Castelnuovo-Mumford Regularity and finiteness of Hilbert Functions

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Introduction

The Castelnuovo-Mumford regularity is a very important invariant of graded modules which arises naturally in the study of finite free resolutions. There have been several results which establish bounds for the Castelnuovo-Mumford regularity of projective schemes in terms of numerical characters. Unfortunately, these invariants are often difficult to handle and the problem of finding good bounds in terms of simpler invariants is a topic featured in much recent research.

The notion of regularity has been used by S. Kleiman in the construction of bounded families of ideals or sheaves with given Hilbert polynomial, a crucial point in the construction of Hilbert or Picard scheme. In a related direction, Kleiman proved that if I is an equidimensional reduced ideal in a polynomial ring S over an algebraically closed field, then the coefficients of

the Hilbert polynomial of $R = S/I$ can be bounded by the dimension and the multiplicity of R (see [10, Corollary 6.11]). Srinivas and Trivedi proved that the corresponding result does not hold for a local domain. However, they proved that there exist a finite number of Hilbert functions for a local Cohen-Macaulay ring of given multiplicity and dimension (see [18]). The proofs of the above results are very difficult and involve deep results from Algebraic Geometry.

The aim of this paper is to introduce a unified approach which gives more general results and easier proofs of the above mentioned results. This approach is based on a classical result of Mumford on the behaviour of the geometric regularity by hyperplane sections in [13] which allows us to bound the regularity once there is a bound for the size of certain component of the first local cohomology modules or a uniform bound for the Hilbert polynomials. The finiteness of the Hilbert functions then follows from the boundness of the regularity. More precisely, we shall see that a class $\mathcal{C}$ of standard graded algebras has a finite number of Hilbert functions if and only if there are upper bounds for the regularity and the embedding dimension of the members of $\mathcal{C}$.

In Section 1 we will prepare some preliminary facts on Castelnuovo-Mumford regularity and related notions such as weak regularity and geometric regularity. In Section 2 we will clarify the relationship between the boundedness of Castelnuovo-regularity (resp. geometric regularity) and the finiteness of the Hilbert functions (resp. Hilbert polynomials). The main technique is the theory of Gröbner basis.

In Section 3 we will show that the boundedness of the regularity of classes of graded algebras with positive depth can be deduced from the boundedness of the dimension of the zero-graded component of the first local cohomology modules. As a consequence, Kleiman's result follows from the well-known fact that for equidimensional reduced schemes, the dimension of the first sheaf cohomology module is equal to the number of the connected components minus one. We also give an example showing that Kleiman's result does not hold if the graded algebras are not equidimensional.

The local case is studied in Section 4. First we present a local version of Mumford's Theorem, which allows us to reduce the problem of bounding the regularity of the tangent cone to the problem of bounding the Hilbert polynomial. Such a bound for the Hilbert polynomials was already established in [16], [21]. From this we can easily deduce Srinivas and Trivedi's result on the finiteness of the Hilbert functions of Cohen-Macaulay local rings with given dimesion and multiplicity. This approach can be used to prove the more general result that the number of numerical functions which can arise as the Hilbert functions of local rings with given dimension and extended degree is finite (see [15] for details).

1 Variations on Castelnuovo-Mumford regularity

Throughout this paper let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k.

Let $M = \oplus_t M_t$ be a finitely generated graded S-module and let

$$0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded free resolution of M as S-module. Write b_i for the maximum of the degrees of the generators of F_i. Following [6], Section 20.5, we say that M is *m-regular* for some integer m if $b_j - j \leq m$ for all j.

The *Castelnuovo-Mumford regularity* $\mathrm{reg}(M)$ of M is defined to be the least integer m for which M is m-regular, that is,

$$\mathrm{reg}(M) = \max\{b_i - i \mid i = 0, \ldots, s\}.$$

It is well known that M is m-regular if and only if $\mathrm{Ext}_S^i(M, S)_n = 0$ for all i and all $n \leq -m - i - 1$ (see [6]). This result is hard to apply because in principle infinitely many conditions must be checked. However, in some cases, it suffices to check just a few.

