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THE STRUCTURE OF THE SALLY MODULE OF INTEGRALLY CLOSED IDEALS

KAZUHO OZEKI AND MARIA EVELINA ROSSI

To Shiro Goto on the occasion of his seventieth birthday

Abstract. The first two Hilbert coefficients of a primary ideal play an important role in commutative algebra and in algebraic geometry. In this paper we give a complete algebraic structure of the Sally module of integrally closed ideals I in a Cohen–Macaulay local ring A satisfying the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$, where Q is a minimal reduction of I, and $\mathrm{e}_0(I)$ and $\mathrm{e}_1(I)$ denote the first two Hilbert coefficients of I, respectively, the multiplicity and the Chern number of I. This almost extremal value of $\mathrm{e}_1(I)$ with respect to classical inequalities holds a complete description of the homological and the numerical invariants of the associated graded ring. Examples are given.

§1. Introduction and notation

Throughout this paper, let A denote a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and positive dimension d. Let I be an $\mathfrak{m}$-primary ideal in A and, for simplicity, we assume the residue class field $A/\mathfrak{m}$ is infinite. Let $\ell_A(N)$ denote, for an A-module N, the length of N. The integers $\{\mathrm{e}_i(I)\}_{0 \leqslant i \leqslant d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

holds true for all integers $n \gg 0$, are called the *Hilbert coefficients* of A with respect to I. This polynomial, known as the Hilbert–Samuel polynomial of I and denoted by $HP_I(n)$, encodes the asymptotic information coming from the Hilbert function $H_I(t)$ of I which is defined as

$$H_I(t) = \ell_A(I^t/I^{t+1}).$$

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The generating function of the numerical function $H_I(t)$ is the power series

$$HS_I(z) = \sum_{t \geqslant 0} H_I(t) z^t.$$

This series is called the Hilbert series of I. It is well known that this series is rational and that, even more, there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$HS_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

Notice that for all $i \geqslant 0$, the Hilbert coefficients can be computed as it follows

$$\mathrm{e}_i(I) := \frac{h_I^{(i)}(1)}{i!}$$

where $h_I^{(i)}(1)$ denotes the ith derivative of $h_I(z)$ evaluated at 1 ($h^{(0)} = h_I$).

Choose a parameter ideal Q of A which forms a reduction of I and let

$$R = \mathrm{R}(I) := A[It] \qquad \text{and} \qquad T = \mathrm{R}(Q) := A[Qt] \subseteq A[t]$$

denote, respectively, the Rees algebras of I and Q. Let

$$R' = \mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}] \quad \text{and}$$
$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geqslant 0} I^n/I^{n+1}.$$

Following Vasconcelos [V], we consider

$$S = \mathrm{S}_Q(I) = IR/IT \cong \bigoplus_{n \geqslant 1} I^{n+1}/Q^n I$$

the Sally module of I with respect to Q.

The notion of *filtration of the Sally module* was introduced by Vaz Pinto [VP] as follows. We denote by $E(\alpha)$, for a graded T-module E and each $\alpha \in \mathbb{Z}$, the graded T-module whose grading is given by $[E(\alpha)]_n = E_{\alpha+n}$ for all $n \in \mathbb{Z}$.

Definition 1.1. [VP] We set, for each $i \geqslant 1$,

$$C^{(i)} = (I^i R/I^i T)(-i+1) \cong \bigoplus_{n \geqslant i} I^{n+1}/Q^{n-i+1} I^i,$$

and let $L^{(i)} = T[C^{(i)}]_i$. Then, because $L^{(i)} \cong \bigoplus_{n \geqslant i} Q^{n-i} I^{i+1}/Q^{n-i+1} I^i$ and $C^{(i)}/L^{(i)} \cong C^{(i+1)}$ as graded T-modules, we have the following natural exact sequences of graded T-modules

$$0 \to L^{(i)} \to C^{(i)} \to C^{(i+1)} \to 0$$

for every $i \geqslant 1$.

We notice that $C^{(1)} = S$, and $C^{(i)}$ are finitely generated graded T-modules for all $i \geqslant 1$, since R is a module-finite extension of the graded ring T.

So, from now on, we set

$$C = \mathrm{C}_Q(I) = C^{(2)} = (I^2 R/I^2 T)(-1)$$

and we shall explore the structure of C. Assume that I is integrally closed. Then, by [EV, GR], the inequality

$$\mathrm{e}_1(I) \geqslant \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$$

holds true and the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds if and only if $I^3 = QI^2$. When this is the case, the associated graded ring G of I is Cohen–Macaulay and the behavior of the Hilbert–Samuel function $\ell_A(A/I^{n+1})$ of I is known (see [EV], Corollary 2.10). Thus the integrally closed ideal I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ enjoys nice properties and it seems natural to ask what happens to the integrally closed ideal I which satisfies the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. The problem is not trivial even if we consider $d = 1$.

We notice here that $\ell_A(I^2/QI) = \mathrm{e}_0(I) + (d-1)\ell_A(A/I) - \ell_A(I/I^2)$ holds true (see for instance [RV3]), so that $\ell_A(I^2/QI)$ does not depend on a minimal reduction Q of I.

Let $B = T/\mathfrak{m}T \cong (A/\mathfrak{m})[X_1, X_2, \ldots, X_d]$ which is a polynomial ring with d indeterminates over the field $A/\mathfrak{m}$. The main result of this paper is stated as follows.

THEOREM 1.2. *Assume that I is integrally closed. Then the following conditions are equivalent:*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$,
(2) $\mathfrak{m}C = (0)$ *and* $\mathrm{rank}_B C = 1$,
(3) $C \cong (X_1, X_2, \ldots, X_c)B(-1)$ *as graded T-modules for some $1 \leqslant c \leqslant d$, where $X_1, X_2, \ldots, X_d$ are linearly independent linear forms of the polynomial ring B.*

When this is the case, $c = \ell_A(I^3/QI^2)$ *and* $I^4 = QI^3$, *and the following assertions hold true:*

(i) depth $G \geqslant d - c$ *and* $\mathrm{depth}_T C = d - c + 1$,
(ii) depth $G = d - c$, *if* $c \geqslant 2$.
(iii) *Suppose* $c = 1 < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 0$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) + 1 & \text{if } i = 2, \\ 1 & \text{if } i = 3 \text{ and } d \geqslant 3, \\ 0 & \text{if } 4 \leqslant i \leqslant d. \end{cases}$$

(iv) *Suppose* $2 \leqslant c < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 0$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \text{if } i = 2, \\ 0 & \text{if } i \neq c+1, c+2, \quad 3 \leqslant i \leqslant d \\ (-1)^{c+1} & \text{if } i = c+1, c+2, \quad 3 \leqslant i \leqslant d \end{cases}$$

(v) *Suppose* $c = d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 2$ *and*

$$\mathrm{e}_i(I) = \begin{cases} \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) & \text{if } i = 2 \text{ and } d \geqslant 2, \\ 0 & \text{if } 3 \leqslant i \leqslant d \end{cases}$$

(vi) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1-z)^{c+1}z}{(1-z)^d}.$$

Let us briefly explain how this paper is organized. We shall prove Theorem 1.2 in Section 3. In Section 2 we will introduce some auxiliary results on the structure of the T-module $C = \mathrm{C}_Q(I) = (I^2R/I^2T)(-1)$, some of them are stated in a general setting. Our hope is that this information will be successfully applied to give new insight to problems related to the structure of the Sally module. In Section 4 we will introduce some consequences of Theorem 1.2. In particular, we shall explore the integrally closed ideals I with $\mathrm{e}_1(I) \leqslant \mathrm{e}_0(I) - \ell_A(A/I) + 3$. In Section 5 we will construct a class of Cohen–Macaulay local rings satisfying condition (1) in Theorem 1.2.

§2. Preliminary steps

The purpose of this section is to summarize some results on the structure of the graded T-module $C = \mathrm{C}_Q(I) = (I^2R/I^2T)(-1)$, which we need throughout this paper. Remark that in this section I is an $\mathfrak{m}$-primary ideal not necessarily integrally closed.

LEMMA 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^\ell C = (0)$ *for integers* $\ell \gg 0$*; hence* $\dim_T C \leqslant d$.
(2) *The homogeneous components* $\{C_n\}_{n\in\mathbb{Z}}$ *of the graded* T*-module* C *are given by*

$$C_n \cong \begin{cases} (0) & \text{if } n \leqslant 1, \\ I^{n+1}/Q^{n-1}I^2 & \text{if } n \geqslant 2. \end{cases}$$

(3) $C = (0)$ *if and only if* $I^3 = QI^2$.
(4) $\mathfrak{m}C = (0)$ *if and only if* $\mathfrak{m}I^{n+1} \subseteq Q^{n-1}I^2$ *for all* $n \geqslant 2$.
(5) $C = TC_2$ *if and only if* $I^4 = QI^3$.

Proof. (1) Let $u = t^{-1}$ and $T' = \mathrm{R}'(Q)$. Notice that we have $C = (I^2R/I^2T)(-1) \cong (I^2R'/I^2T')(-1)$ as graded T-modules. We then have $u^\ell \cdot (I^2R'/I^2T') = (0)$ for some $\ell \gg 0$, because the graded T'-module I^2R'/I^2T' is finitely generated and $[I^2R'/I^2T']_n = (0)$ for all $n \leqslant 0$. Therefore, $\mathfrak{m}^\ell C = (0)$ for $\ell \gg 0$, because $Q^\ell = (Qt^\ell)u^\ell \subseteq u^\ell T' \cap A$ and $\mathfrak{m} = \sqrt{Q}$.

(2) Since $[I^2R]_n = (I^{n+2})t^n$ and $[I^2T]_n = (I^2Q^n)t^n$ for all $n \geqslant 0$, assertion (2) follows from the definition of the module $C = (I^2R/I^2T)(-1)$.

Assertions (3), (4), and (5) readily follow from assertion (2). □

In the following result we need that $Q \cap I^2 = QI$ holds true. This condition is automatically satisfied in the case where I is integrally closed (see [H, I2]).

PROPOSITION 2.2. *Suppose that* $Q \cap I^2 = QI$*. Then we have* $\mathrm{Ass}_T C \subseteq \{\mathfrak{m}T\}$ *so that* $\dim_T C = d$*, if* $C \neq (0)$.

Let $Q = (a_1, \ldots, a_d)$ be a minimal reduction of I. In the proof of Proposition 2.2 we need the following lemmata.

LEMMA 2.3. *Suppose that* $Q \cap I^2 = QI$*. Then we have the equality* $(a_1, a_2, \ldots, a_i) \cap Q^{n+1}I^2 = (a_1, a_2, \ldots, a_i)Q^nI^2$ *for all* $n \geqslant 0$ *and* $1 \leqslant i \leqslant d$*. Therefore,* $a_1t \in T$ *is a nonzero divisor on* T/I^2T.

