The proof of the above result is effective in the sense that in each case we construct a local level K-algebra with a given h-vector verifying the necessary conditions (see Theorem 3.8).

Combining Theorem 1(b) and results in [18] we show that there are level O-sequences which are not realizable in the graded case (see Section 3, Table 1 and Table 2). A similar behaviour was observed in [12] for socle degree 3. Theorem 1(a) is a consequence of the following more general result which holds for any embedding dimension:

Theorem 2.

(a) *If* $(1, h_1, \ldots, h_{s-2}, h_{s-1}, 1)$ *is a Gorenstein O-sequence with* $s > 3$, *then*

$$h_{s-1} \leq h_1 \text{ and } h_{s-2} \leq \binom{h_{s-1}+1}{2} + (h_1 - h_{s-1}). \tag{1.2}$$

(b) *If* $h = (1, h_1, h_2, h_3, 1)$ *is a unimodal O-sequence satisfying* (1.2), *then* h *is a Gorenstein O-sequence.*

Notice that Theorem 2(a) can be obtained as a consequence of a more general result by Iarrobino in [22, Theorem 3.2A].

If A is a Gorenstein local K-algebra with symmetric h-vector, then $gr_{\mathfrak{m}}(A)$ is Gorenstein, see [22, Proposition 1.7]. It is a natural question to ask, in this case, whether A is analytically isomorphic to $gr_{\mathfrak{m}}(A)$. Accordingly with the definition given in [17, Page 408] and in [14], recall that an Artinian local K-algebra $(A, \mathfrak{m})$ is said to be *canonically graded* if there exists a K-algebra isomorphism between A and its associated graded ring $gr_{\mathfrak{m}}(A)$.

For instance J. Elias and M. E. Rossi in [14] proved that every Gorenstein K-algebra with symmetric h-vector and $\mathfrak{m}^4 = 0$ ($s \leq 3$) is canonically graded under the assumption that K is an algebraically closed field of characteristic zero. A local K-algebra A of socle type E is said to be *compressed* if equality holds in (1.1) for all $1 \leq i \leq s$, equivalently the h-vector is maximal given the socle type and embedding dimension (see [21, Definition 2.3]). Compressed Gorenstein local K-algebras enjoy nice homological properties, see for instance [33]. In [15, Theorem 3.1] Elias and Rossi proved that if A is any compressed Gorenstein local K-algebra of socle degree $s \leq 4$, then A is canonically graded under the assumption that K is an algebraically closed field of characteristic zero. In Section 4 we prove that if the socle degree is 4, then the assumption can not be relaxed. More precisely only the maximal h-vector forces every corresponding Gorenstein K-algebra to be canonically graded. We prove that if h is unimodal and not maximal, then there exists a Gorenstein K-algebra with Hilbert function h which is not canonically graded (for arbitrary field K). To prove that a local K-algebra is not canonically graded is in general a very difficult task. See also [25] for interesting discussions.

Theorem 3. *Assume K is an algebraically closed field of characteristic zero. Let $h = (1, h_1, h_2, h_3, 1)$ be an O-sequence with $h_2 \geq h_3$. Then every local Gorenstein K-algebra with Hilbert function h is necessarily canonically graded if and only if $h_1 = h_3$ and $h_2 = \binom{h_1+1}{2}$.*

An analogue of the above result is no longer true for socle degree 5. We prove that there exists a non-canonically graded Gorenstein K-algebra (for arbitrary K) with Hilbert function $h = (1, h_1, h_2, h_2, h_1, 1)$ for every pair (h_1, h_2) satisfying $2 \leq h_1 \leq h_2 \leq \binom{h_1+1}{2}$ (even in the compressed case), see Theorem 4.3.

The main tool of the paper is Macaulay's inverse system [28] which gives a one-to-one correspondence between ideals $I \subseteq R$ such that R/I is an Artinian local ring and finitely generated R-submodules of a polynomial ring. In Section 2 we gather preliminary results needed for our purpose. We prove Theorems 1 and 2 in Section 3, and Theorem 3 in Section 4.

We have used Singular [11], [13] and CoCoA [9] for various computations and examples.

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2. Preliminaries

2.1. Macaulay's inverse system

In this subsection we recall some results on Macaulay's inverse system which we will use in the subsequent sections. This theory is well-known in the literature, especially in the graded setting (see for example [28, Chapter IV] and [24]). However, the local case is not so well explored. We refer the reader to [17], [22] for an extended treatment.

It is known that the injective hull of K as an R-module is isomorphic to a divided power ring $P := K_{DP}[X_1, \ldots, X_r]$ which has a structure of R-module by means of the following action:

$$\begin{array}{cccc} \circ : & R \times P & \longrightarrow & P \\ & (x^\alpha, X^\beta) & \longrightarrow & x^\alpha \circ X^\beta = \begin{cases} X^{\beta-\alpha} & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha \not\leq \beta \end{cases} \end{array}$$

where $\alpha = (\alpha_1, \ldots, \alpha_r),\ \beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r,\ x^\alpha = x_1^{\alpha_1} \ldots x_r^{\alpha_r}, X^\beta = X_1^{\beta_1} \ldots X_r^{\beta_r}$, and by $\alpha \leq \beta$ we mean that $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, r$. For the sake of simplicity from now on we will use x_i instead of the capital letters X_i. If $\{f_1, \ldots, f_t\} \subseteq P$ is a set of polynomials, we will denote by $\langle f_1, \ldots, f_t \rangle_R$ the R-submodule of P generated by $f_1, \ldots, f_t$, i.e., the K-vector space generated by $f_1, \ldots, f_t$ and by the corresponding derivatives of all orders. We consider the exact pairing of K-vector spaces:

$$\begin{array}{rccc} \langle\ ,\ \rangle : & R \times P & \longrightarrow & K \\ & (f, g) & \longrightarrow & (f \circ g)(0). \end{array}$$

For any ideal $I \subset R$ we define the following R-submodule of P called Macaulay's inverse system:

$$I^\perp := \{g \in P \mid \langle f, g \rangle = 0 \ \ \forall f \in I\ \}.$$

Conversely, for every R-submodule M of P we define

$$\operatorname{Ann}_R(M) := \{g \in R \mid \langle g, f \rangle = 0 \ \ \forall f \in M\}$$

which is an ideal of R. If M is generated by polynomials $\underline{f} := f_1, \ldots, f_t$, with $f_i \in P$, then we will write $\operatorname{Ann}_R(M) = \operatorname{Ann}_R(\underline{f})$ and $A_{\underline{f}} = R/\operatorname{Ann}_R(\underline{f})$.

By using Matlis duality one proves that there exists a one-to-one correspondence between ideals $I \subseteq R$ such that R/I is an Artinian local ring and R-submodules M of P which are finitely generated. More precisely, Emsalem in [17, Proposition 2] and Iarrobino in [22, Lemma 1.2] stated the following result.

