Maximal Hilbert Functions

M. E. Rossi*
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16132 Genova, Italy
e-mail: rossim@dima.unige.it

G. Valla†
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16132 Genova, Italy
e-mail: valla@dima.unige.it

W. V. Vasconcelos‡
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd
Piscataway, New Jersey 08854-8019
e-mail: vasconce@math.rutgers.edu

1 Introduction

A standard graded algebra is a graded ring $G = \bigoplus_{n\geq 0} G_n$, finitely generated over G_0 by its elements of degree 1, $G = G_0[G_1]$. When G_0 is an Artinian local ring, we denote the Hilbert function of G by $H_G(n) = \ell(G_n)$, and its Hilbert–Poincaré series by

$$P_G(t) = \sum_{n\geq 0} H_G(n)t^n.$$

This is a rational function $P_G(t) = \frac{h(t)}{(1-t)^d}$, where $h(1) = \deg(G)$, $d = \dim G$ are respectively the *degree* or *multiplicity* of G and its dimension. We note by $\mathcal{P}_G(t)$ the corresponding Hilbert polynomial

$$\mathcal{P}_G(t) = e_0\binom{t+d-1}{d-1} - e_1\binom{t+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}.$$

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There are also iterated versions of these functions and we will make use of $H_G^1(n) = \sum_{i \leq n} H_G(i)$ and the corresponding Hilbert series $P_G^1(t) = \frac{P_G(t)}{1-t}$ and Hilbert polynomial $\mathcal{P}_G^1(t)$.

There is a great deal of interest on the structure of the set $\mathcal{H}$ of these functions. Our approach to them takes into account the underlying ordering:

$$P_G(t) \geq P_{G'}(t) \Leftrightarrow H_G(n) \geq H_{G'}(n), \quad \forall n.$$

Thus for a given a condition $\mathfrak{C}$ on Hilbert functions, we define $H(\mathfrak{C})$ to be the partially ordered set of all Hilbert functions satisfying $\mathfrak{C}$. Two of the main questions are to search for the extremal members of $H(\mathfrak{C})$ to ascertain when it is finite. Among these sets we will consider $H(d, e_0)$, defined by all algebras with a given dimension d and multiplicity e_0, and its subset $H(d, e_0, e_1)$.

One of the most significant classes of these algebras arise as associated graded rings of filtrations of Noetherian local rings, particularly of the following kind. Let $(R, \mathfrak{m})$ be a Noetherian local and let I be an $\mathfrak{m}$–primary ideal. The Hilbert function of the associated graded ring

$$\mathrm{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$$

is significant for its role as a control of the blowup process of $\mathrm{Spec}(R)$ along the subvariety $V(I)$. A challenging problem consists in relating $P_I(t) = P_{\mathrm{gr}_I(R)}(t)$ directly to R and I as $\mathrm{gr}_I(R)$ may fail to inherit some of the arithmetical (e.g. Cohen–Macaulayness) properties of R.

We will now describe some of our results. Each deals with one of the general aspects mentioned above of the set of Hilbert functions of algebras of a fixed dimension. Section 2 deals with general bounds for the set $H(d, e_0)$. It is centered on estimates of the following kind:

Theorem 2.3 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$–primary ideal in R. If $J = (x_1, \ldots, x_d)$ is a system of parameters in I, then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

When R is Cohen–Macaulay, $\ell(R/J) = e_0(I)$, which gives the formula above a convenient expression. In this case it shows that $H(d, e_0)$ has a unique maximal element.

The next section considers $H(d, e_0, e_1)$. One difficulty here is that there may exist hidden relationships between d, e_0 and e_1. After a general comparison between e_0 and e_P1, we turn to an examination of $H(d, e_0, e_1)$, but restricted to tangent cones of Cohen–Macaulay local rings. The maximal Hilbert function in $H(d, e_0, e_1)$ (Theorem 3.2) is then

$$P_R^1(t) \leq \frac{1 + bt + (e - b - 2)t^2 + t^{\delta+2}}{(1-t)^{d+1}},$$

where b and δ are certain functions of e and e_1.