Proof. We have only to show that $(a_1, a_2, \ldots, a_i) \cap Q^{n+1}I^2 \subseteq (a_1, a_2, \ldots, a_i)Q^nI^2$ holds true for all $n \geqslant 0$. We proceed by induction on n and i. We may assume that $i < d$ and that our assertion holds true for $i+1$. Suppose $n = 0$ and take $x \in (a_1, a_2, \ldots, a_i) \cap QI^2$. Then, by the hypothesis of induction on i, we have

$$\begin{aligned}(a_1, a_2, \ldots, a_i) \cap QI^2 &\subseteq (a_1, a_2, \ldots, a_i, a_{i+1}) \cap QI^2 \\ &= (a_1, a_2, \ldots, a_i, a_{i+1})I^2.\end{aligned}$$

Then we write $x = y + a_{i+1}z$ with $y \in (a_1, a_2, \ldots, a_i)I^2$ and $z \in I^2$. Therefore, $z \in [(a_1, a_2, \ldots, a_i) : a_{i+1}] \cap I^2 = (a_1, a_2, \ldots, a_i) \cap I^2 \subseteq (a_1, a_2, \ldots, a_i) \cap QI$. Notice that, since $Q/QI \cong (A/I)^d$, $(a_1, a_2, \ldots, a_i) \cap QI = (a_1, a_2, \ldots, a_i)I$ holds true. Hence, we get $x = y + a_{i+1}z \in (a_1, a_2, \ldots, a_i)I^2$. Therefore, we have $(a_1, a_2, \ldots, a_i) \cap QI^2 = (a_1, a_2, \ldots, a_i)I^2$.

Assume that $n \geqslant 1$ and that our assertion holds true for $n-1$. Take $x \in (a_1, a_2, \ldots, a_i) \cap Q^{n+1}I^2$. Then, by the hypothesis of induction on i, we have

$$\begin{aligned}(a_1, a_2, \ldots, a_i) \cap Q^{n+1}I^2 &\subseteq (a_1, a_2, \ldots, a_i, a_{i+1}) \cap Q^{n+1}I^2 \\ &= (a_1, a_2, \ldots, a_i, a_{i+1})Q^nI^2.\end{aligned}$$

Write $x = y + a_{i+1}z$ with $y \in (a_1, a_2, \ldots, a_i)Q^nI^2$ and $z \in Q^nI^2$. By the hypothesis of induction on n, we have $z \in [(a_1, a_2, \ldots, a_i) : a_{i+1}] \cap Q^nI^2 = (a_1, a_2, \ldots, a_i) \cap Q^nI^2 = (a_1, a_2, \ldots, a_i)Q^{n-1}I^2$. Therefore, we get $x = y + a_{i+1}z \in (a_1, a_2, \ldots, a_i)Q^nI^2$. Thus, $(a_1, a_2, \ldots, a_i) \cap Q^{n+1}I^2 = (a_1, a_2, \ldots, a_i)Q^nI^2$ as required. ∎

LEMMA 2.4. *Suppose that $Q \cap I^2 = QI$. Then $Q^{n+1} \cap Q^nI^2 = Q^{n+1}I$ for all $n \geqslant 0$.*

Proof. We have only to show that $Q^{n+1} \cap Q^nI^2 \subseteq Q^{n+1}I$ for $n \geqslant 0$. Take $f \in Q^{n+1} \cap Q^nI^2$ and write $f = \sum_{|\alpha|=n} x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}$ with $x_\alpha \in I^2$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d$ and $|\alpha| = \sum_{i=1}^d \alpha_i$. Then, we have $\overline{ft^n} = \sum_{|\alpha|=n} \overline{x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^n} = 0$ in $\mathrm{G}(Q) = \mathrm{R}'(Q)/t^{-1}\mathrm{R}'(Q)$, where $\overline{ft^n}$ and $\overline{x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^n} = 0$ denote the images of ft^n and $x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_d^{\alpha_d}t^n$ in $\mathrm{G}(Q)$, respectively. Because $\mathrm{G}(Q) \cong (A/Q)[\overline{a_1t}, \overline{a_2t}, \ldots, \overline{a_dt}]$ is the polynomial ring over the ring A/Q, we have $x_\alpha \in Q \cap I^2 = QI$. Thus $f \in Q^{n+1}I$ so that $Q^{n+1} \cap Q^nI^2 = Q^{n+1}I$ holds true. ∎

LEMMA 2.5. *Suppose that $Q \cap I^2 = QI$. Then the following sequences*

(1) $0 \to T/IT \xrightarrow{a_1} T/I^2T \to T/[I^2T + a_1T] \to 0$, *and*
(2) $0 \to T/[I^2T + QT](-1) \xrightarrow{a_1t} T/[I^2T + a_1T] \to T/[I^2T + a_1tT + a_1T] \to 0$

of graded T-modules are exact.

Proof. (1) Let us consider the homomorphism

$$\phi : T \to T/I^2T$$

of graded T-modules such that $\phi(f)=\overline{a_1 f}$ for $f \in T$ where $\overline{a_1 f}$ denotes the image of $a_1 f$ in T/I^2T. Because $\phi(IT)=(0)$ and $\mathrm{Coker}\phi = T/[I^2T+a_1T]$, we have only to show that $\ker\phi \subseteq IT$. Take $f \in [\ker\phi]_n$ and write $f = xt^n$ with $n \geqslant 0$ and $x \in Q^n$. Then we have $a_1 f = a_1 x t^n \in I^2T$ so that $a_1 x \in Q^n I^2 \cap Q^{n+1} = Q^{n+1}I$ by Lemma 2.4. Because $a_1 t \in T$ forms a nonzero divisor on T/IT (notice that $T/IT \cong (A/I)[\overline{a_1 t}, \overline{a_2 t}, \ldots, \overline{a_d t}]$ is a polynomial ring over the ring A/I), we have $(a_1) \cap Q^{n+1}I = a_1 Q^n I$ so that $x \in Q^n I$. Therefore, $f \in IT$, and hence $\ker\phi \subseteq IT$. Thus, we get the first required exact sequence.

(2) Let us consider the homomorphism

$$\varphi : T(-1) \to T/[I^2T + a_1T]$$

of graded T-modules such that $\varphi(f)=\overline{a_1 t f}$ for $f \in T$ where $\overline{a_1 t f}$ denotes the image of $a_1 t f$ in $T/[I^2T+a_1T]$. Because $\varphi(I^2T+QT)=(0)$ and $\mathrm{Coker}\phi = T/[I^2T+a_1tT+a_1T]$, we need to show that $[\ker\varphi]_n \subseteq I^2T_{n-1}+QT_{n-1}$ for all $n \geqslant 1$. Take $f \in [\ker\varphi]_n$ and write $f = xt^{n-1}$ with $n \geqslant 1$ and $x \in Q^{n-1}$. Then we have $a_1 t f = a_1 x t^n \in I^2T + a_1T$ so that $a_1 x \in Q^n I^2 + a_1 Q^n$. Write $a_1 x = y + a_1 z$ with $y \in Q^n I^2$ and $z \in Q^n$. Then have

$$a_1(x-z) = y \in (a_1) \cap Q^n I^2 = a_1 Q^{n-1} I^2$$

by Lemma 2.3. Hence $x - z \in Q^{n-1}I^2$ so that $x \in Q^{n-1}I^2 + Q^n$. Therefore, $f \in I^2T_{n-1}+QT_{n-1}$ and hence $[\ker\varphi]_n \subseteq I^2T_{n-1}+QT_{n-1}$. Consequently, we get the second required exact sequence. □

The following Lemma 2.6 is the crucial fact in the proof of Proposition 2.2.

LEMMA 2.6. *Assume that* $Q \cap I^2 = QI$. *Then we have* $\mathrm{Ass}_T(T/I^2T) = \{\mathfrak{m}T\}$.

Proof. Take $P \in \mathrm{Ass}_T(T/I^2T)$; then we have $\mathfrak{m}T \subseteq P$, because $\mathfrak{m}^\ell(T/I^2T)=(0)$ for $\ell \gg 0$. Assume that $\mathfrak{m}T \subsetneq P$; then $\mathrm{ht}_T P \geqslant 2$, because $\mathfrak{m}T$ is a height one prime ideal in T (notice that $\dim T = d+1$). Consider the following exact sequences

$$0 \to T_P/IT_P \to T_P/I^2T_P \to T_P/[I^2T_P + a_1T_P] \to 0 \quad (*_1) \quad \text{and}$$

$$0 \to T_P/[I^2T_P + QT_P] \to T_P/[I^2T_P + a_1T_P] \to T_P/[I^2T_P + a_1tT_P + a_1T_P] \to 0 \quad (*_2)$$

of T_P-modules, which follow from the exact sequences in Lemma 2.5. Then, since $\mathrm{depth}_{T_P} T_P/IT_P > 0$ (notice that $T/IT \cong (A/I)[X_1, X_2, \ldots, X_d]$ is the polynomial ring with d indeterminates over the ring A/I) and

$\mathrm{depth}_{T_P} T_P/I^2T_P = 0$, we have $\mathrm{depth}_{T_P} T_P/[I^2T_P + a_1T_P] = 0$ by the exact sequence $(*_1)$. Notice that $T/[I^2T + QT] \cong (A/[I^2 + Q])[X_1, X_2, \ldots, X_d]$ is the polynomial rings with d indeterminates over the ring $A/[I^2 + Q]$. Hence, $\mathrm{depth}_{T_P} T_P/[I^2T_P + a_1tT_P + a_1T_P] = 0$ by the exact sequence $(*_2)$. Then, because $a_1t \in \mathrm{Ann}_T(T/[I^2T + a_1tT + a_1T])$, we have $a_1t \in P$. Therefore, $\mathrm{depth}_{T_P} T_P/I^2T_P > 0$ by Lemma 2.3; however, it is impossible. Thus, $P = \mathfrak{m}T$ as required. ∎

Let us now give a proof of Proposition 2.2.

Proof of Proposition 2.2. Take $P \in \mathrm{Ass}_T C$. Then we have $\mathfrak{m}T \subseteq P$, because $\mathfrak{m}^\ell C = (0)$ for some $\ell \gg 0$ by Lemma 2.1(1). Suppose that $\mathfrak{m}T \subsetneq P$; then $\mathrm{ht}_T P \geqslant 2$, because $\mathfrak{m}T$ is a height one prime ideal in T. We look at the following exact sequences

$$0 \to I^2T_P \to I^2R_P \to C_P \to 0 \quad (*_3) \quad \text{and}$$
$$0 \to I^2T_P \to T_P \to T_P/I^2T_P \to 0 \quad (*_4)$$

of T_P-modules which follow from the canonical exact sequences

$$0 \to I^2T(-1) \to I^2R(-1) \to C \to 0 \quad \text{and} \quad 0 \to I^2T \to T \to T/I^2T \to 0$$

of T-modules. We notice here that $\mathrm{depth}_{T_P} I^2R_P > 0$, because $a_1 \in A$ is a nonzero divisor on I^2R. Thanks to the depth lemma and the exact sequence $(*_3)$, we get $\mathrm{depth}_{T_P} I^2T_P = 1$, because $\mathrm{depth}_{T_P} I^2R_P > 0$ and $\mathrm{depth}_{T_P} C_P = 0$. Then, since $\mathrm{depth}_{T_P} T_P \geqslant 2$, we have $\mathrm{depth}_{T_P} T_P/I^2T_P = 0$ by the exact sequence $(*_4)$. Therefore, we have $P = \mathfrak{m}T$ by Lemma 2.6, which is impossible. Thus, $P = \mathfrak{m}T$ as required. ∎

The following techniques are due to Vaz Pinto [VP, Section 2].