Proposition 2.1. *There is a one-to-one correspondence between ideals I such that R/I is an Artinian level local ring of socle degree s and type τ and R-submodules of P generated by τ polynomials of degree s having linearly independent forms of degree s. The correspondence is defined as follows:*

$$\left\{ \begin{array}{c} I \subseteq R \textit{ such that } R/I \\ \textit{is an Artinian level local ring of} \\ \textit{socle degree } s \textit{ and type } \tau \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} M \subseteq P \textit{ submodule generated by} \\ \tau \textit{ polynomials of degree} \\ s \textit{ with linearly independent leading forms} \end{array} \right\}$$

$$\begin{array}{ccc} I & \longrightarrow & I^\perp \\ \operatorname{Ann}_R(M) & \longleftarrow & M \end{array}$$

The action $\langle\ ,\ \rangle$ induces the following isomorphism of K-vector spaces (see [17, Proposition 2(a)],):

$$(R/I)^* \simeq I^\perp, \tag{2.2}$$

(where $(R/I)^*$ denotes the dual with respect to the pairing $\langle\ ,\ \rangle$ induced on R/I). Hence $\dim_K R/I = \dim_K I^\perp$. As in the graded case, it is possible to compute the Hilbert function of $A = R/I$ via the inverse system. We define the following K-vector space:

$$(I^\perp)_i := \frac{I^\perp \cap P_{\leq i} + P_{<i}}{P_{<i}}.$$

Then, by (2.2), it is known that

$$h_i(R/I) = \dim_K (I^\perp)_i. \tag{2.3}$$

2.2. Q-decomposition

It is well-known that the Hilbert function of an Artinian graded Gorenstein K-algebra is symmetric, which is not true in the local case. The problem comes from the fact that, in general, the associated algebra $G := gr_{\mathfrak{m}}(A)$ of a Gorenstein local algebra A is no longer Gorenstein. However, in [22] Iarrobino proved that the Hilbert function of a Gorenstein local K-algebra A admits a "symmetric" decomposition. To be more precise, consider a filtration of G by a descending sequence of ideals:

$$G = C(0) \supseteq C(1) \supseteq \cdots \supseteq C(s) = 0,$$

where

$$C(a)_i := \frac{(0 : \mathfrak{m}^{s+1-a-i}) \cap \mathfrak{m}^i}{(0 : \mathfrak{m}^{s+1-a-i}) \cap \mathfrak{m}^{i+1}}.$$

Let

$$Q(a) = C(a)/C(a+1).$$

Then

$$\{Q(a) : a = 0, \ldots, s-1\}$$

is called *Q-decomposition* of the associated graded ring G. We have

$$h_i(A) = \dim_K G_i = \sum_{a=0}^{s-1} \dim_K Q(a)_i.$$

Iarrobino [22, Theorem 1.5] proved that if $A = R/I$ is a Gorenstein local ring then for all $a = 0, \ldots, s-1$, $Q(a)$ is a reflexive graded G-module, up to a shift in degree: $\mathrm{Hom}_K(Q(a)_i, K) \cong Q(a)_{s-a-i}$. Hence the Hilbert function of $Q(a)$ is symmetric about $\frac{s-a}{2}$. Moreover, since each partial sum $\sum_{a=0}^{j} \dim_K Q(a)$ is the Hilbert function of

$G/C(j+1)$, $\sum_{a=0}^{j} \dim_K Q(a)$ is also an O-sequence (see [22, Page 69]). Iarrobino also showed that $Q(0) = G/C(1)$ is the unique (up to isomorphism) socle degree s graded Gorenstein quotient of G. Let $f = f[s] + ...$ lower degree terms be a polynomial in P of degree s where $f[s]$ is the homogeneous part of degree s of f and consider A_f the corresponding Gorenstein local K-algebra. Then, $Q(0) \cong R/\operatorname{Ann}_R(f[s])$ (see [17, Proposition 7] and [22, Lemma 1.10]).

Therefore a necessary condition for an O-sequence to be Gorenstein is that it admits a symmetric Q-decomposition by which we mean that (cf. [3]):

Definition 2.4. An O-sequence h is said to *admit a symmetric Q-decomposition* if there exist numerical sequences $h(a) = (h(a)_0, h(a)_1, \ldots, h(a)_s)$ for $a = 0, \ldots, s-1$ such that
(1) each $h(a)$ is symmetric about $\frac{s-a}{2}$;
(2) $h = \sum_{a=0}^{s-1} h(a)$;
(3) each partial sum $\sum_{a=0}^{j} h(a)$ for $j = 0, \ldots, s-1$ is an O-sequence.
If this is the case we also say that $\{h(a) : a = 0, \ldots, s-1\}$ is an "admissible symmetric Q-decomposition" for h.

3. Characterization of level O-sequences

In this section we characterize Gorenstein and level O-sequences of socle degree 4 and embedding dimension 3. First if $h = (1, h_1, \ldots, h_s)$ is a Gorenstein O-sequence (in any embedding dimension), we obtain an upper bound on h_{s-2} in terms of h_{s-1}. This result (the first part of the following theorem) can be obtained as a consequence of a more general result proved by Iarrobino in [22, Theorem 3.2A]. For the sake of completeness we include a direct proof here.

Theorem 3.1.

(a) *If* $(1, h_1, \ldots, h_{s-2}, h_{s-1}, 1)$ *is a Gorenstein O-sequence with* $s > 3$*, then*

$$h_{s-1} \leq h_1 \text{ and } h_{s-2} \leq \left(\binom{h_{s-1}+1}{2} + (h_1 - h_{s-1}) \right). \tag{3.2}$$

(b) *If* $h = (1, h_1, h_2, h_3, 1)$ *is an O-sequence such that* $h_2 \geq h_3$ *and it satisfies* (3.2)*, then* h *is a Gorenstein O-sequence.*

Proof. (a): From (1.1) it follows that $h_{s-1} \leq h_1$. Let A be a Gorenstein local K-algebra with the Hilbert function h and $\{Q(a) : a = 0, \ldots, s-1\}$ be Q-decomposition of $gr_{\mathfrak{m}}(A)$. Since $Q(i)_{s-1} = 0$ for $i > 0$, $\dim_K Q(0)_{s-1} = h_{s-1}$. Hence $\dim_K Q(0)_1 = \dim_K Q(0)_{s-1} = h_{s-1}$ and $\dim_K Q(0)_{s-2} = \dim_K Q(0)_2 \leq h_{s-1}^{\langle 1 \rangle}$. This in turn implies that $\dim_K Q(1)_{s-2} = \dim_K Q(1)_1 \leq h_1 - h_{s-1}$. Therefore

$$\begin{aligned} h_{s-2} &= \dim_K Q(0)_{s-2} + \dim_K Q(1)_{s-2} \\ &\leq \binom{h_{s-1}+1}{2} + (h_1 - h_{s-1}). \end{aligned}$$

(b): Suppose $h_2 \leq h_1$. Set

$$f = x_1^4 + \cdots + x_{h_3}^4 + x_{h_3+1}^3 + \cdots + x_{h_2}^3 + x_{h_2+1}^2 + \cdots + x_{h_1}^2.$$