We say that M is *weakly m-regular* if $\mathrm{Ext}_S^i(M, S)_{-m-i-1} = 0$ for all $i \geq 0$. Due to a result of Mumford, if $\mathrm{depth} M > 0$ and M is weakly m-regular, then M is m-regular (see [6, 20.18]). From this we can easily deduce the following result (see [15, Corollary 1.2]).

PROPOSITION 1.1 *Let $R = S/I$ be a standard graded algebra and m a non-negative integer. If R is weakly m-regular, then R is m-regular.*

Using local duality we can characterize these notions of regularity by means of the local cohomology modules of M.

Let $\Im$ denote the maximal graded ideal of S and let M be a finitely generated graded S-module. For any integer i we denote by $H_\Im^i(M)$ the i-th local cohomology module of M with respect to $\Im$. By local duality (see [6], A4.2) we have

$$H_\Im^i(M)_m \cong \mathrm{Ext}_S^{r-i}(M, S)_{-m-r}$$

for all i and m. Thus, M is m-regular if and only if $H_\Im^i(M)_n = 0$ for all i and $n \geq m - i + 1$, and M is weakly m-regular if and only if $H_\Im^i(M)_{m-i+1} = 0$ for all i. In particular, $\mathrm{reg}(M)$ is the least integer m for which $H_\Im^i(M)_n = 0$ for all i and $n \geq m - i + 1$. Hence the Castelnuovo-Mumford regularity can be defined for any finite S-module regardless of its presentation.

For any integer i we set $a_i(M) := \max\{n \mid H_\Im^i(M)_n \neq 0\}$, where $a_i(M) = -\infty$ if $H_\Im^i(M) = 0$. Then

$$\mathrm{reg}(M) = \max\{a_i(M) + i \mid i \geq 0\}.$$

A relevant remark is that the Castelnuovo-Mumford regularity controls the behaviour of the Hilbert function. We recall that the Hilbert function of M is the numerical function

$$h_M(t) = \dim_k(M_t).$$

The Hilbert polynomial $p_M(X)$ of M is the polynomial of degree $d-1$ such that we have $h_M(t) = p_M(t)$ for $t >> 0$.

The Hilbert function $h_M(n)$ and the Hilbert polynomial $p_M(n)$ are related by the formula $h_M(n) = p_M(n)$ for $n > \text{reg}(M)$. This is a consequence of the Serre formula which holds for every integer n:

$$h_M(n) - p_M(n) = \sum_{i \geq 0} (-1)^i \dim_k H_{\mathfrak{I}}^i(M)_n.$$

For a proof, see for example [3, Theorem 4.4.3].

Inspired by the notion of regularity for sheaves on projective spaces, it is natural to introduce the following weaker notion of regularity.

We say that M is *geometrically m-regular* if $H_{\mathfrak{I}}^i(M)_n = 0$ for all $i > 0$ and $n \geq m - i + 1$, and we define the *geometric regularity* g-reg(M) of M to be the least integer m for which M is geometrically m-regular.

It is clear that

$$\text{g-reg}(M) = \max\{a_i(M) + i \mid i > 0\}.$$

Hence we always have

$$\text{g-reg}(M) = \text{reg}(M/H_{\mathfrak{I}}^0(M)) \leq \text{reg}(M).$$

In particular, $\text{reg}(M) = \text{g-reg}(M)$ if $\text{depth} M > 0$.

For a standard graded algebra $R = S/I$, a theorem of Gotzmann gives an upper bound for the geometric regularity in term of an integer which can be computed from the Hilbert polynomial of R.

THEOREM 1.2 (see [8]) *Assume that*

$$p_R(n) = \binom{n + a_1}{a_1} + \binom{n + a_2 - 1}{a_2} + \cdots + \binom{n + a_s - (s - 1)}{a_s}$$

with $a_1 \geq a_2 \geq \cdots \geq a_s \geq 0$. *Then*

$$\text{g-reg}(R) = \text{reg}(S/I^{sat}) \leq s - 1.$$

For example, if R has dimension 1 and multiplicity e, then its Hilbert polynomial is

$$p_R(n) = e = \binom{n}{0} + \binom{n-1}{0} + \cdots + \binom{n - (e - 1)}{0}$$

so that g-reg$(R) \leq e - 1$. In particular, if R is Cohen-Macaulay of dimension 1 and multiplicity e, then reg$(R) \leq e - 1$.