Finally, in Section 4 we give a bound on the number of Hilbert functions that have a given dimension and given extended multiplicity. This notion is however a strong requirement in comparison to the ordinary multiplicity. In counterpoint it provides effective bounds on the corresponding Hilbert coefficients:

Theorem 4.3 *Let* $\mathrm{Deg}(\cdot)$ *be an extended degree function on graded algebras. Given two positive integers* A, d*, there exist only a finite number of Hilbert functions associated to standard graded algebras* G *over Artinian rings such that* $\dim G = d$ *and* $\mathrm{Deg}(G) \leq A$*. Furthermore, there are integers* b_i *dependent only on* $\dim G$ *such that* $e_i(G) \leq b_i \mathrm{Deg}(G)^{i+1}$*.*

2 Boundedness of Hilbert Functions

In this section we develop general bounds for the Hilbert functions of algebras of a given dimension d and a given multiplicity.

The algebras considered throughout will be either Noetherian local rings or graded algebras $G = \bigoplus_{n \geq 0} G_n$ where G_0 is an Artinian local ring. For simplicity only of expression we will drop 'Noetherian'. It will be harmlessly assumed that the residue fields of these local rings are infinite. This is achieved, without changing the Hilbert functions, in the usual manner: replacing the local ring $(R, \mathfrak{m})$ by $R[X]_{\mathfrak{m}R[X]}$, where X is an indeterminate over R.

Let $(R, \mathfrak{m})$ be a local ring of dimension d and let I be an $\mathfrak{m}-$primary ideal in R. We denote by

$$H_I(n) = \ell(I^n / I^{n+1})$$

the Hilbert function of I. In the case $I = \mathfrak{m}$, we write $H_R(n)$. If we let

$$H_I^1(n) = \sum_{j=0}^{n} H_I(j) = \ell(R/I^{n+1})$$

then $H_I^1(n) - H_I^1(n-1) = H_I(n)$.

Let $P_I(t) = \sum_{n \geq 0} H_I(n) t^n$ be the Hilbert series of I, then

$$P_I^1(t) = \sum_{n \geq 0} H_I^1(n) t^n = \frac{P_I(t)}{(1-t)}.$$

Lemma 2.1 (Singh's inequality) *Let $(R, \mathfrak{m})$ be a local ring and let I be an $\mathfrak{m}-$primary ideal in R. If $x \in I$ and $\overline{I} = I/(x)$, then*

$$H_I(n) = H^1_{\overline{I}}(n) - \ell(I^{n+1} : x/I^n)$$

for every $n \geq 0$.

Proof. We let $\overline{R} = R/xR$, then from the exact sequence

$$0 \to (I^{n+1} : x)/I^n \longrightarrow R/I^n \longrightarrow R/I^{n+1} \longrightarrow \overline{R}/\overline{I}^{n+1} \to 0,$$

induced by multiplication by x we get the desired equality. □

Proposition 2.2 *Let $(R, \mathfrak{m})$ be a local ring and let I be an $\mathfrak{m}-$primary ideal in R. If $x \in I$ and $\overline{I} = I/(x)$, then*

$$P_I(t) \leq \frac{P_{\overline{I}}(t)}{(1-t)}$$

Proof. Since

$$\frac{P_{\overline{I}}(t)}{(1-t)} = P^1_{\overline{I}}(t) = \sum_{n\geq 0} H^1_{\overline{I}}(n)t^n,$$

the conclusion follows by Singh's inequality. □

Theorem 2.3 *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}-$primary ideal in R. If $J = (x_1, \ldots, x_d)$ is a system of parameters in I, then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