Let $L = L^{(1)} = TS_1$ then $L \cong \bigoplus_{n\geqslant 1} Q^{n-1}I^2/Q^nI$ and $S/L \cong C$ as graded T-modules. Then there exists a canonical exact sequence

$$0 \to L \to S \to C \to 0 \quad (\dagger)$$

of graded T-modules (Definition 1.1). We set $D = (I^2/QI) \otimes_A (T/\mathrm{Ann}_A(I^2/QI)T)$. Notice here that D forms a graded T-module and $T/\mathrm{Ann}_A(I^2/QI)T \cong (A/\mathrm{Ann}_A(I^2/QI))[X_1, X_2, \ldots, X_d]$ is the polynomial ring with d indeterminates over the ring $A/\mathrm{Ann}_A(I^2/QI)$. Let

$$\theta : D(-1) \to L$$

denote an epimorphism of graded T-modules such that $\theta(\sum_\alpha \overline{x_\alpha} \otimes X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}) = \sum_\alpha \overline{x_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d} t^{|\alpha|+1}}$ for $x_\alpha \in I^2$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d$ with $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant d)$, where $|\alpha| = \sum_{i=1}^d \alpha_i$, and $\overline{x_\alpha}$ and $\overline{x_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d} t^{|\alpha|+1}}$ denote the images of x_α in I^2/QI and $x_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d} t^{|\alpha|+1}$ in L.

Then we have the following lemma.

LEMMA 2.7. *Suppose that $Q \cap I^2 = QI$. Then the map $\theta : D(-1) \to L$ is an isomorphism of graded T-modules.*

Proof. We have only to show that $\ker \theta = (0)$. Assume that $\ker \theta \neq (0)$ and let $n \geqslant 2$ as the least integer so that $[\ker \theta]_n \neq (0)$ (notice that $[\ker \theta]_n = (0)$ for all $n \leqslant 1$). Take $0 \neq g \in [\ker \theta]_n$ and we set

$$\Gamma = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, 0) \in \mathbb{Z}^d | \alpha_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant d-1, \text{ and } \sum_{i=1}^{d-1} \alpha_i = n-1 \right\},$$

$$\Gamma' = \left\{ (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{Z}^d | \beta_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant d-1, \beta_d \geqslant 1, \text{ and } \sum_{i=1}^{d} \beta_i = n-1 \right\}.$$

Then because

$$\Gamma \cup \Gamma' = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d | \alpha_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant d \text{ and } \sum_{i=1}^{d} \alpha_i = n-1 \right\}$$

we may write

$$\begin{aligned} g &= \sum_{\alpha \in \Gamma \cup \Gamma'} \overline{x_\alpha} \otimes X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d} \\ &= \sum_{\alpha \in \Gamma} \overline{x_\alpha} \otimes X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{d-1}^{\alpha_{d-1}} + \sum_{\beta \in \Gamma'} \overline{x_\beta} \otimes X_1^{\beta_1} X_2^{\beta_2} \cdots X_d^{\beta_d}, \end{aligned}$$

where $\overline{x_\alpha}, \overline{x_\beta}$ denote the images of $x_\alpha, x_\beta \in I^2$ in I^2/QI, respectively. We may assume that $\sum_{\beta \in \Gamma'} \overline{x_\beta} \otimes X_1^{\beta_1} X_2^{\beta_2} \cdots X_d^{\beta_d} \neq 0$ in D. Then we have

$$\theta(g) = \sum_{\alpha \in \Gamma} \overline{x_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{d-1}^{\alpha_{d-1}} t^n} + \overline{\sum_{\beta \in \Gamma'} x_\beta a_1^{\beta_1} a_2^{\beta_2} \cdots a_d^{\beta_d} t^n} = 0$$

so that

$$\sum_{\alpha \in \Gamma} x_\alpha a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{d-1}^{\alpha_{d-1}} + \sum_{\beta \in \Gamma'} x_\beta a_1^{\beta_1} a_2^{\beta_2} \cdots a_d^{\beta_d} \in Q^n I.$$

Because $Q^nI = (a_1, a_2, \ldots, a_{d-1})^nI + a_dQ^{n-1}I$ and $\beta_d \geqslant 1$, we may write

$$\sum_{\alpha\in\Gamma} x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_{d-1}^{\alpha_{d-1}} + a_d\left(\sum_{\beta\in\Gamma'} x_\beta a_1^{\beta_1}a_2^{\beta_2}\cdots a_d^{\beta_d-1}\right) = \tau + a_d\rho$$

with $\tau \in (a_1, a_2, \ldots, a_{d-1})^nI$ and $\rho \in Q^{n-1}I$. Then because

$$a_d\left(\sum_{\beta\in\Gamma'} x_\beta a_1^{\beta_1}a_2^{\beta_2}\cdots a_d^{\beta_d-1} - \rho\right) = \tau - \sum_{\alpha\in\Gamma} x_\alpha a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_{d-1}^{\alpha_{d-1}} \in (a_1, a_2, \ldots, a_{d-1})^{n-1}$$

we have, by Lemma 2.4,

$$\sum_{\beta\in\Gamma'} x_\beta a_1^{\beta_1}a_2^{\beta_2}\cdots a_d^{\beta_d-1} - \rho \in (a_1, a_2, \ldots, a_{d-1})^{n-1} \cap Q^{n-2}I^2 \subseteq Q^{n-1}I.$$

Therefore, $\sum_{\beta\in\Gamma'} x_\beta a_1^{\beta_1}a_2^{\beta_2}\cdots a_d^{\beta_d-1} \in Q^{n-1}I$ and hence $\theta(\sum_{\beta\in\Gamma'} \overline{x_\beta} \otimes X_1^{\beta_1}X_2^{\beta_2}\cdots X_d^{\beta_d-1}) = 0$. Then, because $\sum_{\beta\in\Gamma'} \overline{x_\beta} \otimes X_1^{\beta_1}X_2^{\beta_2}\cdots X_d^{\beta_d-1} \in [\ker\theta]_{n-1} = (0)$, we have $\sum_{\beta\in\Gamma'} \overline{x_\beta} \otimes X_1^{\beta_1}X_2^{\beta_2}\cdots X_d^{\beta_d} = 0$, which is contradiction. Thus $\ker\theta = (0)$. Consequently, the map $\theta : D(-1) \to L$ is an isomorphism. ∎

Thanks to Lemma 2.7, we can prove the following result.

PROPOSITION 2.8. *Suppose that $Q \cap I^2 = QI$. Then we have*

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)\}\binom{n+d-1}{d-1} + \ell_A(I^2/QI)\binom{n+d-2}{d-2} - \ell_A(C_n)$$

for all $n \geqslant 0$.

Proof. We have, for all $n \geqslant 0$,

$$\begin{aligned}\ell_A(S_n) &= \ell_A(L_n) + \ell_A(C_n)\\ &= \ell_A(I^2/QI)\binom{n+d-2}{d-1} + \ell_A(C_n)\\ &= \ell_A(I^2/QI)\binom{n+d-1}{d-1} - \ell_A(I^2/QI)\binom{n+d-2}{d-2} + \ell_A(C_n)\end{aligned}$$

by the exact sequence

$$0 \to L \to S \to C \to 0 \quad (\dagger)$$

and the isomorphisms $L \cong D(-1) \cong (I^2/QI) \otimes_A (A/\mathrm{Ann}_A(I^2/QI))[X_1, X_2, \ldots, X_d]$ of graded T-modules (see Lemma 2.7). Therefore, we have, for all $n \geqslant 0$,

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I)\}\binom{n+d-1}{d-1} - \ell_A(S_n)\\ &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I)\}\binom{n+d-1}{d-1} - \Bigg\{\ell_A(I^2/QI)\\ &\quad \cdot \binom{n+d-1}{d-1} - \ell_A(I^2/QI)\binom{n+d-2}{d-2} + \ell_A(C_n)\Bigg\}\\ &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)\}\\ &\quad \cdot \binom{n+d-1}{d-1} + \ell_A(I^2/QI)\binom{n+d-2}{d-2} - \ell_A(C_n)\end{aligned}$$

by [GNO, Proposition 2.2 (2)]. ∎

The following result specifies [GNO, Proposition 2.2(3)] and, by using Propositions 2.2 and 2.8, the proof takes advantage of the same techniques.

Proposition 2.9. *Suppose that $Q \cap I^2 = QI$. Let $\mathfrak{p} = \mathfrak{m}T$. Then we have*

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + \ell_{T_\mathfrak{p}}(C_\mathfrak{p}).$$

Combining Lemma 2.1(3) and Proposition 2.9 we obtain the following result that was proven by Elias and Valla [EV, Theorem 2.1] in the case where $I = \mathfrak{m}$.

Corollary 2.10. *Suppose that $Q \cap I^2 = QI$. Then we have $\mathrm{e}_1(I) \geqslant \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$. The equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true if and only if $I^3 = QI^2$. When this is the case, $\mathrm{e}_2(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$ if $d \geqslant 2$, $\mathrm{e}_i(I) = 0$ for all $3 \leqslant i \leqslant d$, and G is a Cohen–Macaulay ring.*

Let us introduce the relationship between the depth of the module C and the associated graded ring G of I.

LEMMA 2.11. *Suppose that $Q \cap I^2 = QI$ and $C \neq (0)$. Let $s = \mathrm{depth}_T C$. Then we have $\mathrm{depth} G \geqslant s - 1$. In particular, we have $\mathrm{depth} G = s - 1$, if $s \leqslant d - 2$.*

Proof. Notice that $L \cong D(-1)$ by Lemma 2.7 and D is a d dimensional Cohen–Macaulay T-module. Therefore, we have $s \leqslant \mathrm{depth}_T S$ and, if $s \leqslant d-2$ then $s = \mathrm{depth}_T S$ by the exact sequence

$$0 \to L \to S \to C \to 0 \quad (\dagger).$$

Because $\mathrm{depth} G \geqslant \mathrm{depth}_T S - 1$ and, if $\mathrm{depth}_T S \leqslant d-1$ then $\mathrm{depth} G = \mathrm{depth}_T S - 1$ by [GNO, Proposition 2.2(4)], our assertions follow. □

§3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Throughout this section, let I be an integrally closed $\mathfrak{m}$-primary ideal.