Unlike the regularity, the geometric regularity does not behave well under generic hyperplane sections. Take for example $R = k[x, y, z]/(x^2, xy)$. Then g-reg$(R) = $ reg$(R) = 1$ while g-reg$(R/zR) = 0$, reg$(R/zR) = 1$.

However the following result of Mumford (see [13, page 101, Theorem]) gives us the possibility to control this behaviour. It will be the basic result for our further investigation on the Castelnuovo-Mumford regularity of a standard graded algebra R. A ring theoretic proof can be found in [15, Theorem 1.4].

THEOREM 1.3 *Let $R = S/I$ be a standard graded algebra and $z \in R_1$ a regular linear form in R. If* g-reg$(R/zR) \leq m$, *then*
    (a) *for every $s \geq m + 1$,*

$$\dim_k H^1_{\mathfrak{S}}(R)_m = \dim_k H^1_{\mathfrak{S}}(R)_s + \sum_{j=m+1}^{s} \dim_k H^0_{\mathfrak{S}}(R/zR)_j,$$

    (b) reg$(R) \leq m + \dim_k H^1_{\mathfrak{S}}(R)_m$,
    (c) $\dim_k H^1_{\mathfrak{S}}(R)_t = p_R(t) - h_R(t)$ *for every $t \geq m - 1$.*

2   Finiteness of Hilbert Functions

The aim of this section is to clarify how the finiteness of Hilbert functions for a given class of standard graded algebras is related to the Castelnuovo-Mumford regularity of the members of the class.

In the following we let $R = S/I$ where $S = k[x_1, \ldots, x_r]$ is a polynomial ring over an infinite field k of any characteristic and I an homogeneous ideal of S. We will denote by embdim(R) its embedding dimension, that is $h_R(1)$.

It is well-known that

$$h_{S/I}(t) = h_{S/\mathrm{in}(I)}(t)$$

for all $t \geq 0$ where in(I) denotes the initial ideal of I with respect to some term order. Therefore, we can pass to our study to Hilbert function of factor rings by initial ideal.

On the other hand, we have the following basic result of Bayer and Stillman on the behaviour of the regularity when passing to initial ideals.

PROPOSITION 2.1 (see [1]) *Let $R = S/I$ be a standard graded algebra. Then*

$$\mathrm{reg}(R) = \mathrm{reg}(S/\mathrm{gin}(I)),$$

where $\mathrm{gin}(I)$ is the generic initial ideal of I with respect to the reverse lexicographic order.

Moreover, it follows from a result of Bigatti and Hulett (for $\mathrm{char}(k) = 0$) and of Pardue (for $\mathrm{char}(k) > 0$) that among all the ideals with the same Hilbert functions, the lex-segment ideal has the largest regularity. Recall that the lex-segment ideal $\mathrm{Lex}(I)$ of I is the monomial ideal which is generated in every degree t by the first $h_I(t)$ monomials in the lexicographical order.

PROPOSITION 2.2 (see [2], [9], [14]) *Let $R = S/I$ be a standard graded algebra. Then*

$$\mathrm{reg}(R) \leq \mathrm{reg}(S/\mathrm{Lex}(I)).$$

In the following we will employ the following notations.
Let $\mathcal{C}$ be a class of standard graded algebras. We say:

- $\mathcal{C}$ *is HF-finite* if the number of numerical functions which arise as the Hilbert functions of $R \in \mathcal{C}$ is finite,

- $\mathcal{C}$ is *HP-finite* if the number of polynomials which arise as the Hilbert polynomials of $R \in \mathcal{C}$, is finite,

- $\mathcal{C}$ is *reg-bounded* if there exists an integer t such that $\mathrm{reg}(R) \leq t$ for all $R \in \mathcal{C}$,

- $\mathcal{C}$ is *g-reg-bounded* if there exists an integer t such that $\mathrm{g\text{-}reg}(R) \leq t$ for all $R \in \mathcal{C}$,

- $\mathcal{C}$ is *embdim-bounded* if if there exists an integer t such that $\mathrm{embdim}(R) \leq t$ for all $R \in \mathcal{C}$.