Proof. We induct on d. Let $d = 1$ and $J = (x)$ where x is a parameter in I. Since

$$\frac{\ell(R/I) + \ell(I/J)t}{(1-t)} = \frac{\ell(R/I) + (\ell(R/J) - \ell(R/I))t}{(1-t)} =$$

$$\ell(R/I) + \ell(R/J)t + \ell(R/J)t^2 + \cdots + \ell(R/J)t^n + \cdots$$

and $H_I(0) = \ell(R/I)$, we need only to prove that for every $n \geq 1$ $H_I(n) \leq \ell(R/xR)$. We remark that

$$R \supseteq I^n \supseteq I^{n+1} \supseteq xI^n$$

$$R \supseteq xR \supseteq xI^n,$$

so that

$$\ell(R/xR) + \ell(xR/xI^n) = \ell(R/I^n) + \ell(I^n/xI^n)$$

Since $\ell(R/I^n) \geq \ell(xR/xI^n)$, we get

$$\ell(R/xR) \geq \ell(I^n/xI^n) = H_I(n) + \ell(I^{n+1}/xI^n) \geq H_I(n).$$

Suppose $d \geq 2$, and let x_1 be a parameter in I. We denote by $\overline{R} = R/x_1R$, $\overline{I} = I/x_1R$, $\overline{J} = J/x_1R$. Then $\overline{I}$ is a primary ideal in the local ring $\overline{R}$ which has $\dim \overline{R} = d-1$. By induction we have

$$P_{\overline{I}}(t) \leq \frac{\ell(\overline{R}/\overline{I}) + \ell(\overline{I}/\overline{J})t}{(1-t)^{d-1}} = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^{d-1}}.$$

Since $\frac{1}{(1-t)} \geq 0$, we get

$$\frac{P_{\overline{I}}(t)}{(1-t)} \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

The assertion follows since by Proposition 2.2, $P_I(t) \leq \frac{P_{\overline{I}}(t)}{(1-t)}$. □

Nearly the same treatment applies to standard graded algebras, which we state for later reference.

Proposition 2.4 *Let $G = \bigoplus_{n\geq 0} G_n$ be a standard graded algebra over the Artinian local ring G_0. Suppose $\dim G = d \geq 1$ and $h \in G_1$. Setting $\overline{G} = G/(h)$, one has $P_G(t) \leq \frac{P_{\overline{G}(t)}}{1-t}$.*

Corollary 2.5 *Let $(G_0, \mathfrak{m})$ be an Artinian local ring and let $G = \bigoplus_{n\geq 0} G_n$ be a standard algebra of dimension $d \geq 1$. If J is the ideal generated by an homogeneous system of parameters in G, then*

$$P_G(t) \leq \frac{\ell(G_0) + (\ell(G/J) - \ell(G_0))t}{(1-t)^d}.$$

Proof. Proposition 2.4 could be used to reduce to the dimension 1 case, but instead we derive the assertion from Theorem 2.3.

Set $I = G_+ = \bigoplus_{n\geq 1} G_n$, and note that I is primary for the irrelevant maximal ideal M of G, $\mathrm{gr}_I(G) \simeq G$, and that J is generated by a system of parameters. Note also that the associated graded rings and lengths are not changed if G or the localization G_M are considered. Theorem 2.3 can now be applied directly. □

These results show that the Hilbert function of I is bounded by the rational function

$$\frac{\ell(R/I) + (\ell(R/J) - \ell(R/I))t}{(1-t)^d}$$

for any ideal J generated by a system of parameters that yields minimal length for R/J. It is however not clear which number this turns out to be except when R is Cohen–Macaulay when we have:

Corollary 2.6 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and multiplicity e. If I is an $\mathfrak{m}-$primary ideal in R and $J = (x_1, \ldots, x_d)$ is the ideal generated by a superficial sequence for I, then*

$$P_I(t) \leq \frac{\ell(R/I) + (e - \ell(R/I))t}{(1-t)^d}. \tag{1}$$

Proof. Under our assumptions, if $x_1, \ldots, x_d$ is a superficial sequence for I, then it is a system of parameters in I and $\ell(R/J) = e(R/J) = e$. □

This formula, in case $I = \mathfrak{m}$, was obtained in [2] through different means. We shall now explain the difference between the two sides of (1). Let J be a minimal reduction of the ideal I. Since R has an infinite residue field, J is generated by a regular sequence. Now consider the construction of the Sally module of I relative to J: it is simply the $R[Jt]$–module $S_J(I)$ defined by the natural exact sequence

$$0 \to I \cdot R[Jt] \longrightarrow I \cdot R[It] \longrightarrow S_J(I) \to 0.$$

$S_J(I) = 0$ exactly when $I^2 = JI$, that is when I has so–called *minimal multiplicity*. In all the other cases $\dim S_J(I) = \dim R = d$.