THEOREM 3.1. *Suppose that I is integrally closed. Then the following conditions are equivalent:*

(1) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$,
(2) $\mathfrak{m}C = (0)$ *and* $\mathrm{rank}_B C = 1$,
(3) *there exists a non-zero graded ideal $\mathfrak{a}$ of B such that $C \cong \mathfrak{a}(-1)$ as graded T-modules.*

To prove Theorem 3.1, we need the following bound on $\mathrm{e}_2(I)$.

LEMMA 3.2. ([I1, Theorem 12], [S2, Corollary 2.5], [RV3, Corollary 3.1]) *Suppose $d \geqslant 2$ and let I be an integrally closed ideal. Then* $\mathrm{e}_2(I) \geqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$.

Proof of Theorem 3.1. Let $\mathfrak{p} = \mathfrak{m}T$; then we see that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + \ell_{T_\mathfrak{p}}(C_\mathfrak{p})$ by Lemma 2.9 and $\mathrm{Ass}_T C = \{\mathfrak{p}\}$ by Proposition 2.2.

(1) $\Rightarrow$ (2) Since $\ell_{T_\mathfrak{p}}(C_\mathfrak{p}) = 1$ and $\mathrm{Ass}_T C = \{\mathfrak{p}\}$, we have $\mathfrak{m}C = (0)$ and $\mathrm{rank}_B C = 1$.

(2) $\Rightarrow$ (1) This is clear, because assertion (1) is equivalent to saying that $\ell_{T_\mathfrak{p}}(C_\mathfrak{p}) = 1$.

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Because $\mathrm{Ass}_T C = \{\mathfrak{p}\}$, C is a torsion free B-module. If C is B-free, then we have $C \cong B(-2)$ as graded B-modules, because $C_2 \neq (0)$

and $C_n = (0)$ for $n \leqslant 1$. Hence $C \cong X_1 B(-1)$ as graded B-modules with $0 \neq X_1 \in B_1$.

Suppose that C is not B-free. Then we have $d = \dim A \geqslant 2$. Because $\mathrm{rank}_B C = 1$, there exists a graded ideal $\mathfrak{a}$ of B such that

$$C \cong \mathfrak{a}(r)$$

as graded B-modules for some $r \in \mathbb{Z}$. Since every height one prime in the polynomial ring B is principal, we may choose $\mathfrak{a}$ with $\mathrm{ht}_B \mathfrak{a} \geqslant 2$. Then since $\mathfrak{a}_{r+2} \cong \mathfrak{a}(r)_2 \cong C_2 \neq (0)$ and $\mathfrak{a}_n = (0)$ for all $n \leqslant 0$, we have $r + 2 \geqslant 1$ so that $r \geqslant -1$. It is now enough to show that $r = -1$. Applying the exact sequence

$$0 \to C \to B(r) \to (B/\mathfrak{a})(r) \to 0$$

of graded B-modules, we have

$$\begin{aligned}
\ell_A(C_n) &= \ell_A(B_{r+n}) - \ell_A([B/\mathfrak{a}]_{r+n}) \\
&= \binom{n+r+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{r+n}) \\
&= \binom{n+d-1}{d-1} + r\binom{n+d-2}{d-2} + (\text{lower terms})
\end{aligned}$$

for all $n \gg 0$, because $\mathrm{ht}_B \mathfrak{a} \geqslant 2$. Therefore, thanks to Proposition 2.8, we have

$$\begin{aligned}
\ell_A(A/I^{n+1}) &= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)\} \\
&\quad \cdot \binom{n+d-1}{d-1} + \ell_A(I^2/QI)\binom{n+d-2}{d-2} - \ell_A(C_n) \\
&= \mathrm{e}_0(I)\binom{n+d}{d} - \{\mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1\} \\
&\quad \cdot \binom{n+d-1}{d-1} + \{\ell_A(I^2/QI) - r\}\binom{n+d-2}{d-2} + (\text{lower terms})
\end{aligned}$$

for all $n \gg 0$. Therefore, $\mathrm{e}_2(I) = \ell_A(I^2/QI) - r$. Then, because $\mathrm{e}_2(I) \geqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) = \ell_A(I^2/QI) + 1$ by Lemma 3.2 we have $r \leqslant -1$. Thus $r = -1$ and so $C \cong \mathfrak{a}(-1)$ as graded B-modules. ∎

As a direct consequence of Theorem 3.1 the following result holds true.

Proposition 3.3. *Assume that I is integrally closed. Suppose that $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$ and $I^4 = QI^3$. Let $c = \ell_A(I^3/QI^2)$. Then*

(1) $1 \leqslant c \leqslant d$ *and* $\mu_B(C) = c$.
(2) depth $G \geqslant d - c$ *and* $\mathrm{depth}_T C = d - c + 1$,
(3) depth $G = d - c$, *if* $c \geqslant 2$.
(4) *Suppose* $c = 1 < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 0$ *and*

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) + 1 & \text{if } i = 2, \\ 1 & \text{if } i = 3 \text{ and } d \geqslant 3, \\ 0 & \text{if } 4 \leqslant i \leqslant d. \end{cases}$$

(5) *Suppose* $2 \leqslant c < d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 0$ *and*

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2, \\ 0 & \text{if } i \neq c+1, c+2, 3 \leqslant i \leqslant d, \\ (-1)^{c+1} & \text{if } i = c+1, c+2, 3 \leqslant i \leqslant d, \end{cases}$$

(6) *Suppose* $c = d$. *Then* $HP_I(n) = \ell_A(A/I^{n+1})$ *for all* $n \geqslant 2$ *and*

$$e_i(I) = \begin{cases} e_1(I) - e_0(I) + \ell_A(A/I) & \text{if } i = 2 \text{ and } d \geqslant 2, \\ 0 & \text{if } 3 \leqslant i \leqslant d, \end{cases}$$

(7) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{e_0(I) - \ell_A(A/I) - \ell_A(I^2/QI) - 1\}z + \{\ell_A(I^2/QI) + 1\}z^2 + (1-z)^{c+1}z}{(1-z)^d}.$$

Proof. (1) We have $C = TC_2$, since $I^4 = QI^3$ (cf. Lemma 2.1(5)). Therefore, thanks to Theorem 3.1, $C \cong \mathfrak{a}(-1)$ as graded T-modules, where $\mathfrak{a} = (X_1, X_2, \ldots, X_c)B$ is an ideal generated by linear forms $\{X_i\}_{1 \leqslant i \leqslant c}$ of B. Hence, we get $1 \leqslant c \leqslant d$ and $\mu_B(C) = c$.

(4), (5), (6) Let us consider the exact sequence

$$0 \to C \to B(-1) \to (B/\mathfrak{a})(-1) \to 0 \quad (*_5)$$

of graded B-modules. Then, we have

$$\begin{aligned} \ell_A(C_n) &= \ell_A(B_{n-1}) - \ell_A([B/\mathfrak{a}]_{n-1}) \\ &= \binom{n-1+d-1}{d-1} - \binom{n-1+d-c-1}{d-c-1} \end{aligned}$$

$$= \binom{n+d-1}{d-1} - \binom{n+d-2}{d-2} - \binom{n+d-c-1}{d-c-1} + \binom{n+d-c-2}{d-c-2}$$

for all $n \geqslant 0$ (resp. $n \geqslant 2$) if $1 \leqslant c \leqslant d-1$ (resp. $c = d$). Therefore, our assertions (4), (5), and (6) follow by Proposition 2.8.

(7) We have

$$HS_C(z) = HS_B(z)z - HS_{B/\mathfrak{a}}(z)z = \frac{z - (1-z)^c z}{(1-z)^d}$$

by the above exact sequence $(*_5)$, where $HS_*(z)$ denotes the Hilbert series of the graded modules. We also have

$$HS_S(z) = HS_L(z) + HS_C(z) = \frac{\{\ell_A(I^2/QI)+1\}z - (1-z)^c z}{(1-z)^d}$$

by the exact sequence

$$0 \to L \to S \to C \to 0 \quad (\dagger)$$

and the isomorphism $L \cong D(-1)$ of graded T-modules (Lemma 2.7). Then, because

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I)\}z}{(1-z)^d} - (1-z)HS_S(z)$$

([VP], [RV3, Proposition 6.3]), we can get the required result.

We prove now (2), (3). We have $\mathrm{depth}_T C = d - c + 1$ by the exact sequence $(*_5)$ so that $\mathrm{depth} G \geqslant d - c$ and, if $c \geqslant 3$, then $\mathrm{depth} G = d - c$ by Lemma 2.11. Let us consider the case where $c = 2$ and we need to show $\mathrm{depth} G = d - 2$. Assume $\mathrm{depth} G \geqslant d - 1$; then S is a Cohen–Macaulay T-module by [GNO, Proposition 2.2]. Taking the local cohomology functors $\mathrm{H}^i_M(*)$ of T with respect to the graded maximal ideal $M = \mathfrak{m}T + T_+$ to the above exact sequence $(\dagger)$ of graded T-modules, we get a monomorphism

$$\mathrm{H}^{d-1}_M(C) \hookrightarrow \mathrm{H}^d_M(L)$$

of graded T-modules. Because $C \cong (X_1, X_2)B(-1)$, we have $\mathrm{H}^{d-1}_M(C) \cong \mathrm{H}^{d-2}_M(B/(X_1, X_2)B)(-1)$ as graded T-modules so that $[\mathrm{H}^{d-1}_M(C)]_{-d+3} \neq (0)$ (notice that $B/(X_1, X_2)B \cong (A/\mathfrak{m})[X_3, X_4, \ldots, X_d]$). On the other hand, we have $[\mathrm{H}^d_M(L)]_n = (0)$ for all $n \geqslant -d+2$, because $L \cong D(-1) \cong (I^2/QI) \otimes_A (A/\mathrm{Ann}_A(I^2/QI))[X_1, X_2, \ldots, X_d](-1)$ by Lemma 2.7. However, this is impossible. Therefore, depth $G = d - 2$ if $c = 2$. ∎

We prove now Theorem 1.2. Assume assertion (1) in Theorem 1.2. Then we have an isomorphism $C \cong \mathfrak{a}(-1)$ as graded B-modules for a graded ideal $\mathfrak{a}$ in B by Theorem 3.1. Once we are able to show $I^4 = QI^3$, then, because $C = TC_2$ by Lemma 2.1(5), the ideal $\mathfrak{a}$ is generated by linearly independent linear forms $\{X_i\}_{1 \leqslant i \leqslant c}$ of B with $c = \ell_A(I^3/QI^2)$ (recall that $\mathfrak{a}_1 \cong C_2 \cong I^3/QI^2$ by Lemma 2.1(2)). Therefore, the implication (1) ⇒ (3) in Theorem 1.2 follows. We also notice that, the last assertions of Theorem 1.2 follow by Proposition 3.3.

Thus our Theorem 1.2 has been proven modulo the following theorem.

THEOREM 3.4. *Assume that I is integrally closed. Suppose that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. Then $I^4 = QI^3$.*

Proof. We proceed by induction on d. Suppose that $d = 1$. Then the result follows by [HM, Proposition 4.6] since $\mathrm{e}_1(I) = \sum_{i \geqslant 0} \ell_A(I^{i+1}/QI^i)$.