Moreover, we will denote by $\overline{\mathcal{C}}$ the class of graded algebras of the form $\bar{R} := R/H^0_{\mathfrak{I}}(R)$ for $R \in \mathcal{C}$. Note that

$$h_R(t) = h_{\bar{R}}(t)$$

for $t \gg 0$ and that

$$\mathrm{g\text{-}reg}(R) = \mathrm{g\text{-}reg}(\bar{R}) = \mathrm{reg}(\bar{R}).$$

Then $\mathcal{C}$ is HP-finite if and only if $\overline{\mathcal{C}}$ is HP-finite and $\mathcal{C}$ is g-reg-bounded if and only if $\overline{\mathcal{C}}$ is reg-bounded.

THEOREM 2.3 *Let $\mathcal{C}$ be a class of standard graded algebras. Then*
  (a) *$\mathcal{C}$ is HF-finite if and only if $\mathcal{C}$ is reg-bounded and embdim-bounded,*
  (b) *$\mathcal{C}$ is HP-finite if and only if $\mathcal{C}$ is g-reg-bounded and $\overline{\mathcal{C}}$ is embdim-bounded.*

Proof: We may assume that the base field is infinite by tensoring it with a transcendental extension.

(a) If $\mathcal{C}$ is HF-finite, then $\mathcal{C}$ is embdim-bounded. By Proposition 2.2, for every $S/I \in \mathcal{C}$ we have $\mathrm{reg}(S/I) \le \mathrm{reg}(S/Lex(I))$. Since there are a finite number of possible Hilbert functions for S/I, there are also a finite number of lex-segment ideals for I, so that $\mathcal{C}$ is reg-bounded by Proposition 2.2.

Conversely, assume $\mathcal{C}$ is reg-bounded and embdim-bounded. By Proposition 2.1, for all $R = S/I \in \mathcal{C}$ we have

$$\mathrm{reg}(S/I) = \mathrm{reg}(S/\mathrm{gin}(I)) \ge D - 1,$$

where D is the maximum degree of the monomials in the standard set of generators of $\mathrm{gin}(I)$. Since the embedding dimension is bounded, the number of monomials of degree smaller than or equal to D in S are finite. Thus, there are only a finite number of possibilities for $\mathrm{gin}(I)$ and hence also for the Hilbert functions of S/I because $h_{S/I}(t) = h_{S/\mathrm{gin}(I)}(t)$.

(b) By Theorem 1.2 we have $\mathrm{g\text{-}reg}(S/I) \le s - 1$ where the integer s depends only on the Hilbert polynomial of S/I. Therefore, if $\mathcal{C}$ is HP-finite, then $\mathcal{C}$ is g-reg-bounded. As remarked above, $\overline{\mathcal{C}}$ is reg-bounded. Furthermore, for every $R \in \overline{\mathcal{C}}$ we have $h_R(1) \le h_R(n) = p_R(n)$ for $n = \mathrm{reg}(R)$. Since $\overline{\mathcal{C}}$ is HP-finite, there are only a finite number of Hilbert polynomials $p_R(n)$. Let $t = \max\{p_R(n) | n = \mathrm{reg}(R), R \in \overline{\mathcal{C}}\}$. Then $h_R(1) \le t$ for all $R \in \overline{\mathcal{C}}$. Hence $\overline{\mathcal{C}}$ is embdim-bounded.

Conversely, if $\mathcal{C}$ is g-reg-bounded, then $\overline{\mathcal{C}}$ is reg-bounded. If moreover $\overline{\mathcal{C}}$ is embdim-bounded, then $\overline{\mathcal{C}}$ is HF-finite by (a). This implies that $\mathcal{C}$ is HP-finite. □

COROLLARY 2.4 *Let $\mathcal{C}$ be a class of standard graded algebras. Then $\mathcal{C}$ is HP-finite if and only if $\overline{\mathcal{C}}$ is HF-finite.*

The following example shows that $\overline{\mathcal{C}}$ is embdim-bounded does not imply $\mathcal{C}$ is embdim-bounded.

Let $\mathcal{C}$ be the class of algebras of the form

$$R_n = k[x_1, ..., x_n]/(x_1^2, ..., x_{n-1}^2, x_1 x_n, ..., x_{n-1} x_n)$$

for $n > 0$. Then $\overline{R}_n \cong k[x_n]$. Hence $\overline{\mathcal{C}}$ is embdim-bounded while $\mathcal{C}$ is not because $\mathrm{embdim} R_n = n$.