A calculation in [15] shows that

$$P_I(t) = \frac{\ell(R/I) + (e(I) - \ell(R/I))t}{(1-t)^d} - (1-t)P_{S_J(I)}(t). \tag{2}$$

In particular this formula gives the following equivalence of the inequality (1):

Corollary 2.7 *The Hilbert function $H_S(t)$ of $S_J(I)$ is non–decreasing.*

This answers a question raised in [14, p. 385].

Remark 2.8 One application of this formula is to employ the technique of [2] to obtain estimates for the reduction number of the ideal I. In other words, to find find a minimal reduction L of I and an integer r for which an equality $I^{r+1} = LI^r$ holds.

For simplicity of notation, set $a := \ell(R/I)$ and $b := \ell(I/J)$. From the inequality of Hilbert functions

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d},$$

we have that for each positive integer n,

$$\ell(I^n/\mathfrak{m}I^n) \leq \ell(I^n/I^{n+1}) \leq a\binom{n+d-1}{d-1} + b\binom{n+d-2}{d-1}.$$

According to [3], if for some integer n we bound the right hand side of this inequality by $\binom{n+d}{d}$, we have found a reduction L of I with reduction number $< n$. This is easy to work out since the inequality is quadratic:

$$(n+d)(n+d-1) > ad(n+d-1) + bdn,$$

will be satisfied for (set $c = a + b = \ell(R/J)$):

$$r \leq cd - 2d + 1 + \sqrt{(a-1)(d-1)d}.$$

The inequality $r \leq cd - 2d + 1$ is the bound in [2] for the Cohen–Macaulay case, so that $\sqrt{(a-1)(d-1)d}$ is a penalty for the lack of that condition.

3 Maximal Functions of $H(d, e, e_1)$

Throughout this section $(R, \mathfrak{m})$ will be a Cohen–Macaulay local ring of dimension $d > 0$. We denote by e and e_1 the first two coefficients of the Hilbert polynomial of R. We will now consider the set $H(d, e, e_1)$ of the Hilbert functions defined by these parameters.

A first difficulty presents itself by the fact that e and e_1 are loosely related ([8]):

$$e - 1 \leq e_1 \leq \binom{e-1}{2}.$$

Actually there are more strict relations when Hilbert functions of primary ideals are considered. Here is an instance:

Proposition 3.1 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring, let I be an $\mathfrak{m}$–primary ideal and let $e(I)$ and $e_1(I)$ be the first two coefficients of the Hilbert polynomial of I. If $e(I) \neq e(\mathfrak{m})$ then*

$$e_1(I) \leq \binom{e(I)-2}{2}.$$

Proof. The condition $e(I) \neq e(\mathfrak{m})$ means, by the theorem of Rees (see [9]), that $\mathfrak{m}$ is not the integral closure of I. This implies that for each positive integer n, $I^{n+1} \neq \mathfrak{m}I^n$, and therefore $\ell(I^n/I^{n+1}) < \ell(I^n/\mathfrak{m}I^n)$.

We may assume $\dim R = 1$. If (x) is a minimal reduction of I, the Hilbert function of I can be written

$$H_I(n) = \ell(I^n/I^{n+1}) = e(I) - \ell(I^{n+1}/xI^n),$$

$I^{r+1} = xI^r$. We claim that for all $n \leq r$, $\ell(I^n/I^{n+1}) \geq n + 2$. Indeed otherwise we would have $\ell(I^n/\mathfrak{m}I^n) \leq n$, which by the main theorem of [3] would lead to an equality $I^n = yI^{n-1}$, contradicting the definition of r. This means that we have

$$\begin{aligned} e_1(I) &= \sum_{n=0}^{r}(e(I) - \ell(I^n/I^{n+1})) \\ &\leq e(I) - \ell(R/I) + \sum_{n=1}^{r}(e(I) - (n+2)) \\ &= e(I) - \ell(R/I) + r(e(I) - 2) - \binom{r+1}{2}. \end{aligned}$$