Then A_f has the Hilbert function h. Now assume $h_2 > h_1$. Denote $h_3 := n$ and define monomials $g_i \in K_{DP}[x_1, \ldots, x_n]$ as follows:

$$g_i = \begin{cases} x_i^2 & \text{if } 1 \leq i \leq n \\ x_{i-n}x_{i-n+1} & \text{if } n+1 \leq i \leq 2n-1 \\ x_n x_1 & \text{if } i = 2n. \end{cases}$$

For $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, let $x^{\underline{i}} := x_1^{i_1} \ldots x_n^{i_n}$. Let T be the set of monomials $x^{\underline{i}}$ of degree 2 in $K_{DP}[x_1, \ldots, x_n]$ such that

$$x^{\underline{i}} \notin \{g_i : 1 \leq i \leq 2n\}.$$

Then $|T| = \binom{n+1}{2} - 2n$. We write $T = \{g_i : 2n < i \leq \binom{n+1}{2}\}$. Define

$$f = \begin{cases} \displaystyle\sum_{i=1}^{n} x_i^2 g_i + \sum_{i=1}^{h_2-h_1} x_i^2 g_{n+i} + x_{n+1}^3 + \cdots + x_{h_1}^3 & \text{if } h_2 - h_1 \leq n \\ \displaystyle\sum_{i=1}^{n} x_i^2 g_i + \sum_{i=1}^{n} x_i^2 g_{n+i} + \sum_{i=2n+1}^{h_2-h_1+n} g_i^2 + x_{n+1}^3 + \cdots + x_{h_1}^3 & \text{if } h_2 - h_1 > n. \end{cases}$$

Then $h_3 = \dim_K \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) = n$ and

$$h_2 = \dim_K \left(\{g_i : 1 \leq i \leq h_2 - h_1 + n\} \bigcup \{x_{n+1}^2, \ldots, x_{h_1}^2\} \right).$$

Thus A_f has the Hilbert function $(1, h_1, h_2, h_3, 1)$. $\square$

Remark 3.3. If $h_3 = h_1$, then f in the proof of Theorem 3.1(b) is homogeneous and hence A_f is a graded Gorenstein K-algebra.

If the socle degree is 4 and $h_1 \leq 12$, then the converse holds in 3.1(a).

Corollary 3.4. *An O-sequence $h = (1, h_1, h_2, h_3, 1)$ with $h_1 \leq 12$ is a Gorenstein O-sequence if and only if h satisfies* (3.2).

Proof. By Theorem 3.1, it suffices to show that $h_2 \geq h_3$ if $h_1 \leq 12$. Let A be a local Gorenstein K-algebra with the Hilbert function h. Considering the symmetric Q-decomposition of $gr_{\mathfrak{m}}(A)$, we observe that in this case $Q(0)$ has the Hilbert function $(1, h_3, h_2 - k, h_3, 1)$ for some non-negative integer k. Since $h_1 \leq 12$, by (3.2) $h_3 \leq 12$. As $Q(0)$ is a graded Gorenstein K-algebra, by [30, Theorem 3.2] we conclude that $h_2 - k \geq h_3$ since $Q(0)$ has unimodal Hilbert function, hence $h_2 \geq h_3$. □

Remark 3.5. A Gorenstein O-sequence $(1, h_1, h_2, h_3, 1)$ does not necessarily satisfy $h_2 \geq h_3$. For example, consider the sequence $h = (1, 13, 12, 13, 1)$. By [36, Example 4.3] there exists a graded Gorenstein K-algebra with h as h-vector.

It would be interesting to know which Gorenstein sequences appearing in Theorem 3.1(b) are admissible for complete intersections. In [26], jointly with J. Jelisiejew, we discuss this problem in codimension 3 for any socle degree.

Remark 3.6. We do not know a characterization of the Gorenstein O-sequences of socle degree 5, even if h is unimodal.

Clearly, (3.2) is not sufficient for an O-sequence to be Gorenstein when $s = 5$. For instance, $h = (1, 3, 3, 4, 3, 1)$ satisfies (3.2), but it is not admissible for a Gorenstein K-algebra because it does not admit symmetric Q-decomposition (see Definition 2.4). Hence an extension of Theorem 3.1(b) to $s = 5$ is not straightforward. As the referee suggests, the information given by [22, Theorem 3.2A] could be useful. But we feel that Q-decomposition is not enough to characterize Gorenstein sequences of higher socle degree and new ideas will be needed.

Discussion 3.7. Let $h = (1, h_1, h_2, h_3, 1)$ be an O-sequence as in Theorem 3.1(b). It can be verified that following is a complete list of admissible symmetric Q-decompositions (see Definition 2.4) for h:

$$\begin{array}{rcccccc} Q(0) & = & (1, & h_3, & \alpha, & h_3, & 1) \\ Q(1) & = & (0, & h_2 - \alpha, & h_2 - \alpha, & 0, & 0) \\ Q(2) & = & (0, & h_1 - h_3 - h_2 + \alpha, & 0, & 0, & 0) \\ \hline h & = & (1, & h_1, & h_2, & h_3, & 1) \end{array}$$

where $h_2 + (h_3 - h_1) \leq \alpha \leq \min\{h_2, \binom{h_3+1}{2}\}$. We claim that if $Q(0)$ is an admissible graded Gorenstein algebra, then each Q-decomposition is realizable. Indeed, suppose that $Q(0)$ is admissible for graded Gorenstein algebra. Then there exists a homogeneous polynomial $F \in K_{DP}[x_1, \ldots, x_{h_3}]$ of degree 4 such that A_F has the Hilbert function $(1, h_3, \alpha, h_3, 1)$. Define

$$f = F + x_{h_3+1}^3 + \cdots + x_{h_3+h_2-\alpha}^3 + x_{h_3+h_2-\alpha+1}^2 + \cdots + x_{h_1}^2.$$

Then A_f has the Hilbert function h and Q-decomposition as above (see [22, Section 4C]). Notice that f in the proof of Theorem 3.1(b) corresponds to the Q-decomposition where

$$\alpha = \begin{cases} h_3 & \text{if } h_2 \leq h_1 \\ h_2 + (h_3 - h_1) & \text{if } h_2 > h_1. \end{cases}$$

However, there are admissible symmetric Q-decompositions that are not realizable. For instance, consider $h = (1, 16, 14, 13, 1)$. Then

$$\begin{array}{ccccccc} Q(0) & = & (1, & 13, & 11, & 13, & 1) \\ Q(1) & = & (0, & 3, & 3, & 0, & 0) \\ \hline h & = & (1, & 16, & 14, & 13, & 1) \end{array}$$

is an admissible symmetric Q-decomposition for h. But this Q-decomposition is not realizable since $(1, 13, 11, 13, 1)$ does not occur as a graded Gorenstein O-sequence by [30, Theorem 3.2]. However, since $(1, 13, 12, 13, 1)$ is a graded Gorenstein O-sequence by [36, Example 4.3], the above argument shows that the following Q-decomposition is realizable for $h = (1, 16, 14, 13, 1)$:

$$\begin{array}{ccccccc} Q(0) & = & (1, & 13, & 12, & 13, & 1) \\ Q(1) & = & (0, & 2, & 2, & 0, & 0) \\ Q(2) & = & (0, & 1, & 0, & 0, & 0) \\ \hline h & = & (1, & 16, & 14, & 13, & 1) \end{array}$$

□

In the following theorem we characterize the h-vector of local level algebras of socle degree 4 and embedding dimension 3. We remark that the first part of the following theorem was already known due to [22, Page 91].