3   Reg-bounded algebras and Kleiman's Theorem.

The aim of this section is to present a relevant class of algebras which are reg-bounded and HF-finite. As an application, we give a proof of a theorem of Kleiman (see [10, Corollary 6.11]), which says that the class of graded reduced and equidimensional algebras with given multiplicity and dimension is reg-bounded and HF-finite. The main tool is the afore mentioned result of Mumford (Theorem 1.3).

For every integer $p \geq 1$ we define recursively the following polynomials $F_p(X)$ with rational coefficients. We let

$$F_1(X) := X$$

and, if $p \geq 2$, then we let

$$F_p(X) := F_{p-1}(X) + X \left(\frac{F_{p-1}(X) + p - 1}{p - 1} \right).$$

THEOREM 3.1 *Let $\mathcal{C}$ be a class of standard graded algebras with the following properties:*

1. *$\mathrm{depth}(R) > 0$ for every $R \in \mathcal{C}$,*

2. *for every $R \in \mathcal{C}$ with $\dim R \geq 2$, there exists some regular linear form $x \in R$ such that $R/(x)^{sat} \in \mathcal{C}$,*

3. *there exists an integer t such that $\dim_k H^1_{\mathfrak{S}}(R)_0 \leq t$ for every $R \in \mathcal{C}$.*

Then for every $R \in \mathcal{C}$ with $d = \dim R$ we have

$$\mathrm{reg}(R) \leq F_d(t).$$

It would be interesting the problem of finding the complexity of the regularity bound. Explicit bounds can be found in [15].

We need the following observation for the proof of the above theorem.

LEMMA 3.2 *Let $\mathcal{C}$ be a class of standard graded algebras as in the above theorem. Then for all $j \geq 0$ and for all $R \in \mathcal{C}$ we have*

$$\dim_k H^1_{\mathfrak{S}}(R)_j \leq t \binom{j + d - 1}{d - 1}.$$

Proof: If $j = 0$, the conclusion holds by the assumption. If $d = 1$, then R is a one-dimensional standard graded algebra of positive depth. Hence $H^0_{\mathfrak{S}}(R) = 0$ so that by Serre formula we get for every j,

$$\dim_k H^1_{\mathfrak{S}}(R)_j = p_R(j) - h_R(j) = e - h_R(j),$$

where e denotes the multiplicity of R. In particular, if $R \in \mathcal{C}$ and $j \geq 0$, we have

$$\dim_k H^1_{\mathfrak{S}}(R)_j = e - h_R(j) \leq e - 1 = \dim_k H^1_{\mathfrak{S}}(R)_0 \leq t.$$

Hence the conclusion holds for every $R \in \mathcal{C}$ of dimension 1.

Now let $j \geq 1$ and $R \in \mathcal{C}$ of dimension $d \geq 2$. We may identify R with the graded flat extension R' of Theorem 3.1 (2). Then there exists a regular element $x \in R_1$ such that $B = R/(x)^{sat} \in \mathcal{C}$. It is clear that $H^i(R/xR) = H^i(B)$ for every $i > 0$. From the short exact sequence

$$0 \longrightarrow R(-1) \overset{x}{\longrightarrow} R \longrightarrow R/xR \longrightarrow 0$$

we get for every j an exact sequence

$$\cdots \longrightarrow H^1_{\mathfrak{S}}(R)_{j-1} \longrightarrow H^1_{\mathfrak{S}}(R)_j \longrightarrow H^1_{\mathfrak{S}}(R/xR)_j \longrightarrow \cdots$$

Hence

$$\dim_k H^1_{\mathfrak{S}}(R)_j \leq \dim_k H^1_{\mathfrak{S}}(R)_{j-1} + \dim_k H^1_{\mathfrak{S}}(R/xR)_j = \dim_k H^1_{\mathfrak{S}}(R)_{j-1} + \dim_k H^1_{\mathfrak{S}}(B)_j.$$

Since $\dim B = d - 1$, we can use the inductive assumption and we get

$$\dim_k H^1_{\mathfrak{S}}(R)_j \leq t\binom{j + d - 2}{d - 1} + t\binom{j + d - 2}{d - 2} = t\binom{j + d - 1}{d - 1}.$$