Since $r \leq e(I) - 1$ (as we may assume that $e(I) \geq 3$ as otherwise $I = (x)$), substituting we have the desired inequality. □

In case $I = \mathfrak{m}$ there is another relationship between $e = e(\mathfrak{m})$ and $e_1 = e_1(\mathfrak{m})$ that improves the basic inequality of Corollary 2.6. the following inequalities hold (see [6] and [4])

$$2e - h - 2 \leq e_1 \leq \binom{e}{2} - \binom{h}{2}$$

We define

$$\begin{aligned} b &= \max\{n : \binom{n}{2} \leq \binom{e}{2} - e_1\} \\ \delta &= e_1 - 2e + b + 2. \end{aligned}$$

Since $\binom{h}{2} \leq \binom{e}{2} - e_1$, we have $b \geq h$. In particular $\delta = e_1 - 2e + b + 2 \geq e_1 - 2e + h + 2 \geq 0$.

Theorem 3.2 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$. We denote by e and e_1 the first two coefficients of the Hilbert polynomial of R, and b and δ the integers defined above. Then*

$$P_R^1(t) \leq \frac{1 + bt + (e - b - 2)t^2 + t^{\delta+2}}{(1-t)^{d+1}}.$$

Proof. We induct on $d \geq 1$. Suppose $d = 1$ and we prove that

$$\begin{aligned} P_R^1(t) &\leq \frac{1 + bt + (e - b - 2)t^2 + t^{\delta+2}}{(1-t)^2} \\ &= 1 + \sum_{n \geq 1}[(n-1)e + b + 2 - \min\{n-1, \delta\}]t^n. \end{aligned}$$

We recall that if y is a superficial element for R, then for every n

$$H_R(n) = e - v_n$$

where $v_n = \ell(m^{n+1}/ym^n)$. In particular $v_0 = e - 1$, $v_1 = e - h - 1$. Moreover $v_n \geq 0$ for every n, and if $v_j = 0$ for some integer j, then $v_n = 0$ for every $n \geq j$.

We have $H_R^1(0) = 1$, $H_R^1(1) = h + 2 \leq b + 2$ and for every $n \geq 2$

$$H_R^1(n) = (n+1)e - \sum_{j=0}^{n} v_j = (n-1)e + h + 2 - \sum_{j=2}^{n} v_j$$

We have to prove that for every $n \geq 2$

$$h - \sum_{j=2}^{n} v_j \leq b - \min\{n-1, \delta\} = b - \min\{n-1, e_1 - 2e + b + 2\}$$

We recall that

$$e_1 = \sum_{j \geq 0} v_j = 2e - h - 2 + \sum_{j \geq 2} v_j,$$

and we remark that

$$h - \sum_{j=2}^{n} v_j \leq b - \min\{n-1, e_1 - 2e + b + 2\} = b - \min\{n-1, b - h + \sum_{j \geq 2} v_j\},$$

or equivalently

$$b - h + \sum_{j=2}^{n} v_j \geq \min\{n-1, b - h + \sum_{j \geq 2} v_j\}.$$

In fact if $v_n = 0$, then $\sum_{j=2}^{n} v_j = \sum_{j \geq 2} v_j \geq n-1$. Otherwise $\min\{n-1, b-h+\sum_{j \geq 2} v_j\} = n - 1 \leq b - h + \sum_{j=2}^{n} v_j$.

Suppose now $d \geq 2$ and let x be a superficial element in R, then $\overline{R} = R/xR$ is a local Cohen-Macaulay ring of dimension $d-1$. In particular

$$e(R) = e(\overline{R}), \quad e_1(R) = e_1(\overline{R}), \quad h(R) = h(\overline{R}).$$