Theorem 3.8. *Let $h = (1, 3, h_2, h_3, h_4)$ be an O-sequence.*

(a) *Let $h_4 = 1$. Then h is a Gorenstein O-sequence if and only if $h_3 \leq 3$ and $h_2 \leq \binom{h_3+1}{2} + (3 - h_3)$.*
(b) *Let $h_4 \geq 2$. Then h is a level O-sequence if and only if $h_3 \leq 3h_4$.*

Proof. (a): Follows from Corollary 3.4.
(b): The "only if" part follows from (1.1). The converse is constructive and we prove it by induction on h_4. First we consider the cases $h_4 = 2, 3, 4$ and then $h_4 \geq 5$. In each case we define the polynomials $\underline{f} := f_1, \ldots, f_{h_4} \in P = K_{DP}[x_1, x_2, x_3]$ of degree 4 such that $A_{\underline{f}}$ has the Hilbert function h. We set $g'_1 = x_3^3, g'_2 = x_2^2x_3, g'_3 = x_1^2x_2, g'_4 = x_1x_3^2$.

For short, in this proof we use the following notation: $m := h_2$ and $n := h_3$.

Case 1: $h_4 = 2$.

In this case $n \leq 6$ as $h_3 \leq 3h_4$ by assumption. Suppose $m = 2$. Then h is an O-sequence implies that $n = 2$. In this case, let $f_1 = x_1^4 + x_3^2$ and $f_2 = x_2^4$. Then $A_{\underline{f}}$ has the Hilbert function $(1, 3, 2, 2, 2)$. Now assume that $m \geq 3$.

Subcase 1: $m \geq n$. We set $g_i = \begin{cases} x_{4-i}g_i' & \text{if } 1 \leq i \leq n-2 \\ g_i' & \text{if } n-1 \leq i \leq m-2 \\ 0 & \text{if } m-1 \leq i \leq 4. \end{cases}$

(Here $x_0 = x_3$). Define

$$f_1 = x_1^4 + g_1 + g_2 \text{ and } f_2 = x_2^4 + g_3 + g_4.$$

Then

$$h_3 = \dim_K\{x_1^3, x_2^3, g_1', \ldots, g_{n-2}'\} = n \text{ and}$$

$$h_2 = \dim_K\{x_1^2, x_2^2, \frac{g_i'}{x_{4-i}} : 1 \leq i \leq m-2\} = m$$

and hence $A_{\underline{f}}$ has the required Hilbert function h.

Subcase 2: $m < n$. The only possible ordered tuples (m, n) with $m < n \leq 6$ such that h is an O-sequence are $\{(3, 4), (4, 5), (5, 6)\}$. For each 2-tuple (m, n) we define f_1, f_2 as:

a.$(m, n) = (3, 4)$: $f_1 = x_1^4 + x_1^2x_2^2 + x_3^2$; $f_2 = x_2^4 + x_1^2x_2^2$.

b.$(m, n) = (4, 5)$: $f_1 = x_1^4 + x_1^2x_2^2 + x_3^4$; $f_2 = x_2^4 + x_1^2x_2^2$.

c.$(m, n) = (5, 6)$: $f_1 = x_1^4 + x_1^2x_2^2 + x_3^4$; $f_2 = x_2^4 + x_1^2x_2^2 + x_2^3x_3$.

Case 2: $h_4 = 3$.

In this case $n \leq 9$. We consider the following subcases:

Subcase 1: $n \leq 6$. Let $\underline{f}' = f_1, f_2$ be polynomials defined as in Case 1 such that $A_{\underline{f}'}$ has the Hilbert function $(1, 3, m, n, 2)$. Now define $f_3 = \begin{cases} x_3^4 & \text{if } m \geq n \\ x_1^2x_2^2 & \text{if } m < n. \end{cases}$

Then $A_{\underline{f}}$ has the required Hilbert function h.

Subcase 2: $7 \leq n \leq 9$. Let $\underline{f}' = f_1, f_2$ be polynomials defined as in Case 1 such that $A_{\underline{f}'}$ has the Hilbert function $(1, 3, m, 6, 2)$. We set $p_1 = x_2^2x_3^2, p_2 = x_1^2x_2^2$ and $p_3 = x_1^2x_3^2$. Since h is an O-sequence and $n \geq 7$, we get $m \geq 5$. Now define $f_3 = \begin{cases} \sum_{i=1}^{n-6} p_i & \text{if } m = 6 \\ x_2^2x_3^2 & \text{if } m = 5. \end{cases}$

Then $A_{\underline{f}}$ has the required Hilbert function h.

Case 3: $h_4 = 4$.

Since h is an O-sequence, $n \leq 10$. We consider the following subcases:

Subcase 1: $n \leq 9$. Let $\underline{f}' = f_1, f_2, f_3$ be polynomials defined as in Case 2 such that $A_{\underline{f}'}$ has the Hilbert function $(1, 3, m, n, 3)$. Define

$$f_4 = \begin{cases} x_2^3x_3 & \text{if } \{m \geq n \text{ and } n \leq 6\} \text{ OR } \{n \geq 7 \text{ and } m = 6\} \\ x_1^3x_2 & \text{if } \{m < n \leq 6\} \text{ OR } \{(m,n) = (5,7)\}. \end{cases}$$

Then $A_{\underline{f}}$ has the Hilbert function $(1, 3, m, n, 4)$.
Subcase 2: $\underline{n = 10.}$ As h is an O-sequence, we conclude that $m = 6$. Let $\underline{f}' = f_1, f_2, f_3$ be polynomials defined as in Case 2 such that $A_{\underline{f}'}$ has the Hilbert function $(1, 3, 6, 9, 3)$. Define $f_4 = x_1^2x_2x_3$. Then $A_{\underline{f}}$ has the Hilbert function $(1, 3, 6, 10, 4)$.

Case 4: $h_4 \geq 5$.