$\square$

Proof: *of Theorem 3.1.* If $R \in \mathcal{C}$ is one-dimensional, then R is Cohen-Macaulay. Using Theorem 1.2 and the proof of Lemma 3.2 we have

$$\mathrm{reg}(R) = \text{g-reg}(R) \leq e - 1 = \dim_k H^1_{\mathfrak{S}}(R)_0 \leq t = F_1(t).$$

If $R \in \mathcal{C}$ is of dimension $d \geq 2$, we replace R by the graded flat extension R' of (2). Then there exists a regular element $x \in R_1$ such that $B = R/(x)^{sat} \in \mathcal{C}$. We have $\mathrm{depth}(B) \geq 1$ and if we let $m := \mathrm{reg}(B)$, then

$$m = \text{g-reg}(B) = \text{g-reg}(R/xR).$$

By Theorem 1.3 and Lemma 3.2, this implies

$$\mathrm{reg}(R) \leq m + \dim_k H^1_{\mathfrak{S}}(R)_m \leq m + t\binom{m + d - 1}{d - 1}.$$

Since $\dim B = d - 1$, by induction we have $m \leq F_{d-1}(t)$. Thus,

$$\operatorname{reg}(R) \leq F_{d-1}(t) + t\binom{F_{d-1}(t) + d - 1}{d - 1} = F_d(t).$$

$\square$

We want to apply now the above result to the class $\mathcal{C}$ of reduced equidimensional graded algebras with given multiplicity e and dimension $d \geq 1$.

It is clear that every $R \in \mathcal{C}$ has positive depth. Moreover, we have the following easy lemma.

LEMMA 3.3 *If R is a reduced equidimensional graded algebra over an algebraically closed field with multiplicity e, then $\dim_k H^1_{\mathfrak{S}}(R)_0 \leq e - 1$.*

Proof: If $\dim R = 1$, then R is Cohen-Macaulay and, as above, $\dim_k H^1_{\mathfrak{S}}(R)_0 = e - 1$. If $\dim R \geq 2$, then $\dim_k H^1_{\mathfrak{S}}(R)_0 = N - 1$ where N is the number of the connected components of the corresponding scheme [12, Theorem 1.2.6 (b)]. Since $N \leq e$, we get $\dim_k H^1_{\mathfrak{S}}(R)_0 \leq e - 1$. $\square$

We can prove now that Kleiman's result is a particular case of our theorem.

THEOREM 3.4 *[10, Corollary 6.11] Let $\mathcal{C}$ be the class of reduced equidimensional graded algebras over an algebraically closed field with given multiplicity e and dimension $\leq d$, $d \geq 1$. Then $\mathcal{C}$ is HF-finite.*

Proof: By Bertini theorem [7, Corollary 3.4.14], if $\dim R \geq 2$, there exists a regular linear form $x \in R$ such that $R/(x)^{sat}$ is a reduced equidimensional algebra with $\dim R/(x)^{sat} = \dim R - 1$. Therefore, we may apply Theorem 3.1 and Lemma 3.3 to get $\operatorname{reg}(R) \leq F_d(e - 1)$ for every R in the class $\mathcal{C}$. So $\mathcal{C}$ is reg-bounded. Now we need only to prove that $\mathcal{C}$ is embdim-bounded since by Theorem 2.1 (a), these conditions will imply that $\mathcal{C}$ is finite.

Let $R \in \mathcal{C}$ be arbitrary. Write $R = S/I$ where S is a polynomial ring over an algebraically closed field and $I = \bigcap_{i=1}^{r} \mathfrak{p}_i$ is an intersection of equidimensional ideals $\mathfrak{p}_i$. It is obvious that $\operatorname{embdim}R \leq \sum_{i=1}^{r} \operatorname{embdim}S/\mathfrak{p}_i$ and $r \leq e$. On the other hand, we know that $\operatorname{embdim}S/\mathfrak{p}_i \leq e_i + d - 1$, where e_i is the multiplicity of $S/\mathfrak{p}_i$. Therefore,

$$\operatorname{embdim}R \leq \sum_{i=1}^{r} e_i + d - 1 = e + r(d - 1) \leq ed.$$