Then

$$P_{\overline{R}}(t) \leq \frac{1 + bt + (e-b-2)t^2 + t^{\delta+2}}{(1-t)^d}$$

and the assertion follows by Proposition 2.2. □

Corollary 3.3 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. We denote by e and e_1 the first two coefficients of the Hilbert polynomial. Then*

$$P_R(t) \leq \frac{1 + bt + (e-b-2)t^2 + t^{\delta+2}}{(1-t)^d}.$$

Proof. Let x be a superficial element for R, then R/xR is a local Cohen-Macaulay ring of dimension $d-1 \geq 1$, in particular $e(R) = e(R/xR)$, $e_1(R) = e_1(R/xR)$, $h(R) = h(R/xR)$ and, by Proposition 2.2, we have

$$P_R(t) \leq P^1_{R/xR}(t)$$

The assertion follows by Theorem 3.2. □

Remark 3.4 If we apply Corollary 2.6 with $I = \mathfrak{m}$, we obtain

$$P_R(t) \leq \frac{1+(e-1)t}{(1-t)^d}$$

We remark that, if $d \geq 2$, Corollary 3.3 improves this bound. To prove this, note that $\frac{1}{(1-t)^{d-2}} \geq 0$, so that we only need to prove that

$$\frac{1+bt+(e-b-2)t^2+t^{\delta+2}}{(1-t)^2} \leq \frac{1+(e-1)t}{(1-t)^2}$$

that is

$$(n-1)e+b+2-\min\{n-1,\delta\} \leq ne+1$$

or equivalently

$$e-b-1+\min\{n-1,\delta\} \geq 0$$

Since $\binom{b}{2} \leq \binom{e}{2} - e_1$ we have $e \geq b+1$ and the assertion follows.

Remark 3.5 Corollary 3.3 does not hold if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension one.

We recall that if $e_1 = e+1$, then by [5, Proposition 2.4] there are only two possible Hilbert series

$$\frac{1+(e-3)t+2t^2}{(1-t)^d} \quad or \quad \frac{1+(e-2)t+t^3}{(1-t)^d}$$

Note that $e \geq 4$, hence $b = e-2$ and $\delta = 1$. If $d = 1$, these series are not comparable, while

$$P^1_R(t) \leq \frac{1+(e-2)t+t^3}{(1-t)^2}.$$

With the usual notations, for every Cohen–Macaulay local ring R of dimension $d \geq 1$ it is possible to prove that

$$P_R(t) \leq \frac{1+bt+(e-b-1)t^2}{(1-t)^d}$$

If $e_1 = e+1$, we get

$$P_R(t) \leq \frac{1+(e-2)t+t^2}{(1-t)^d}$$

but this is not satisfactory since e_1 of the above series is not necessarily the given e_1.

4 Finiteness of Hilbert Functions

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local of dimension d and let I be an $\mathfrak{m}$–primary ideal of multiplicity $e_0 = e(I)$. Denote by $H(d, e_0)$ the set of all Hilbert functions of the algebras $\mathrm{gr}_I(R)$. In [10] and [11] it is proved that $H(d, e_0)$ is a finite set. Its difficult proof is accomplished by providing very large bounds on the coefficients of the Hilbert polynomials and on the Castelnuovo–Mumford regularity of the algebra $\mathrm{gr}_I(R)$ in terms of e_0. As their authors point out, the assertion fails if R is not Cohen–Macaulay.

Our result in this section shows that using a different notion of multiplicity one obtains a weaker finiteness theorem which applies to arbitrary graded algebras.

We recall the notion of *extended multiplicity* introduced in [2]. Let S be either a graded algebra generated by its elements of degree 1 or a local ring. An *extended degree* is a function $\mathrm{Deg}(\cdot)$ on finitely generated S–modules (graded in the case of the former ring) satisfying the following conditions:

(i) If $L = \Gamma_{\mathfrak{m}}(M)$ is the submodule of elements of M which are annihilated by a power of the maximal ideal (maximal irrelevant ideal in the graded case) and $\overline{M} = M/L$, then

$$\mathrm{Deg}(M) = \mathrm{Deg}(\overline{M}) + \ell(L),$$

where $\ell(\cdot)$ is the ordinary length function.

(ii) (Bertini's rule) If S has positive depth and $h \in S$ is a generic hyperplane section on M, then

$$\mathrm{Deg}(M) \geq \mathrm{Deg}(M/hM).$$

(iii) (The calibration rule) If M is a Cohen–Macaulay module, then

$$\mathrm{Deg}(M) = \deg(M),$$

where $\deg(M)$ is the ordinary multiplicity of the module M.