Since h is an O-sequence, $n \leq 10$ and $h_4 \leq 15$.
Subcase 1: $n \geq h_4$ **OR** $h_4 \geq 11$. Let $\underline{f}' = f_1, f_2, f_3, f_4$ be defined as in Case 3 such that $A_{\underline{f}'}$ has the Hilbert function $(1, 3, m, n, 4)$. For $5 \leq i \leq 15$, define f_i as follows:

$$f_5 = \begin{cases} x_1^3x_2 & \text{if } \{m \geq n \text{ and } n \leq 6\} \text{ OR } \{n \geq 7 \text{ and } m = 6\} \\ x_1x_2^3 & \text{if } \{m < n \leq 6\} \text{ OR } \{(m,n) = (5,7)\}, \end{cases}$$

$$f_6 = \begin{cases} x_1x_3^3 & \text{if } \{m \geq n \text{ and } n \leq 6\} \text{ OR } \{n \geq 7 \text{ and } m = 6\} \\ x_3^4 & \text{if } \{m < n \leq 6\} \text{ OR } \{(m,n) = (5,7)\}, \end{cases}$$

$$f_7 = \begin{cases} x_3^4 & \text{if } n \geq 7 \text{ and } m = 6 \\ x_2^3x_3 & \text{if } \{(m,n) = (5,7)\}. \end{cases}$$

(Note that in the last case, $h_4 \geq 7$ implies that $n \geq 7$). If $h_4 \geq 8$, then $n \geq 8$ which implies that $m = 6$. We set

$$f_8 = x_1^2x_2^2, f_9 = x_1^2x_3^2, f_{10} = x_2x_3^3, f_{11} = x_1x_2^3, f_{12} = x_1^3x_3, f_{13} = x_2^3x_3,$$
$$f_{14} = x_1x_2^2x_3, f_{15} = x_1x_2x_3^2.$$

Now $A_{\underline{f}}$ has the Hilbert function $(1, 3, m, n, h_4)$.
Subcase 2: $n < h_4 \leq 10$. The smallest ordered tuple (n, h_4) such that h is an O-sequence and $n < h_4$ is $(4, 5)$. (Here smallest ordered tuple means smallest with respect to the order $\leq$ defined as: $(n_1, n_2) \leq (m_1, m_2)$ if and only if $n_1 \leq m_1$ and $n_2 \leq m_2$). Let

$$q_1 = \begin{cases} x_3^2 & \text{if } m = 3 \\ x_3^3 & \text{if } m \geq 4, \end{cases} \quad q_2 = \begin{cases} 0 & \text{if } m < 5 \\ x_2^2x_3 & \text{if } m \geq 5 \end{cases} \text{ and } q_3 = \begin{cases} 0 & \text{if } m < 6 \\ x_1x_3^2 & \text{if } m = 6. \end{cases}$$

We define

$$f_1 = x_1^4 + q_1 + q_2, f_2 = x_2^4 + q_3, f_3 = x_1^3x_2, f_4 = x_1x_2^3, f_5 = x_1^2x_2^2.$$

Then $A_{\underline{f}}$ has the Hilbert function $(1, 3, m, 4, 5)$.

Let $h_4 \geq 6$. We set

$$f_6 = x_3^4, f_7 = x_2^3 x_3, f_8 = x_2 x_3^3, f_9 = \begin{cases} x_2^2 x_3^2 & \text{if } n = 7 \\ x_1 x_3^3 & \text{if } n \geq 8, \end{cases} f_{10} = \begin{cases} x_2^2 x_3^2 & \text{if } n = 8 \\ x_1^3 x_3 & \text{if } n = 9. \end{cases}$$

Then $A_{\underline{f}}$ has the Hilbert function $(1, 3, m, n, h_4)$. □

Using [18, Appendix D] and Theorem 3.8(b) we list in Table 1 all the O-sequences with $h_1 = 3,\ s = 4$ and $h_4 \geq 2$ which are realizable for local level K-algebras, but not for graded level K-algebras.

Table 1
Non-graded level O-sequences with $s = 4$ and $h_1 = 3$.

$(1,3,2,2,2)$	$(1,3,3,2,2)$	$(1,3,4,2,2)$	$(1,3,5,2,2)$	$(1,3,6,2,2)$	$(1,3,5,3,2)$
$(1,3,6,3,2)$	$(1,3,3,4,2)$	$(1,3,4,3,3)$	$(1,3,5,3,3)$	$(1,3,6,3,3)$	$(1,3,3,4,3)$
$(1,3,6,4,3)$	$(1,3,3,4,4)$	$(1,3,5,4,4)$	$(1,3,6,4,4)$	$(1,3,3,4,5)$	$(1,3,4,4,5)$
$(1,3,5,4,5)$	$(1,3,6,4,5)$	$(1,3,6,5,5)$	$(1,3,5,5,6)$	$(1,3,6,5,6)$	$(1,3,6,6,7)$
$(1,3,6,7,9)$					

Analogously, by Theorem 3.8(a), the following are all the Gorenstein O-sequences with $h_1 = 3$ and $s = 4$ that are not graded Gorenstein sequences since they are not symmetric. This list agrees with the list [22, 5F.i.b. and 5F.i.c in Page 91]. If an O-sequence h is symmetric with $h_1 = 3$ then h is also a graded Gorenstein O-sequence by Corollary 3.4.

Table 2
Non-graded Gorenstein O-sequences with $s = 4$ and $h_1 = 3$.

$(1,3,1,1,1)$	$(1,3,2,1,1)$	$(1,3,3,1,1)$	$(1,3,2,2,1)$	$(1,3,3,2,1)$	$(1,3,4,2,1)$

We remark that in Table 2 $(1, 3, 3, 2, 1)$ is the only sequence with two admissible symmetric decompositions. By Discussion 3.7 we know that each Q-decomposition is realizable. Among the Gorenstein sequences appearing in Table 2 it is easy to see that $(1, 3, 1, 1, 1)$, $(1, 3, 2, 1, 1)$, $(1, 3, 2, 2, 1)$ are not admissible for complete intersections. It is not difficult to show that even the sequences $(1, 3, 4, 2, 1), (1, 3, 5, 3, 1), (1, 3, 6, 3, 1)$ are not admissible for complete intersections and the sequence $(1, 3, 4, 3, 1)$ is admissible for a complete intersection (for instance, consider R/I where $I = (x_1^2, x_2^2, x_3^3)$). In [26] we show that any O-sequence of the form $(1, 3, 3, h_3 \dots, \dots)$ with $h_3 \leq 3$ is admissible for local complete intersection and hence in particular, the sequences $(1, 3, 3, 1, 1)$, $(1, 3, 3, 2, 1)$, $(1, 3, 3, 3, 1)$ are admissible for complete intersections.

The following example shows that Theorem 3.8(b) can not be extended to $h_1 \geq 4$ because the necessary condition $h_3 \leq h_1 h_s$ is no longer sufficient for characterizing level O-sequences of socle degree 4.

Example 3.9. The O-sequence $h = (1, 4, 9, 2, 2)$ is not a level O-sequence.