Assume that $d \geqslant 2$ and that our assertion holds true for $d-1$. Since the residue class field $A/\mathfrak{m}$ of A is infinite, without loss of generality, we may assume that a_1 is a superficial element of I and $I/(a_1)$ is integrally closed (cf. [I1, p. 648], [RV3, Proposition 1.1]). We set $A' = A/(a_1), I' = I/(a_1), Q' = Q/(a_1)$. We then have $\mathrm{e}_1(I') = \mathrm{e}_0(I') - \ell_{A'}(A'/I') + \ell_{A'}(I'^2/Q'I') + 1$ because $\mathrm{e}_i(I') = \mathrm{e}_i(I)$ for $0 \leqslant i \leqslant d-1$, $\ell_{A'}(A'/I') = \ell_A(A/I)$, and since $(a_1) \cap I^2 = a_1 I$, $\ell_{A'}(I'^2/Q'I') = \ell_A(I^2/QI)$. Then the inductive assumption on d says that $I'^4 = Q'I'^3$ holds true. If $\operatorname{depth} \mathrm{G}(I') > 0$ then, thanks to Sally's technique (cf. [S1], [HM, Lemma 2.2]), $a_1 t$ is a non-zero divisor on G. Then we have $I^4 = QI^3$.

Assume that $\operatorname{depth} \mathrm{G}(I') = 0$. Then because $\mathrm{e}_1(I') = \mathrm{e}_0(I') - \ell_{A'}(A'/I') + \ell_{A'}(I'^2/Q'I') + 1$ and $I'^4 = Q'I'^3$, we have $\ell_{A'}(I'^3/Q'I'^2) = d-1$ by Proposition 3.3(2). Since $I'^3/Q'I'^2$ is a homomorphic image of I^3/QI^2, we have $\ell_A(C_2) = \ell_A(I^3/QI^2) \geqslant d-1$. Let us now take an isomorphism

$$\varphi : C \to \mathfrak{a}(-1)$$

of graded B-modules, where $\mathfrak{a}$ is a graded ideal of B (cf. Theorem 3.1). Then since $\ell_A(\mathfrak{a}_1) = \ell_A(C_2) \geqslant d-1$, we have $Y_1, Y_2, \ldots, Y_{d-1} \in \mathfrak{a}$ where $Y_1, Y_2, \ldots, Y_{d-1}$ denote linearly independent linear forms of B, which we enlarge to a basis $Y_1, Y_2, \ldots, Y_{d-1}, Y_d$ of B_1. If $\mathfrak{a} = (Y_1, Y_2, \ldots, Y_{d-1})B$ then, because $C = TC_2$, we have $I^4 = QI^3$ by Lemma 2.1(5). Assume that $\mathfrak{a} \neq (Y_1, Y_2, \ldots, Y_{d-1})B$. Then, since $B = k[Y_1, Y_2, \ldots, Y_d]$, the ideal $\mathfrak{a}/(Y_1, Y_2, \ldots, Y_{d-1})$ is principal so that $\mathfrak{a} = (Y_1, Y_2, \ldots, Y_{d-1}, Y_d^{\ell})B$ for some $\ell \geqslant 1$.

We need the following.

Claim 1. We have $\ell = 1$ or $\ell = 2$.

Proof of Claim 1. Assume that $\ell \geqslant 3$. Then $I^4/[QI^3 + \mathfrak{m}I^4] \cong [C/MC]_3 = (0)$, where $M = \mathfrak{m}T + T_+$ denotes the graded maximal ideal of T. Therefore, $I^4 = QI^3$ by Nakayama's lemma so that $C = TC_2$, which is impossible. Thus $\ell = 1$ or $\ell = 2$. ∎

We have to show that $\ell = 1$. Assume that $\ell = 2$. Let us write, for each $1 \leqslant i \leqslant d$, $Y_i = \overline{b_i t}$ with $b_i \in Q$, where $\overline{b_i t}$ denotes the image of $b_i t \in T$ in B. We notice here that $Q = (b_1, b_2, \ldots, b_d)$, because $Y_1, Y_2, \ldots, Y_d$ forms a k-basis of B_1.

Let us choose elements $f_i \in C_2$ for $1 \leqslant i \leqslant d-1$ and $f_d \in C_3$ so that $\varphi(f_i) = Y_i$ for $1 \leqslant i \leqslant d-1$ and $\varphi(f_d) = Y_d^2$. Let $z_i \in I^3$ for $1 \leqslant i \leqslant d-1$ and $z_d \in I^4$ so that $\{f_i\}_{1 \leqslant i \leqslant d-1}$ and f_d are, respectively, the images of $\{z_i t^2\}_{1 \leqslant i \leqslant d-1}$ and $z_d t^3$ in C. Let us now consider the relations $Y_d^2 f_i = Y_i f_d$ in C for $1 \leqslant i \leqslant d-1$, that is

$$b_d^2 z_i - b_i z_d \in Q^3 I^2$$

for $1 \leqslant i \leqslant d-1$. Notice that

$$(b_1, b_d) \cap Q^3 I^2 = (b_1, b_d) Q^2 I^2$$

by Lemma 2.3 and write

$$b_d^2 z_1 - b_1 z_d = b_1 \tau_1 + b_d \tau_d \tag{1}$$

with $\tau_1, \tau_d \in Q^2 I^2$. Then we have

$$b_d(b_d z_1 - \tau_d) = b_1(z_d + \tau_1)$$

so that $b_d z_1 - \tau_d \in (b_1)$ because b_1, b_d forms a regular sequence on A. Since $\tau_d \in (b_1, b_d) \cap Q^2 I^2 = (b_1, b_d) Q I^2$ by Lemma 2.3, there exist elements $\tau_1', \tau_d' \in Q I^2$ such that $\tau_d = b_1 \tau_1' + b_d \tau_d'$. Then by the equality (1) we have

$$b_d^2(z_1 - \tau_d') = b_1(z_d + \tau_1 + b_d \tau_1') \tag{2}$$

so that we have $z_1 - \tau_d' \in (b_1)$. Hence $z_1 \in QI^2 + (b_1)$. The same argument works for each $1 \leqslant i \leqslant d-1$ to see that $z_i \in QI^2 + (b_i)$. Therefore, because $I^3 = QI^2 + (z_1, z_2, \ldots, z_{d-1})$, we have $I^3 \subseteq b_d I^2 + (b_1, b_2, \ldots, b_{d-1})$ and hence

$$I^4 \subseteq b_d^2 I^2 + (b_1, b_2, \ldots, b_{d-1}).$$

Then, because $z_d + \tau_1 + b_d\tau_1' \in I^4$, there exist elements $h \in I^2$ and $\eta \in (b_1, b_2, \ldots, b_{d-1})$ such that $z_d + \tau_1 + b_d\tau_1' = b_d^2 h + \eta$. Then we have

$$b_d^2(z_1 - \tau_d' - b_1 h) = b_1\eta \tag{3}$$

by the equality (2). Since $b_1, b_2, \ldots, b_d$ is a regular sequence on A, $\eta \in (b_d^2) \cap (b_1, b_2, \ldots, b_{d-1}) = b_d^2(b_1, b_2, \ldots, b_{d-1})$. Write $\eta = b_d^2\eta'$ with $\eta' \in (b_1, b_2, \ldots, b_{d-1})$, then we have

$$b_d^2(z_1 - \tau_d' - b_1 h) = b_1 b_d^2 \eta'$$

by the equality (3), so that $z_1 - \tau_d' - b_1 h = b_1\eta'$. Then we have $b_1\eta' = z_1 - \tau_d' - b_1 h \in Q^2 \cap I^3 = Q^2 I$, since $Q^2 \cap I^3 \subseteq Q^2 \cap \overline{Q^3} = Q^2\overline{Q} = Q^2 I$ (cf. [H, I2]), where $\overline{J}$ denotes the integral closure of an ideal J. Hence $z_1 \in QI^2$, because $\tau_d', b_1 h, b_1\eta' \in QI^2$. Therefore, $f_1 = 0$ in C, which is impossible. Thus $\ell = 1$ so that we have $\mathfrak{a} = (X_1, X_2, \ldots, X_d)B$. Therefore, $C = TC_2$, that is $I^4 = QI^3$. This completes the proof of Theorem 3.4 and that of Theorem 1.2 as well. ∎

§4. Consequences

The purpose of this section is to present some consequences of Theorem 1.2. Let us begin with the following which is exactly the case where $c = 1$ in Theorem 1.2.

THEOREM 4.1. *Assume that I is integrally closed. Then the following conditions are equivalent.*

(1) $C \cong B(-2)$ *as graded T-modules.*
(2) $e_1(I) = e_0(I) - \ell_A(A/I) + \ell_A(I^2/IQ) + 1$, *and if $d \geqslant 2$ then* $e_2(I) \neq e_1(I) - e_0(I) + \ell_A(A/I)$.
(3) $\ell_A(I^3/QI^2) = 1$ *and* $I^4 = QI^3$.

When this is the case, the following assertions follow.

(i) $\operatorname{depth} G \geqslant d - 1$.
(ii) $e_2(I) = e_1(I) - e_0(I) + \ell_A(A/I) + 1$ *if* $d \geqslant 2$.
(iii) $e_3(I) = 1$ *if* $d \geqslant 3$, *and* $e_i(I) = 0$ *for* $4 \leqslant i \leqslant d$.

(iv) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - \ell_A(I^2/QI)\}z + \{\ell_A(I^2/QI) - 1\}z^2 + z^3}{(1-z)^d}.$$

Proof. For (1) $\Leftrightarrow$ (2), (1) $\Rightarrow$ (3) and the last assertions see Theorem 1.2 with $c = 1$.

(3) $\Rightarrow$ (1): Since $\mathfrak{m}I^{n+1} = Q^{n-1}I^2$ for all $n \geqslant 2$, we have $\mathfrak{m}C = (0)$ by Lemma 2.1(4). Then we have an epimorphism $B(-2) \to C \to 0$ of graded T-modules, which must be an isomorphism because $\dim_T C = d$ (Proposition 2.2). ∎

Let $\widetilde{J} = \bigcup_{n\geqslant 1}[J^{n+1} :_A J^n] = \bigcup_{n\geqslant 1} J^{n+1} :_A (a_1^n, a_2^n, \ldots, a_d^n)$ denote the Ratliff–Rush closure of an $\mathfrak{m}$-primary ideal J in A, which is the largest $\mathfrak{m}$-primary ideal in A such that $J \subseteq \widetilde{J}$ and $\mathrm{e}_i(\widetilde{J}) = \mathrm{e}_i(J)$ for all $0 \leqslant i \leqslant d$ (cf. [RR]).

Let us note the following remark.