$\square$

Theorem 3.1 does not hold if we delete the assumption that every element of the class is reduced. Take for example the class $\mathcal{C}$ of the graded algebras

$$R_n := k[x, y, z, t]/(y^2, xy, x^2, xz^n - yt^n) \ (n \geq 1).$$

Note that $(y^2, xy, x^2, xz^n - yt^n)$ is a primary ideal. We have $\dim(R_n) = 2$ and $e(R_r) = 2$. The minimal free resolution of R_r over $S = k[x, y, z, t]$ is given by

$$0 \longrightarrow S(-n-3) \longrightarrow S(-3)^2 \oplus S(-n-2)^2 \longrightarrow S(-2)^3 \oplus S(-n-1) \longrightarrow S \longrightarrow R_n \longrightarrow 0.$$

Hence $\operatorname{reg}(R_n) = n$. Therefore, $\mathcal{C}$ is not reg-bounded and hence not HF-finite.

Theorem 3.1 does not hold if we consider reduced graded algebras which are not necessarily equidimensional. Let us consider the class of standard graded algebras

$$R_n := k[x, y, z_1, \ldots, z_r]/(x) \cap (y, f_n)$$

where $f_n \in k[z_1, \ldots, z_r]$ is an irreducible form of degree n. We have $\dim R_n = r + 1$, $e(R_n) = 1$, but $\operatorname{reg}(R_n) = n$.

4 The local version of Mumford Theorem

Let $(A, \mathfrak{m})$ be a local ring of dimension d and multiplicity e. Let

$$G = gr_{\mathfrak{m}}(A) = \oplus_{n \geq 0}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$$

be the associated graded ring of A. The Hilbert function and the Hilbert polynomial of A is by definition the Hilbert function and the Hilbert polynomial of the standard graded algebra G, namely $h_A(t) := h_G(t) = \lambda(\mathfrak{m}^t/\mathfrak{m}^{t+1})$ and $p_A(t) := p_G(t)$.

We will need also the first iterated of these functions and polynomials. So we let

$$h_A^1(t) := h_G^1(t) = \sum_{j=0}^{t} h_G(j) = \lambda(A/\mathfrak{m}^{t+1})$$

and we denote by $p_G^1(t)$ the corresponding polynomial, that is the polynomial which verifies the equality $h_G^1(t) = p_G^1(t)$ for $t \gg 0$.

Let x be a superficial element in $\mathfrak{m}$ and let

$$\bar{G} := gr_{\mathfrak{m}/xA}(A/xA)$$

be the associated graded ring of the local ring A/xA. Let $x^* = \bar{x} \in (\mathfrak{m}/\mathfrak{m}^2)$ be the initial form of x in G. We can consider the standard graded algebra G/x^*G and compare it with $\overline{G}$. These two algebras are not the same, unless x^* is a regular element in G, but they have the same geometric regularity, namely

$$\text{g-reg}(G/x^*G) = \text{g-reg}(\overline{G}).$$

This has been proved in [15, Lemma 2.2].

If A has positive depth, then it is well known that x is a regular element in A and furthermore

$$p_G(t) = p_{\overline{G}}^1(t).$$

In that case, one can prove (see [15, Lemma 3.2]) that $\text{reg}(G) = \text{g-reg}(G)$ (even when G does not necessarily have positive depth).

With the above notations we present now a local version of Mumford's Theorem 1.3.

THEOREM 4.1 *Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ and positive depth. Let x be a superficial element in $\mathfrak{m}$ and $m := \text{reg}(\overline{G})$. Then*

$$\text{reg}(G) \leq m + h_{\overline{G}}^1(m).$$

Proof: We have

$$m = \text{reg}(\overline{G}) \geq \text{g-reg}(\overline{G}) = \text{g-reg}(G/x^*G) = \text{g-reg}(G/(H_{G_+}^0(G) + x^*G)),$$

where the last equality follows because $(H_{G_+}^0(G) + x^*G)/x^*G)$ has finite length in G/x^*G. Now we remark that x^* is a regular element in $G/H_{G_+}^0(G)$, hence, by using Mumford theorem, we get

$$\text{reg}(G) = \text{g-reg}(G) = \text{g-reg}(G/(H_{G_+}^0(G)) \leq m + p_{G/H_{G_+}^0(G)}(m)$$

$$= m + p_G(m) = m + p_{\overline{G}}^1(m) = m + h_{\overline{G}}^1(m)$$

where the equality $p_{\overline{G}}^1(m) = h_{\overline{G}}^1(m)$ is a consequence of the fact that $m = \text{reg}(\overline{G})$. $\square$

Using this local version of Mumford's theorem we can easily deduce the result of Srinivas and Trivedi which says that the number of Hilbert functions of Cohen-Macaulay local rings with fixed dimension and multiplicity is finite (see [18]). For that we need the following inequality proved in [16] and [21].