Proof. Let $A = R/I$ be a local level K-algebra with the Hilbert function h. The lex-ideal $L \in S := K[x_1, \dots, x_4]$ with the Hilbert function h is

$$L = (x_1^2, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, \\ x_3^3, x_3^2x_4, x_3x_4^4, x_4^5).$$

A minimal graded S-free resolution of S/L is:

$$0 \longrightarrow S(-6)^7 \oplus S(-8)^2 \longrightarrow S(-5)^{26} \oplus S(-7)^6 \longrightarrow S(-4)^{33} \oplus S(-6)^6 \\ \longrightarrow S(-2) \oplus S(-3)^{14} \oplus S(-5)^2 \longrightarrow S \longrightarrow 0.$$

By [34, Theorem 4.1] the Betti numbers of A can be obtained from the Betti numbers of S/L by a sequence of negative and zero consecutive cancellations. This implies that $\beta_4(A) \geq 3$ and hence A has type at least 3, which leads to a contradiction. □

4. Canonically graded algebras

It is clear that a necessary condition for a Gorenstein local K-algebra A being canonically graded is that the Hilbert function of A must be symmetric. Hence we investigate whether a Gorenstein K-algebra A with the Hilbert function $(1, h_1, h_2, h_1, 1)$ is necessarily canonically graded. If $h_2 = \binom{h_1+1}{2}$ (equiv. A is compressed), then by [15, Theorem 3.1] A is canonically graded. In this section we prove that if $h = (1, h_1, h_2, h_1, 1)$ is an O-sequence with $h_1 \leq h_2 < \binom{h_1+1}{2}$, then there exists a polynomial F of degree 4 such that A_F has the Hilbert function h and it is *not* canonically graded (Theorem 4.1). We prove that an analogue of this result is no longer true for socle degree 5. In fact, in Theorem 4.3 we construct a non-canonically graded Gorenstein K-algebra of socle degree 5 with unimodal and symmetric Hilbert function whenever $h_1 > 1$ (even in the compressed case).

Theorem 4.1. *Let K be an algebraically closed field of characteristic zero and let $h = (1, h_1, h_2, h_3, 1)$ be an O-sequence with $h_2 \geq h_3$. Then every local Gorenstein K-algebra with Hilbert function h is necessarily canonically graded if and only if $h_1 = h_3$ and $h_2 = \binom{h_1+1}{2}$.*

Proof. The assertion is clear for $h_1 = 1$. Hence we assume $h_1 > 1$. The "if" part of the theorem follows from [15, Theorem 3.1]. We prove the converse, that is we show that if $h_3 < h_1$ or $h_2 < \binom{h_1+1}{2}$, then there exists a polynomial G of degree 4 such that A_G has the Hilbert function h and it is not canonically graded. If $h_3 < h_1$, then the result is clear by Theorem 3.1(b). Hence we assume that $h_3 = h_1$. For simplicity in the notation we put $h_1 := n$ and $h_2 := m$.

First we prove the assertion for $n \leq 3$. We define

$$F = \begin{cases} x_1^3 x_2 & \text{if } n = m = 2 \\ x_1^4 + x_2^4 + x_2^3 x_3 & \text{if } n = 3 \text{ and } m = 3 \\ x_1^4 + x_2^4 + x_2^3 x_3 + x_1^3 x_2 & \text{if } n = 3 \text{ and } m = 4 \\ x_1^4 + x_2^4 + x_2^3 x_3 + x_1^3 x_2 + x_1 x_2^2 x_3 & \text{if } n = 3 \text{ and } m = 5. \end{cases}$$

Let $G = F + x_n^3$. It is easy to check that A_G has the Hilbert function h. We claim that A_G is not canonically graded. Suppose that A_G is canonically graded. Then $A_F \cong gr_{\mathfrak{m}}(A) \cong A_G$. Let $\varphi : A_F \longrightarrow A_G$ be a K-algebra automorphism. Since $x_n^2 \circ F = 0$, $x_n^2 \in \operatorname{Ann}_R(F)$. This implies that $\varphi(x_n)^2 \in \operatorname{Ann}_R(G)$ and hence $\varphi(x_n)^2 \circ G = 0$. For $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, let $|\underline{i}| = i_1 + \cdots + i_n$. Suppose

$$\varphi(x_n) = u_1 x_1 + \cdots + u_n x_n + \sum_{\underline{i} \in \mathbb{N}^n,\ |\underline{i}| \geq 2} a_{\underline{i}} x^{\underline{i}}$$

where u_i for $i = 1, \ldots, n$ and $a_{\underline{i}} \in K$ for all $\underline{i} \in \mathbb{N}^n$ such that $|\underline{i}| \geq 2$. Comparing the coefficients of the monomials of degree ≤ 2 in $\varphi(x_n)^2 \circ G = 0$, it is easy to verify that $u_1 = \cdots = u_n = 0$. This implies that $\varphi(x_n)$ has no linear terms and thus φ is not an automorphism, a contradiction.

Suppose $n > 3$. First we define a homogeneous polynomial $F \in P$ of degree 4 such that A_F has the Hilbert function h and x_n^2 does not divide any monomial in F (in other words, if $x^{\underline{i}}$ is a monomial that occurs in F with nonzero coefficient, then $i_n \leq 1$).

Let T be a monomial basis of P_2. We split the set $T \setminus \{x_n^2\}$ into a disjoint union of monomials as follows. We set

$$p_i = \begin{cases} x_i^2 & \text{for } 1 \leq i \leq n-1 \\ x_2 x_n & \text{for } i = n \\ x_{i-n} x_{i+1-n} & \text{for } n+1 \leq i < 2n \\ x_1 x_n & \text{for } i = 2n. \end{cases}$$

Let $E = \{p_i : 1 \leq i \leq n\}, B = \{p_i : n+1 \leq i \leq 2n\}, C = \{x_i x_j : 1 \leq i < j < n \text{ such that } j - i > 1\}$ and $D := \{x_i x_n : 3 \leq i \leq n-2\}$. Then

$$T \setminus \{x_n^2\} = E \dot{\bigcup} B \dot{\bigcup} C \dot{\bigcup} D.$$

Denote by $|\cdot|$ the cardinality, then $|C| = \binom{n+1}{2} - 2n - (n-4) - 1$ and $|D| = n - 4$. Hence we write $C = \{p_i : 2n < i \leq \binom{n+1}{2} - (n-4) - 1\}$ and $D = \{p_i : \binom{n+1}{2} - (n-4) - 1 < i \leq \binom{n+1}{2} - 1\}$. We set

$$g_i = \begin{cases} x_i^4 & \text{for } 1 \leq i \leq n-1 \\ x_2^3 x_n & \text{for } i = n \\ x_{i-n}^2 p_i & \text{for } n+1 \leq i < 2n \text{ and } i \neq n+2 \\ x_2^2 x_3^2 & \text{for } i = n+2 \\ x_1 x_2^2 x_n & \text{for } i = 2n \\ p_i^2 & \text{for } 2n < i \leq \binom{n+1}{2} - (n-4) - 1 \\ \frac{x_2}{x_n} p_i^2 & \text{for } \binom{n+1}{2} - (n-4) - 1 < i < \binom{n+1}{2}. \end{cases}$$

Define

$$F = \sum_{i=1}^{m} g_i.$$

Since $m \geq n$, $\dim_K(\langle F \rangle_R)_i = n$ for $i = 1, 3$. Also, $\dim_K(\langle F \rangle_R)_2 = \dim_K\{p_i : 1 \leq i \leq m\} = m$. Hence A_F has the Hilbert function h.