REMARK 4.2. Assume that I is integrally closed. Then, by [H, I1], $Q^n \cap \overline{I^{n+1}} = Q^n \cap \overline{Q^{n+1}} = Q^n\overline{Q} = Q^n I$ holds true for $n \geqslant 1$, where $\overline{J}$ denotes the integral closure of an ideal J. Thus, we have $Q^n \cap \widetilde{I^{n+1}} = Q^n I$ for all $n \geqslant 1$.

The following result corresponds to the case where $c = d$ in Theorem 1.2. In Section 5 we give an example of the maximal ideal which satisfies assertion (1) in Theorem 4.3.

THEOREM 4.3. *Suppose that* $d \geqslant 2$ *and assume that* I *is integrally closed. Then the following conditions are equivalent:*

(1) $C \cong B_+(-1)$ *as graded* T*-modules.*
(2) $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1,\ \mathrm{e}_2(I) = \ell_A(I^2/QI) + 1,$ *and* $\mathrm{e}_i(I) = 0$ *for all* $3 \leqslant i \leqslant d$.
(3) $\ell_A(\widetilde{I^2}/I^2) = 1$ *and* $\widetilde{I^{n+1}} = Q\widetilde{I^n}$ *for all* $n \geqslant 2$.

When this is the case, the associated graded ring G *of* I *is a Buchsbaum ring with* depth $G = 0$ *and the Buchsbaum invariant* $\mathbb{I}(G) = d$.

Proof. We set $c = \ell_A(I^3/QI^2)$ and $\mathcal{F} = \{\widetilde{I^n}\}_{n\geqslant 0}$. Let $\mathrm{R}'(\mathcal{F}) = \sum_{n\in\mathbb{Z}} \widetilde{I^n}t \subseteq A[t, t^{-1}]$ and $\mathrm{G}(\mathcal{F}) = \mathrm{R}'(\mathcal{F})/t^{-1}\mathrm{R}'(\mathcal{F})$. Let $\mathrm{e}_i(\mathcal{F})$ denote the ith Hilbert coefficients of the filtration $\mathcal{F}$ for $0 \leqslant i \leqslant d$.

(1) $\Rightarrow$ (2) follows from Theorem 1.2, because $c = \ell_A(C_2) = d$.

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$(2) \Rightarrow (1)$ Because $\mathrm{e}_2(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$ and $\mathrm{e}_i(I) = 0$ for all $3 \leqslant i \leqslant d$, we have $c = d$ by Theorem 1.2. Therefore $C \cong B_+(-1)$ as graded T-modules.

$(1) \Rightarrow (3)$ Since $c = d$, we have $\operatorname{depth} G = 0$ by Theorem 1.2(ii). We apply local cohomology functors $\mathrm{H}^i_M(*)$ of T with respect to the graded maximal ideal $M = \mathfrak{m}T + T_+$ of T to the exact sequences

$$0 \to I^2R(-1) \to IR(-1) \to G_+ \to 0 \quad \text{and}$$
$$0 \to I^2T(-1) \to I^2R(-1) \to C \to 0$$

of graded T-modules and get derived monomorphisms

$$\mathrm{H}^0_M(G_+) \hookrightarrow \mathrm{H}^1_M(I^2R)(-1) \quad \text{and} \quad \mathrm{H}^1_M(I^2R)(-1) \hookrightarrow \mathrm{H}^1_M(C)$$

because $\operatorname{depth}_T IR > 0$ and $\operatorname{depth}_T I^2T \geqslant 2$ (recall that T is a Cohen–Macaulay ring with $\dim T = d+1$ and $\operatorname{depth}_T T/I^2T \geqslant 1$ by Lemma 2.3). We furthermore have $\mathrm{H}^1_M(C) \cong (B/B_+)(-1)$ since $C \cong B_+(-1)$. Since I is integrally closed, we have $[\mathrm{H}^0_M(G)]_0 = (0)$ so that $\mathrm{H}^0_M(G) \cong \mathrm{H}^0_M(G_+) \neq (0)$. Then because $\ell_A(B/B_+) = 1$, we have isomorphisms

$$\mathrm{H}^0_M(G) \cong \mathrm{H}^1_M(I^2R)(-1) \cong \mathrm{H}^1_M(C) \cong B/B_+(-1)$$

of graded B-modules and hence $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_1 \cong B/B_+$. Then since $[\mathrm{H}^0_M(G)]_1 \cong \widetilde{I^2}/I^2$ we have $\ell_A(\widetilde{I^2}/I^2) = 1$. Hence we have

$$\begin{aligned} \mathrm{e}_1(\mathcal{F}) &= \mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1 \\ &= \mathrm{e}_0(\mathcal{F}) - \ell_A(A/I) + \ell_A(\widetilde{I^2}/QI) + 1 - \ell_A(\widetilde{I^2}/I^2) \\ &= \mathrm{e}_0(\mathcal{F}) - \ell_A(A/I) + \ell_A(\widetilde{I^2}/Q \cap \widetilde{I^2}) \end{aligned}$$

because $\widetilde{I} = I$, $Q \cap \widetilde{I^2} = QI$ by Remark 4.2, and $\mathrm{e}_i(\mathcal{F}) = \mathrm{e}_i(I)$ for $i = 0, 1$. Therefore, $\widetilde{I^{n+1}} = Q\widetilde{I^n}$ for all $n \geqslant 2$ by [GR, Theorem 2.2].

$(3) \Rightarrow (2)$ Because $Q \cap \widetilde{I^2} = QI$ by Remark 4.2 and $\widetilde{I^{n+1}} = Q\widetilde{I^n}$ for all $n \geqslant 2$, we have $\mathrm{e}_1(\mathcal{F}) = \mathrm{e}_0(\mathcal{F}) - \ell_A(A/I) + \ell_A(\widetilde{I^2}/QI)$ and $\mathrm{G}(\mathcal{F})$ is a Cohen–Macaulay ring by [GR, Theorem 2.2]. Then, since $\ell_A(\widetilde{I^2}/I^2) = 1$, we have $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. We furthermore have $\mathrm{e}_2(I) = \mathrm{e}_2(\mathcal{F}) = \ell_A(\widetilde{I^2}/QI) = \ell_A(I^2/QI) + 1 = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I)$, and $\mathrm{e}_i(I) = \mathrm{e}_i(\mathcal{F}) = 0$ for $3 \leqslant i \leqslant d$, because $\mathrm{G}(\mathcal{F})$ is a Cohen–Macaulay ring (cf. [HM, Proposition 4.6]).

Assume one of the equivalent conditions. We have $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_1$ by the proof of the implication (1) ⇒ (3). Let $n \geqslant 3$ be an integer. We then have

$$\widetilde{I^n}/I^n = \widetilde{I^n} \cap I^{n-1}/I^n \cong [\mathrm{H}^0_M(G)]_{n-1} = (0)$$

because $\widetilde{I^n} = Q^{n-2}\widetilde{I^2} \subseteq I^{n-1}$. Therefore, we have $\widetilde{I^n} = I^n$ for all $n \geqslant 3$.

We set $W = \mathrm{R}'(\mathcal{F})/R'$ and look at the exact sequence

$$0 \to R' \to \mathrm{R}'(\mathcal{F}) \to W \to 0 \quad (*')$$

of graded R'-modules. Since $\widetilde{I^n} = I^n$ for all $n \neq 2$, we have $W = W_2 = \widetilde{I^2}/I^2$ so that $\ell_A(W) = 1$. Then, because $\mathrm{G}(\mathcal{F}) = \mathrm{R}'(\mathcal{F})/t^{-1}\mathrm{R}'(\mathcal{F})$ is a Cohen–Macaulay ring, so is $\mathrm{R}'(\mathcal{F})$. Let $M' = (\mathfrak{m}, R_+, t^{-1})R'$ be the unique graded maximal ideal in R'. Then applying local cohomology functors $\mathrm{H}^i_{M'}(*)$ to the exact sequence $(*')$ yields $\mathrm{H}^i_{M'}(R') = (0)$ for all $i \neq 1, d+1$ and $\mathrm{H}^1_{M'}(R') = W$. Since $\mathfrak{m}W = (0)$, we have $\mathfrak{m}\mathrm{H}^1_{M'}(R') = (0)$. Thus, R' is a Buchsbaum ring with the Buchsbaum invariant

$$\mathbb{I}(R') = \sum_{i=0}^{d} \binom{d}{i} \ell_A(\mathrm{H}^i_{M'}(R')) = d$$

and hence so is the graded ring $G = R'/t^{-1}R'$. This completes the proof of Theorem 4.3. □

In the rest of this section, we explore the relationship between the inequality of Northcott [N] and the structure of the graded module C of an integrally closed ideal.

It is well known that the inequality $\mathrm{e}_1(I) \geqslant \mathrm{e}_0(I) - \ell_A(A/I)$ holds true [N] and the equality holds if and only if $I^2 = QI$ [H, Theorem 2.1]. When this is the case, the associated graded ring G of I is Cohen–Macaulay.

Suppose that I is integrally closed and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1$ then, thanks to [I1, Corollary 14], we have $I^3 = QI^2$ and the associated graded ring G of I is Cohen–Macaulay. Thus the integrally closed ideal I with $\mathrm{e}_1(I) \leqslant \mathrm{e}_0(I) - \ell_A(A/I) + 1$ seems understood in a satisfactory way. In this section, we briefly study the integrally closed ideals I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$, and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$.

Let us begin with the following.

THEOREM 4.4. *Assume that I is integrally closed. Suppose that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$ and $I^3 \neq QI^2$. Then the following assertions hold true.*

(1) $\ell_A(I^2/QI) = \ell_A(I^3/QI^2) = 1$, *and* $I^4 = QI^3$.
(2) $C \cong B(-2)$ *as graded* T*-modules.*
(3) depth $G = d - 1$.
(4) $\mathrm{e}_2(I) = 3$ *if* $d \geqslant 2$, $\mathrm{e}_3(I) = 1$ *if* $d \geqslant 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leqslant i \leqslant d$.
(5) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \frac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 1\}z + z^3}{(1-z)^d}.$$

Proof. Because $I^3 \neq QI^2$, it follows from Corollary 2.10 that

$$0 < \ell_A(I^2/QI) < \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) = 2.$$

Thus, $\ell_A(I^2/QI) = 1$ and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. Let $I^2 = QI + (xy)$ with x, $y \in I \backslash Q$. Then $I^3 = QI^2 + (x^2y)$, so that $\ell_A(I^3/QI^2) = 1$ since $I^3 \neq QI^2$ and $\mathfrak{m}I^2 \subseteq QI$. Thanks to Theorem 1.2, $C \cong B(-2)$ as graded T-modules, so that assertions (1), (2), (4), and (5) follow, and depth $G \geqslant d - 1$ by Theorem 4.1. Since $I^3 \subseteq QI$, G is not a Cohen–Macaulay ring, for otherwise $I^3 = Q \cap I^3 = QI^2$, so that depth$G = d - 1$. This completes the proof of Theorem 4.4. □

Notice that the following result also follows by [RV3, Theorem 4.6].