In [13] an instance of such functions was constructed:

Definition 4.1 Let M be a finitely generated graded module over the graded algebra A and let S be a Gorenstein graded algebra mapping onto A, with maximal graded ideal $\mathfrak{m}$. Assume that $\dim S = r$, $\dim M = d$. The *homological degree* of M is the integer

$$\mathrm{hdeg}(M) \quad = \quad \deg(M) + \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M, S))$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\begin{aligned} \operatorname{hdeg}(M) &= \deg(M) + \\ & \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \operatorname{hdeg}(\operatorname{Ext}^i_S(M,S)). \end{aligned}$$

Given one such function, [7] proposed a method to construct another extended degree function where equality holds in the Bertini's condition (3). We now restate Corollary 2.5 in the language of these functions.

Corollary 4.2 *Let $G = \bigoplus_{n\geq 0} G_n$ be a standard graded algebra over an Artinian ring and let $\operatorname{Deg}(\cdot)$ be any extended degree function defined on G. If $\dim G = d \geq 1$ then*

$$P_G(t) \leq \frac{\ell(G_0) + (\operatorname{Deg}(G) - \ell(G_0)) \cdot t}{(1-t)^d}.$$

In other words, for all $n \geq 0$

$$\ell(G_n) \leq \operatorname{Deg}(G)\binom{d+n-2}{d-1} + \ell(G_0)\binom{d+n-2}{d-2}.$$

Proof. Let J be an ideal that is generated by a system of parameters of degree 1 that is generic for the function $\operatorname{Deg}(\cdot)$ chosen; according to [2, Proposition 2.3], $\ell(G/J) \leq \operatorname{Deg}(G)$. Now replace $\ell(G/J)$ by $\operatorname{Deg}(G)$ in the estimate of Corollary 2.5. □

Theorem 4.3 *Fix an extended degree function $\operatorname{Deg}(\cdot)$. Given two positive integers A, d, there exists only a finite number of Hilbert functions associated to standard graded algebras G over Artinian rings such that $\dim G = d$ and $\operatorname{Deg}(G) \leq A$. Furthermore, there are integers b_i dependent only on $\dim G$ such that $e_i(G) \leq b_i \operatorname{Deg}(G)^{i+1}$.*

Proof. We are first going to show that the number of Hilbert polynomials of these algebras

$$H^1_G(n) = \ell(G/G_{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d, \quad n \gg 0, \qquad (3)$$

is finite by finding bounds for the e_i in terms of $\operatorname{Deg}(G)$.

We induct on d. The assertion is clear if $d = 0$. For $d \geq 1$, we set $\widetilde{G} = G/H^0_{G_+}(G)$. Note that G and $\widetilde{G}$ have the same Hilbert polynomial and $\operatorname{Deg}(\widetilde{G}) \leq \operatorname{Deg}(G)$. Let $h \in \widetilde{G}_1$ be a generic hyperplane section for the function $\operatorname{Deg}(\cdot)$ and set $\overline{G} = \widetilde{G}/(h)$. Since

$$\ell(\overline{G}/\overline{G}_{n+1}) = e_0\binom{n+d-1}{d-1} - e_1\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}, \quad n \gg 0,$$

and $\mathrm{Deg}(\overline{G}) \leq \mathrm{Deg}(\widetilde{G}) \leq \mathrm{Deg}(G)$, by the induction hypothesis the e_i, for $i < d$, are all bounded as functions of $\mathrm{Deg}(G)$.

We may return to the algebra G. We recall that the difference between the Hilbert function of G and its Hilbert polynomial is given by ([1, Theorem 4.3.5(b)])

$$H_G(n) - \mathcal{P}_G(n) = \sum_{i=0}^{d} (-1)^i \ell(H^i_{G_+}(G)_n). \tag{4}$$

We now make a key point on the vanishing of $H^i_{G_+}(G)_n$, for $n \geq 0$. If we denote by $a_i(G)$ the largest n for which this group does not vanish, we have the well-known description of the Castelnuovo–Mumford regularity of the algebra G,

$$\mathrm{reg}(G) = \sup\{a_i(G) + i \mid i \geq 0\}.$$

However, according to [2] $\mathrm{Deg}(G) > \mathrm{reg}(G)$ so that $H_G(n) = \mathcal{P}_G(n)$ for $n \geq \mathrm{Deg}(G)$.