Let $G = F + x_n^3$. We prove that A_G is not canonically graded. Suppose that A_G is canonically graded. Then, as before, $A_G \cong A_F$. Let $\varphi : A_F \longrightarrow A_G$ be a K-algebra automorphism. Since F does not contain a monomial that is multiple of x_n^2, $x_n^2 \circ F = 0$ and hence $x_n^2 \in \mathrm{Ann}_R(F)$ which implies that $\varphi(x_n)^2 \in \mathrm{Ann}_R(G)$. Let

$$\varphi(x_n) = u_1 x_1 + \cdots + u_n x_n + \sum_{\underline{i} \in \mathbb{N}^n,\ |\underline{i}| \geq 2} a_{\underline{i}} x^{\underline{i}}$$

where u_i for $i = 1, \ldots, n$ and $a_{\underline{i}} \in K$ for all $\underline{i} \in \mathbb{N}^n$ such that $|\underline{i}| \geq 2$. We claim that $u_1 = \cdots = u_{n-1} = 0$.

Case 1: $m = n$. Comparing the coefficients of $x_1^2, x_2 x_n, x_3^2, \ldots, x_{n-1}^2$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

Case 2: $m = n+1$ OR $m = n+2$. Comparing the coefficients of $x_1 x_2, x_2 x_n, x_3^2, \ldots, x_{n-1}^2$ in $\varphi(x_n)^2 \circ G = 0$, to get $u_1 = \cdots = u_{n-1} = 0$.

Case 3: $n+2 < m < 2n$. Comparing the coefficients of $x_1 x_2, x_2 x_n, x_3 x_4, \ldots, x_{m-n} x_{m-n+1}, x_{m-n+1}^2, x_{m-n+2}^2 \ldots, x_{n-1}^2$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

Case 4: $m \geq 2n$. Comparing the coefficients of $x_1 x_2, x_1 x_n, x_3 x_4, \ldots, x_{n-1} x_n$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

This proves the claim. Now, comparing the coefficients of x_n in $\varphi(x_n)^2 \circ G = 0$, we get $u_n = 0$ (since F does not contain a monomial divisible by x_n^2). This implies that $\varphi(x_n)$ has no linear terms and hence φ is not an automorphism, a contradiction. □

We expect that the Theorem 4.1 holds true without the assumption $h_2 \geq h_3$. The problem is that, as far as we know, the admissible Gorenstein non-unimodal h-vectors are not classified even if $s = 4$. However, starting from an example by Stanley, we are able to construct a non-canonically graded Gorenstein K-algebra with (non-unimodal) h-vector $(1, 13, 12, 13, 1)$.

Corollary 4.2. *Assume that K is an algebraically closed field of characteristic zero. Let $h = (1, h_1, h_2, h_3, 1)$ where $h_3 = h_1 \leq 13$ be a Gorenstein O-sequence. Then every Gorenstein K-algebra with the Hilbert function h is necessarily canonically graded if and only if $h_2 = \binom{h_1+1}{2}$.*

Proof. If a local Gorenstein K-algebra A has Hilbert function $h = (1, h_1, h_2, h_1, 1)$, then by considering Q-decomposition of $gr_{\mathfrak{m}}(A)$ we conclude that $gr_{\mathfrak{m}}(A) \cong Q(0)$. This implies that h is also the Hilbert function of a graded Gorenstein K-algebra. By [30, Theorem 3.2] if $h_1 \leq 12$, then the Hilbert function of a graded Gorenstein K-algebra is unimodal. Hence by Theorem 4.1 the result follows.

If $h_1 = 13$ and h is unimodal, then the assertion follows from Theorem 4.1. Now, by [30, Theorem 3.2] the only non-unimodal graded Gorenstein O-sequence with $h_1 = 13$ is $h = (1, 13, 12, 13, 1)$. In this case we write $P = [x_1, \ldots, x_{10}, x, y, z]$. Let

$$F = \sum_{i=1}^{10} x_i \mu_i,$$

where $\mu = \{x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3\} = \{\mu_1, \ldots, \mu_{10}\}$. Let $G = F + x_1^3 + \cdots + x_{10}^3$. Then A_G has the Hilbert function h. We claim that A_G is not canonically graded. Suppose A_G is canonically graded. Then $A_G \cong A_F$. Let $\varphi : A_F \longrightarrow A_G$ be a K-algebra automorphism. Since $x_1^2 \in \mathrm{Ann}_R(F)$, $\varphi(x_1)^2 \in \mathrm{Ann}_R(G)$. Let

$$\varphi(x_1) = u_1x_1 + \cdots + u_{10}x_{10} + u_{11}x + u_{12}y + u_{13}z + \text{non-linear terms in } x_1, \ldots, x_{10}, x, y, z$$

where $u_i \in K$ for $i = 1, \ldots, 13$. Comparing the coefficients of $x_1x, x_7y, x_{10}z$ in $\varphi(x_1)^2 \circ G = 0$, we get $u_{11} = u_{12} = u_{13} = 0$. Now, comparing the coefficients of $x_1, \ldots, x_{10}$ in $\varphi(x_1)^2 \circ G = 0$, we get $u_1 = \cdots = u_{10} = 0$. This implies that $\varphi(x_1)$ has no linear terms and thus φ is not an automorphism, a contradiction. $\square$

We remark that the "only if" part of Theorem 4.1 holds for any arbitrary field K. An analogue of Theorem 4.1 is no longer true for $s = 5$. Notice that in [15, Example 3.4] the authors gave an example of a non-canonically graded Gorenstein compressed algebra of socle degree 5 and codimension 2. However, by a slight modification of the dual polynomial F in the proof of Theorem 4.1 we can show that for a restricted set of local Gorenstein sequences of socle degree five, there exist non-canonically graded Gorenstein algebras.

Theorem 4.3. *For every $1 < h_1 \leq h_2 \leq \binom{h_1+1}{2}$ there exists a Gorenstein K-algebra with Hilbert function $h = (1, h_1, h_2, h_2, h_1, 1)$ which is not canonically graded.*

Proof. For simplicity in the notation we put $h_1 := n$ and $h_2 := m$. We define

$$F = \begin{cases} x_1^4 x_2 & \text{if } n = m = 2 \\ x_1^3 x_2^2 & \text{if } n = 2 \text{ and } m = 3 \\ x_1^5 + x_2^5 + x_2^4 x_3 & \text{if } n = 3 \text{ and } m = 3 \\ x_1^5 + x_2^5 + x_2^4 x_3 + x_1^4 x_2 & \text{if } n = 3 \text{ and } m = 4 \\ x_1^5 + x_2^5 + x_2^4 x_3 + x_1^4 x_2 + x_1 x_2^3 x_3 & \text{if } n = 3 \text{ and } m = 5 \\ x_1^5 + x_2^5 + x_2^4 x_3 + x_1^4 x_2 + x_1 x_2^3 x_3 + x_1^2 x_2 x_3^2 & \text{if } n = 3 \text{ and } m = 6. \end{cases}$$

Then A_F has the Hilbert function h. Let

$$G = \begin{cases} F + x_n^3 \text{ if } A_F \text{ is not compressed} \\ F + x_n^4 \text{ if } A_F \text{ is compressed.} \end{cases}$$

Then A_G also has the Hilbert function h. By a similar argument as in the proof of Theorem 4.1 it can be verified that A_G is not canonically graded.