COROLLARY 4.5. *Assume that* I *is integrally closed and suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 2$. *Then* depth $G \geqslant d - 1$ *and* $I^4 = QI^3$, *and the graded ring* G *is Cohen–Macaulay if and only if* $I^3 = QI^2$.

Before closing this section, we briefly study the integrally closed ideal I with $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$. Suppose that $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$ then we have

$$0 < \ell_A(I^2/QI) \leqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) = 3$$

by Corollary 2.10. If $\ell_A(I^2/QI) = 1$ then we have depth$G \geqslant d - 1$ by [RV1, W]. If $\ell_A(I^2/QI) = 3$ then the equality $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI)$ holds true, so that $I^3 = QI^2$ and the associated graded ring G of I is Cohen–Macaulay by Corollary 2.10. Thus we need to consider the following.

THEOREM 4.6. *Suppose that* $d \geqslant 2$. *Assume that* I *is integrally closed and* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$ *and* $\ell_A(I^2/QI) = 2$. *Let* $c = \ell_A(I^3/QI^2)$. *Then the following assertions hold true.*

(1) *Either* $C \cong B(-2)$ *as graded* T*-modules or there exists an exact sequence*

$$0 \to B(-3) \to B(-2) \oplus B(-2) \to C \to 0$$

of graded T*-modules.*

(2) $1 \leqslant c \leqslant 2$ *and* $I^4 = QI^3$.

(3) *Suppose* $c = 1$*; then* $\mathrm{depth} G \geqslant d-1$ *and* $\mathrm{e}_2(I) = 4$, $\mathrm{e}_3(I) = 1$ *if* $d \geqslant 3$, *and* $\mathrm{e}_i(I) = 0$ *for* $4 \leqslant i \leqslant d$.

(4) *Suppose* $c = 2$*; then* $\mathrm{depth} G = d-2$ *and* $\mathrm{e}_2(I) = 3$, $\mathrm{e}_3(I) = -1$ *if* $d \geqslant 3$, $\mathrm{e}_4(I) = -1$ *if* $d \geqslant 4$, *and* $\mathrm{e}_i(I) = 0$ *for* $5 \leqslant i \leqslant d$.

(5) *The Hilbert series* $HS_I(z)$ *is given by*

$$HS_I(z) = \begin{cases} \dfrac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 2\}z + z^2 + z^3}{(1-z)^d} & \text{if } c = 1, \\ \dfrac{\ell_A(A/I) + \{\mathrm{e}_0(I) - \ell_A(A/I) - 2\}z + 3z^3 - z^4}{(1-z)^d} & \text{if } c = 2. \end{cases}$$

Proof. Since $\ell_A(I^2/QI) = 2$ and $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$, we have $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_A(I^2/QI) + 1$. We also have $1 \leqslant \ell_A(I^3/QI^2) \leqslant 2$ (see the proof of [RV2, Propositions 2.1 and 2.2]). Then, thanks to Theorem 1.2, $C \cong X_1B(-1)$ or $C \cong (X_1, X_2)B(-1)$ as graded T-modules, where X_1, X_2 denote the linearly independent linear forms of B. Thus all assertions follow by Theorem 1.2. ∎

We remark that $\ell_A(I^2/QI)$ measures how far is the multiplicity of I from the minimal value (see [RV3, Corollary 2.1]). If $\ell_A(I^2/QI) \leqslant 1$, then depth $G \geqslant d-1$, but it is still open the problem whether $\mathrm{depth} G \geqslant d-2$, assuming $\ell_A(I^2/QI) = 2$. Theorem 4.6 confirms the conjectured bound.

COROLLARY 4.7. *Assume that* I *is integrally closed. Suppose that* $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 3$. *Then* depth $G \geqslant d-2$.

Proof. We have $0 < \ell_A(I^2/QI) \leqslant \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_A(A/I) = 3$ by Corollary 2.10. If $\ell_A(I^2/QI) = 1$ or $\ell_A(I^2/QI) = 3$ then we have $\mathrm{depth} G \geqslant d-1$ as above. Suppose that $\ell_A(I^2/QI) = 2$ then we have $\mathrm{depth} G \geqslant d-2$ by Theorem 4.6(3), (4). This completes a proof of Corollary 4.7. ∎

§5. An example

The goal of this section is to construct an example of a Cohen–Macaulay local ring with the maximal ideal $\mathfrak{m}$ satisfying the equality in

Theorem 1.2(1). The class of examples we exhibit includes an interesting example given by Wang, see [RV3, Example 3.2].

THEOREM 5.1. *Let $d \geqslant c \geqslant 1$ be integers. Then there exists a Cohen–Macaulay local ring $(A, \mathfrak{m})$ such that*

$$d = \dim A, \quad \mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{m}) + \ell_A(\mathfrak{m}^2/Q\mathfrak{m}), \quad \textit{and} \quad c = \ell_A(\mathfrak{m}^3/Q\mathfrak{m}^2)$$

for some minimal reduction $Q = (a_1, a_2, \ldots, a_d)$ of $\mathfrak{m}$.

To construct necessary examples we may assume that $c = d$. In fact, suppose that $0 < c < d$ and assume that we have already chosen a certain Cohen–Macaulay local ring $(A_0, \mathfrak{m}_0)$ such that $c = \dim A_0$, $\mathrm{e}_1(\mathfrak{m}_0) = \mathrm{e}_0(\mathfrak{m}_0) + \ell_{A_0}(\mathfrak{m}_0^2/Q_0\mathfrak{m}_0)$, and $c = \ell_{A_0}(\mathfrak{m}_0^3/Q_0\mathfrak{m}_0^2)$ with $Q_0 = (a_1, a_2, \ldots, a_c)A_0$ a minimal reduction of $\mathfrak{m}_0$. Let $n = d - c$ and let $A = A_0[[X_1, X_2, \ldots, X_n]]$ be the formal power series ring over the ring A_0. We set $\mathfrak{m} = \mathfrak{m}_0 A + (X_1, X_2, \ldots, X_n)A$ and $Q = Q_0 A + (X_1, X_2, \ldots, X_n)A$. Then A is a Cohen–Macaulay local ring with $\dim A = d$ and maximal ideal $\mathfrak{m} = \mathfrak{m}_0 A + (X_1, X_2, \ldots, X_n)A$. The ideal Q is a reduction of $\mathfrak{m}$ and because $X_1, X_2, \ldots, X_n$ forms a super regular sequence in A with respect to $\mathfrak{m}$ (recall that $\mathrm{G}(\mathfrak{m}) = \mathrm{G}(\mathfrak{m}_0)[Y_1, Y_2, \ldots, Y_n]$ is the polynomial ring, where Y_i's are the initial forms of X_i's), we have $\mathrm{e}_i(\mathfrak{m}) = \mathrm{e}_i(\mathfrak{m}_0)$ for $i = 0, 1$, $\mathfrak{m}^2/Q\mathfrak{m} \cong \mathfrak{m}_0^2/Q_0\mathfrak{m}_0$, and $\mathfrak{m}^3/Q\mathfrak{m}^2 \cong \mathfrak{m}_0^3/Q_0\mathfrak{m}_0^2$. Thus we have $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{m}) + \ell_A(\mathfrak{m}^2/Q\mathfrak{m})$ and $\ell_A(\mathfrak{m}^3/Q\mathfrak{m}^2) = c$. This observation allows us to concentrate our attention on the case where $c = d$.

Let $m \geqslant 0$ and $d \geqslant 1$ be integers. Let

$$D = k[[\{X_j\}_{1 \leqslant j \leqslant m}, Y, \{V_i\}_{1 \leqslant i \leqslant d}, \{Z_i\}_{1 \leqslant i \leqslant d}]]$$

be the formal power series ring with $m + 2d + 1$ indeterminates over an infinite field k, and let

$$\begin{aligned}\mathfrak{a} = {} & [(X_j \mid 1 \leqslant j \leqslant m) + (Y)] \cdot [(X_j \mid 1 \leqslant j \leqslant m) + (Y) + (V_i \mid 1 \leqslant i \leqslant d)] \\ & + (V_i V_j \mid 1 \leqslant i, j \leqslant d, i \neq j) + (V_i^3 - Z_i Y \mid 1 \leqslant i \leqslant d).\end{aligned}$$

We set $A = D/\mathfrak{a}$ and denote the images of X_j, Y, V_i, and Z_i in A by x_j, y, v_i, and a_i, respectively. Then, since $\sqrt{\mathfrak{a}} = (X_j \mid 1 \leqslant j \leqslant m) + (Y) + (V_i \mid 1 \leqslant i \leqslant d)$, we have $\dim A = d$. Let $\mathfrak{m} = (x_j \mid 1 \leqslant j \leqslant m) + (y) + (v_i \mid 1 \leqslant i \leqslant d) + (a_i \mid 1 \leqslant i \leqslant d)$ be the maximal ideal in A and we set $Q = (a_i \mid 1 \leqslant i \leqslant d)$. Then, $\mathfrak{m}^2 = Q\mathfrak{m} + (v_i^2 \mid 1 \leqslant i \leqslant d)$, $\mathfrak{m}^3 = Q\mathfrak{m}^2 + Qy$, and $\mathfrak{m}^4 = Q\mathfrak{m}^3$. Therefore Q is a minimal reduction of $\mathfrak{m}$, and $a_1, a_2, \ldots, a_d$ is a system of parameters for A.

We are now interested in the Hilbert coefficients $\mathrm{e}_i(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ as well as the structure of the associated graded ring $\mathrm{G}(\mathfrak{m})$ and the module $\mathrm{C}_Q(\mathfrak{m})$ of $\mathfrak{m}$.

THEOREM 5.2. *The following assertions hold true.*

(1) *A is a Cohen–Macaulay local ring with* $\dim A = d$.
(2) $\mathrm{C}_Q(\mathfrak{m}) \cong B_+(-1)$ *as graded T-modules. Therefore,* $\ell_A(\mathfrak{m}^3/Q\mathfrak{m}^2) = d$.
(3) $\mathrm{e}_0(\mathfrak{m}) = m + 2d + 2$, $\mathrm{e}_1(\mathfrak{m}) = m + 3d + 2$.
(4) $\mathrm{e}_2(\mathfrak{m}) = d + 1$ *if* $d \geqslant 2$, *and* $\mathrm{e}_i(\mathfrak{m}) = 0$ *for all* $3 \leqslant i \leqslant d$.
(5) $\mathrm{G}(\mathfrak{m})$ *is a Buchsbaum ring with* $\mathrm{depth}\,\mathrm{G}(\mathfrak{m}) = 0$ *and* $\mathbb{I}(\mathrm{G}(\mathfrak{m})) = d$.
(6) *The Hilbert series $HS_{\mathfrak{m}}(z)$ of A is given by*

$$HS_{\mathfrak{m}}(z) = \frac{1 + \{m + d + 1\}z + \sum_{j=3}^{d+2} (-1)^{j-1} \binom{d+1}{j-1} z^j}{(1-z)^d}.$$

Notice that Wang's example before quoted corresponds to the particular case $m = 0$ and $d = 2$.