From this it is clear how to bound the coefficient e_d of the Hilbert polynomial of G. One way to proceed is to add the terms in (4) up to $n = r = \mathrm{Deg}(G)$ to get

$$\ell(G/G_{r+1}) - \sum_{i=0}^{d-1} (-1)^i e_i \binom{d+r-i}{d-i} = K,$$

where we set

$$K = \sum_{i=0}^{d} (-1)^i \ell(H^i_{G_+}(G)_{\geq 0}).$$

Note that $(-1)^d e_d = K$. A crude estimate for K is obtained from the inequality

$$|K| \leq |\ell(G/G_{r+1})| + |\sum_{i=0}^{d-1} (-1)^i e_i \binom{d+r-i}{d-i}|,$$

where now we replace $\ell(G/G_{r+1})$ by the estimate given in Corollary 4.2,

$$\ell(G/G_{r+1}) \leq \mathrm{Deg}(G)\binom{d+r-1}{d} + \ell(G_0)\binom{d+r-1}{d-1}.$$

It follows that $|e_d|$ is bounded by a polynomial (of degree $d+1$ when the degrees of the e_i are tracked carefully) in $\mathrm{Deg}(G)$.

The finiteness of the number of Hilbert functions now follows from the finiteness of the possible Hilbert polynomials, the bound on the postulation numbers and of another application of Corollary 4.2. □

References

[1] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[2] L. R. Doering, T. Gunston and W. V. Vasconcelos, Cohomological degrees and Hilbert functions of graded modules, American J. Math. **120** (1998), 493–504.

[3] P. Eakin and A. Sathaye, Prestable ideals, J. Algebra **41** (1976), 439–454.

[4] J. Elias, Characterization of the Hilbert-Samuel polynomial of curve singularities, Compositio Math. **74** (1990), 135-155

[5] J. Elias, M.E. Rossi and G. Valla, On the coefficients of the Hilbert polynomial, J. Pure Applied Algebra **108** (1996), 35-60

[6] J. Elias and G. Valla, Rigid Hilbert functions, J. Pure Applied Algebra **71** (1991), 19-41

[7] T. Gunston, *Cohomological degrees, Dilworth numbers and linear resolution*, Ph.D. Thesis, Rutgers University, 1998.

[8] D. Kirby and H. A. Mehran, A note on the coefficients of the Hilbert–Samuel polynomial for a Cohen–Macaulay module, J. London Math. Soc. **25** (1982), 449–457.

[9] D. Rees, α–transforms of local rings and a theorem on multiplicities of ideals, Proc. Camb. Phil. Soc. **57** (1961), 8–17.

[10] V. Srinivas and V. Trivedi, On the Hilbert function of a Cohen-Macaulay local ring, J. Algebraic Geom. **6** (1997), 733–751.

[11] V. Trivedi, Hilbert functions, Castelnuovo–Mumford regularity and uniform Artin-Rees numbers, Manuscripta Math. **94** (1997), 485–499.

[12] W. V. Vasconcelos, Hilbert functions, analytic spread and Koszul homology, A. M. S. Contemporary Mathematics **159** (1994), 401–422.

[13] W. V. Vasconcelos, The homological degree of a module, Trans. Amer. Math. Soc. **350** (1998), 1167–1179.

[14] W. V. Vasconcelos, Cohomological degrees of graded modules, in *Six Lectures on Commutative Algebra* (Eds. J. Elias, J. M. Giral, R. M. Miró–Roig and S. Zarzuela), Progress in Mathematics **166**, Birkhäuser, Boston, 1998, 345–392.

[15] M. T. R. Vaz Pinto, *Structure of Sally modules and Hilbert functions*, Ph.D. Thesis, Rutgers University, 1995.