Let $n > 3$ and p_i be as in the proof of Theorem 4.1. We modify g_i as

$$g_i = \begin{cases} x_i^5 & \text{for } 1 \leq i \leq n-1 \\ x_2^4 x_n & \text{for } i = n \\ x_{i-n}^3 p_i & \text{for } n+1 \leq i < 2n \text{ and } i \neq n+2 \\ x_2^3 x_3^2 & \text{for } i = n+2 \\ x_1 x_2^3 x_n & \text{for } i = 2n \\ x_j^3 x_k^2 & \text{for } 2n < i \leq \binom{n+1}{2} - (n-4) - 1 \text{ where } p_i = x_j x_k \text{ with } j < k \\ \frac{x_2^2}{x_n} p_i^2 & \text{for } \binom{n+1}{2} - (n-4) - 1 < i < \binom{n+1}{2}. \end{cases}$$

Define

$$F = \begin{cases} \sum_{i=1}^{m} g_i & \text{if } m < \binom{n+1}{2} \\ \sum_{i=1}^{\binom{n+1}{2}-1} g_i + x_1 x_2 x_3 x_n^2 & \text{if } m = \binom{n+1}{2}. \end{cases}$$

Then A_F has the Hilbert function h. We define

$$G = \begin{cases} F + x_n^3 & \text{if } m < \binom{n+1}{2} \\ F + x_n^4 & \text{if } m = \binom{n+1}{2}. \end{cases}$$

We claim that A_G is not canonically graded. Suppose that A_G is canonically graded. Then, as before, $A_G \cong A_F$. Let $\varphi : A_F \longrightarrow A_G$ be a K-algebra automorphism. Let

$$\varphi(x_n) = u_1 x_1 + \cdots + u_n x_n + \sum_{\underline{i} \in \mathbb{N}^n,\ |\underline{i}| \geq 2} a_{\underline{i}} x^{\underline{i}}$$

where u_i for $i = 1, \dots, n$ and $a_{\underline{i}} \in K$ for all $\underline{i} \in \mathbb{N}^n$ such that $|\underline{i}| \geq 2$. First assume that $m < \binom{n+1}{2}$. Then $x_n^2 \circ F = 0$ and hence $\varphi(x_n)^2 \circ G = 0$. We first show that $u_1 = \cdots = u_{n-1} = 0$.

Case 1: $m = n$. Comparing the coefficients of $x_1^3, x_2^2 x_n, x_3^3, \dots, x_{n-1}^3$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

Case 2: $m = n + 1$ OR $m = n + 2$. Comparing the coefficients of $x_1^2 x_2, x_2^2 x_n, x_3^3, \dots, x_{n-1}^3$ in $\varphi(x_n)^2 \circ G = 0$, to get $u_1 = \cdots = u_{n-1} = 0$.

Case 3: $n + 2 < m < 2n$. Comparing the coefficients of $x_1^2 x_2, x_2^2 x_n, x_3^2 x_4, \dots, x_{m-n}^2 x_{m-n+1}, x_{m-n+1}^3, x_{m-n+2}^3 \dots, x_{n-1}^3$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

Case 4: $2n \leq m < \binom{n+1}{2}$. Comparing the coefficients of $x_1^2 x_2, x_1 x_2 x_n, x_3^2 x_4, \dots, x_{n-1}^2 x_n$ in $\varphi(x_n)^2 \circ G = 0$, we get $u_1 = \cdots = u_{n-1} = 0$.

Thus

$$\varphi(x_n) = u_n x_n + \sum_{1 \leq i \leq j \leq n} a_{i,j} x_i x_j + \sum_{\underline{i} \in \mathbb{N}^n,\ |\underline{i}| \geq 3} a_{\underline{i}} x^{\underline{i}}.$$

Now to show that $u_n = 0$ we argue as follows:

Case 1: $n \leq m < 2n - 1$. Comparing the coefficients of x_2^2 and x_n in $\varphi(x_n)^2 \circ G = 0$, we get $u_n a_{2,2} = u_n^2 + (a_{2,2})^2 = 0$. Hence $u_n = 0$.

Case 2: $m = 2n - 1$. Comparing the coefficients of x_2^2, x_{n-1}^2 and x_n in $\varphi(x_n)^2 \circ G = 0$, we get $u_n a_{2,2} = u_n a_{n-1,n-1} = u_n^2 + (a_{2,2})^2 + (a_{n-1,n-1})^2 = 0$. Hence $u_n = 0$.

Case 3: $2n \leq m < \binom{n+1}{2} - (n - 4) - 1$. Comparing the coefficients of $x_1 x_2, x_{n-1}^2$ and x_n in $\varphi(x_n)^2 \circ G = 0$, we get $u_n a_{2,2} = u_n a_{n-1,n-1} = u_n^2 + (a_{2,2})^2 + (a_{n-1,n-1})^2 + 2a_{1,2} a_{2,2} = 0$. Hence $u_n = 0$.

Case 4: $\binom{n+1}{2} - (n - 4) \leq m < \binom{n+1}{2}$. Comparing the coefficients of $x_1 x_2, x_{n-1}^2, x_2 x_j$ $(3 \leq j \leq m - \binom{n+1}{2} + (n - 4) + 3)$, we get $u_n a_{2,2} = u_n a_{n-1,n-1} = u_n a_{2,j} = 0$. Suppose $u_n \neq 0$. Then $a_{2,2} = a_{n-1,n-1} = \cdots = a_{2,j} = 0$. Now by comparing the coefficient of x_n we conclude that $u_n = 0$.

This implies that $\varphi(x_n)$ has no linear terms and hence φ is not an automorphism, a contradiction.

Suppose $m = \binom{n+1}{2}$ and $n > 4$. Then $x_n^3 \circ F = 0$. Hence $\varphi(x_n)^3 \circ G = 0$. Therefore comparing the coefficients of $x_3 x_4, x_4 x_5, \dots, x_{n-1} x_n$ in $\varphi(x_n)^3 \circ G = 0$, we get $u_3 = \cdots = u_{n-1} = 0$. Now, comparing the coefficient of $x_1 x_n$, we get $u_2 = 0$. Hence comparing the coefficient of $x_1 x_2$ we conclude that $u_1 = 0$ which on comparing the coefficient of x_n gives that $u_n = 0$. Thus A_G is not canonically graded. By a similar argument it can be verified that A_G is not canonically graded also for $n = 4$. □

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