Let us divide the proof of Theorem 5.2 into two steps. Let us begin with the following.

PROPOSITION 5.3. *Let* $\mathfrak{p} = (X_j | 1 \leqslant j \leqslant m) + (Y) + (V_i | 1 \leqslant i \leqslant d)$ *in D. Then* $\ell_{D_{\mathfrak{p}}}(A_{\mathfrak{p}}) = m + 2d + 2$.

Proof. Let $\widetilde{k} = k[\{Z_i\}_{1 \leqslant i \leqslant d}, \{\frac{1}{Z_i}\}_{1 \leqslant i \leqslant d}]$ and $\widetilde{D} = D[\{\frac{1}{Z_i}\}_{1 \leqslant i \leqslant d}]$. We set $X'_j = \frac{X_j}{Z_1}$ for $1 \leqslant j \leqslant m$, $V'_i = \frac{V_i}{Z_1}$ for $1 \leqslant i \leqslant d$, and $Y' = \frac{Y}{Z_1}$. Then we have

$$\begin{aligned}
\widetilde{D} &= \widetilde{k}[\{X'_j | 1 \leqslant j \leqslant m\}, Y', \{V'_i | 1 \leqslant i \leqslant d\}],\\
\mathfrak{a}\widetilde{D} &= [(X'_j | 1 \leqslant j \leqslant m) + (Y')] \cdot [(X'_j | 1 \leqslant j \leqslant m) + (Y') + (V'_i | 1 \leqslant i \leqslant d)]\\
&\quad + (V'_i V'_j | 1 \leqslant i, j \leqslant d,\ i \neq j) + \left(\frac{Z_1^2}{Z_i} {V'_i}^3 - Y' | 1 \leqslant i \leqslant d \right),
\end{aligned}$$

and $\{X'_j\}_{1 \leqslant j \leqslant m}$, Y', and $\{V'_i\}_{1 \leqslant i \leqslant d}$ are algebraically independent over $\widetilde{k}$. Let

$$W = \widetilde{k}[\{X'_j | 1 \leqslant j \leqslant m\}, \{V'_i | 1 \leqslant i \leqslant d\}]$$

in $\widetilde{D}$ and

$$\begin{aligned}
\mathfrak{b} &= [(X'_j | 1 \leqslant j \leqslant m) + ({V'_1}^3)] \cdot [(X'_j | 1 \leqslant j \leqslant m) + (V'_i | 1 \leqslant i \leqslant d)]\\
&\quad + (V'_i V'_j | 1 \leqslant i, j \leqslant d,\ i \neq j) + \left(\frac{Z_1^2}{Z_i} {V'_i}^3 - Z_1 {V'_1}^3 | 2 \leqslant i \leqslant d \right)
\end{aligned}$$

in W. Then substituting Y' with $Z_1{V_1'}^3$ in $\widetilde{D}$, we get the isomorphism

$$\widetilde{D}/\mathfrak{a}\widetilde{D} \cong W/\mathfrak{b}$$

of $\widetilde{k}$ algebras. Then the prime ideal $\mathfrak{p}\widetilde{D}/\mathfrak{a}\widetilde{D}$ corresponds to the prime ideal $P/\mathfrak{b}$ of $W/\mathfrak{b}$, where $P = W_+ = (X_j' | 1 \leqslant j \leqslant m) + (V_i' | 1 \leqslant i \leqslant d)$. Then because

$$\begin{aligned} \mathfrak{b} + ({V_1'}^3) = (X_j' | 1 \leqslant j \leqslant m) \cdot [(X_j' | 1 \leqslant j \leqslant m) + (V_i' | 1 \leqslant i \leqslant d)] \\ + (V_i' V_j' | 1 \leqslant i, j \leqslant d,\ i \neq j) + ({V_i'}^3 | 1 \leqslant i \leqslant d) \end{aligned}$$

and $\ell_{W_P}([\mathfrak{b} + ({V_1'}^3)]W_P/\mathfrak{b}W_P) = 1$, we get

$$\begin{aligned} \ell_{W_P}(W_P/\mathfrak{b}W_P) &= \ell_{W_P}(W_P/[\mathfrak{b} + ({V_1'}^3)]W_P) + \ell_{W_P}([\mathfrak{b} + ({V_1'}^3)]W_P/\mathfrak{b}W_P) \\ &= (m + 2d + 1) + 1 = m + 2d + 2. \end{aligned}$$

Thus $\ell_{A_\mathfrak{p}}(A_\mathfrak{p}) = \ell_{W_P/\mathfrak{b}W_P}(W_P/\mathfrak{b}W_P) = m + 2d + 2$. □

Thanks to the associative formula of multiplicity, we have

$$\mathrm{e}_0(Q) = \ell_{A_\mathfrak{p}}(A_\mathfrak{p}) \cdot \mathrm{e}_0^{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}([Q + \mathfrak{p}A]/\mathfrak{p}A) = m + 2d + 2,$$

because $\mathfrak{p} = \sqrt{\mathfrak{a}}$ and $A/\mathfrak{p}A = D/\mathfrak{p} \cong k[[Z_i | 1 \leqslant i \leqslant d]]$. On the other hand, we have

$$A/Q \cong k[[\{X_j\}_{1 \leqslant j \leqslant m}, Y, \{V_i\}_{1 \leqslant i \leqslant d}]]/\mathfrak{c}$$

where

$$\begin{aligned} \mathfrak{c} = ([(X_j | 1 \leqslant j \leqslant m) + (Y)] \cdot [(X_j | 1 \leqslant j \leqslant m) + (Y) + (V_i | 1 \leqslant i \leqslant d)] \\ + (V_i V_j | 1 \leqslant i, j \leqslant d,\ i \neq j) + ({V_i}^3 | 1 \leqslant i \leqslant d). \end{aligned}$$

Therefore, $\ell_A(A/Q) = m + 2d + 2$. Thus $\mathrm{e}_0(Q) = \ell_A(A/Q)$ so that A is a Cohen–Macaulay local ring with $\mathrm{e}_0(Q) = m + 2d + 2$.

Let $K_0 = A$, $K_1 = \mathfrak{m}$, and $K_n = \mathfrak{m}^n + y\mathfrak{m}^{n-2}$ for $n \geqslant 2$, and we set $\mathcal{K} = \{K_n\}_{n \geqslant 0}$. Let $\mathrm{e}_i(\mathcal{K})$ denote the ith Hilbert coefficients of the filtration $\mathcal{K}$ for $0 \leqslant i \leqslant d$.

Lemma 5.4. *The following assertions hold true.*

(1) $\ell_A(K_2/\mathfrak{m}^2) = 1$ *and* $K_n = \mathfrak{m}^n$ *for all* $n \geqslant 3$.

(2) *$Q \cap K_2 = QK_1$ and $K_{n+1} = QK_n$ for all $n \geqslant 2$. Therefore,* $\mathrm{e}_1(\mathcal{K}) = \mathrm{e}_0(\mathcal{K}) - \ell_A(A/K_1) + \ell_A(K_2/QK_1)$, $\mathrm{e}_2(\mathcal{K}) = \ell_A(K_2/QK_1)$ *if $d \geqslant 2$, and* $\mathrm{e}_i(\mathcal{K}) = 0$ *for $3 \leqslant i \leqslant d$.*

Proof. (1) Since $K_2 = \mathfrak{m}^2 + (y)$, we have $\ell_A(K_2/\mathfrak{m}^2) = 1$. We have $K_n = \mathfrak{m}^n + y\mathfrak{m}^{n-2} = \mathfrak{m}^n$ for all $n \geqslant 3$, because $y\mathfrak{m} = (yv_i | 1 \leqslant i \leqslant d) = ({v_i}^3 | 1 \leqslant i \leqslant d) \subseteq \mathfrak{m}^3$.

(2) Since $K_n = \mathfrak{m}^n$ for all $n \geqslant 3$ by assertion (1), we have $K_2 \subseteq \widetilde{\mathfrak{m}^2}$. Therefore, $Q \cap K_2 \subseteq Q \cap \widetilde{\mathfrak{m}^2} = Q\mathfrak{m} = QK_1$ by Remark 4.2. It is routine to check that $K_{n+1} = QK_n$ for all $n \geqslant 2$. Thus $\mathrm{e}_1(\mathcal{K}) = \mathrm{e}_0(\mathcal{K}) - \ell_A(A/K_1) + \ell_A(K_2/QK_1)$ by [GR, Theorem 2.2]. We also have $\mathrm{e}_2(\mathcal{K}) = \ell_A(K_2/QK_1)$ if $d \geqslant 2$, and $\mathrm{e}_i(\mathcal{K}) = 0$ for $3 \leqslant i \leqslant d$ by [HM, Proposition 4.6]. □

We prove now Theorem 5.2.

Proof of Theorem 5.2. Since $K_n = \mathfrak{m}^n$ for all $n \geqslant 3$ by Lemma 5.4(1), we have $\mathrm{e}_i(\mathcal{K}) = \mathrm{e}_i(\mathfrak{m})$ for $0 \leqslant i \leqslant d$. Therefore, $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{m}) + \ell_A(\mathfrak{m}^2/Q\mathfrak{m})$, $\mathrm{e}_2(\mathfrak{m}) = \ell_A(\mathfrak{m}^2/Q\mathfrak{m}) + 1$ if $d \geqslant 2$, and $\mathrm{e}_i(\mathfrak{m}) = 0$ for all $3 \leqslant i \leqslant d$, because $\ell_A(K_2/\mathfrak{m}^2) = 1$, $\mathrm{e}_1(\mathcal{K}) = \mathrm{e}_0(\mathcal{K}) + \ell_A(K_2/QK_1) - 1$, $\mathrm{e}_2(\mathcal{K}) = \ell_A(K_2/QK_1)$ if $d \geqslant 2$, and $\mathrm{e}_i(\mathcal{K}) = 0$ for $3 \leqslant i \leqslant d$ by Lemma 5.4. Then we have $\mathrm{e}_1(\mathfrak{m}) = m + 3d + 2$ and $\mathrm{e}_2(\mathfrak{m}) = d + 1$ because $\mathrm{e}_0(\mathfrak{m}) = m + 2d + 2$ and $\ell_A(\mathfrak{m}^2/Q\mathfrak{m}) = d$. The ring $\mathrm{G}(\mathfrak{m})$ is Buchsbaum ring with $\operatorname{depth}\mathrm{G}(\mathfrak{m}) = 0$ and $\mathbb{I}(\mathrm{G}(\mathfrak{m})) = d$ by Theorem 4.3. This completes the proof of Theorem 5.2. □

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Kazuho Ozeki
Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
1677-1 Yoshida
Yamaguchi 753-8512
Japan
ozeki@yamaguchi-u.ac.jp

Maria Evelina Rossi
Dipartimento di Matematica
Universita di Genova
Via Dodecaneso 35
16146-Genova
Italy
rossim@dima.unige.it