PROPOSITION 4.2 *Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 1$ and J an ideal generated by a system of parameters in $\mathfrak{m}$. Then*

$$H_A(n) \leq \ell(A/J)\binom{n + d - 2}{d - 1} + \binom{n + d - 2}{d - 2}.$$

If A is a d-dimensional Cohen-Macaulay ring of multiplicity e, then from the above Proposition we immediately get the inequality

$$H_A(n) \le e\binom{n+d-2}{d-1} + \binom{n+d-2}{d-2}.$$

For every $d \ge 1$ we define recursively the following polynomials $Q_d(X)$ with rational coefficients. We let

$$Q_1(X) := X - 1$$

and, if $d \ge 2$, then we let

$$Q_d(X) := Q_{d-1}(X) + X\binom{Q_{d-1}(X) + d - 2}{d-1} + \binom{Q_{d-1}(X) + d - 2}{d-2}.$$

THEOREM 4.3 *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \ge 1$ and multiplicity e. Then*

$$\mathrm{reg}(G) \le Q_d(e).$$

Proof: If $d = 1$, then $\mathrm{reg}(G) \le e - 1 = Q_1(e)$. Let $d \ge 2$ and x be a superficial element in $\mathfrak{m}$. Then A/xA is a Cohen-Macaulay local ring of dimension $d - 1$ and multiplicity e. By Proposition 4.2 we get

$$H^1_{\overline{G}}(n) \le e\binom{n+d-2}{d-1} + \binom{n+d-2}{d-2}.$$

By Theorem 4.1 it follows that, if $m = \mathrm{reg}(\overline{G})$, then

$$\mathrm{reg}(G) \le m + h^1_{\overline{G}}(m) \le m + e\binom{m+d-2}{d-1} + \binom{m+d-2}{d-2}.$$

By induction we have $m \le Q_{d-1}(e)$, so that

$$\mathrm{reg}(G) \le Q_{d-1}(e) + e\binom{Q_{d-1}(e) + d - 2}{d-1} + \binom{Q_{d-1}(e) + d - 2}{d-2} = Q_d(e).$$

$\square$

COROLLARY 4.4 *The number of numerical functions which can arise as the Hilbert functions of Cohen-Macaulay local rings with given dimension and multiplicity is finite.*

Proof: By a classical result of Abhyankar we have $v(\mathfrak{m}) \leq e + d - 1$, where $v(\mathfrak{m})$ is the embedding dimension of G. Now we need only to apply Proposition 2.3 and the above theorem. $\qquad\square$

The analogous of Kleiman result does not hold in the local case. Srinivas and Trivedi gave the following example showing that classes of local domains of fixed dimension and multiplicity need not to be HF-finite.

Let

$$A_r := k[[x, y, z, t]]/(z^r t^r - xy, x^3 - z^{2r}y, y^3 - t^{2r}x, x^2 t^r - y^2 z^r).$$

It is easy to see that A_r is a local domain and the associated graded ring of A_r is the standard graded algebra

$$G_r = k[x, y, z, t]/(xy, x^3, y^3, x^2 t^r - y^2 z^r).$$

We have $\text{reg}(G_r) = r + 1$ and

$$H_{A_r}(n) = \begin{cases} 5n - 1 & \text{for} \quad n \leq r, \\ 4n + r & \text{for} \quad n > r. \end{cases}$$

Finally, we remark that the above approach can be used to prove that the number of numerical functions which can arise as the Hilbert functions of local rings with given dimension and extended degree is finite. Note that extended degree coincide with the usual multiplicity for Cohen-Macaulay local rings. We refer to [15] for details. Furthermore, one can prove similar results for Hilbert functions of finitely generated modules over local rings with respect to $\mathfrak{m}$-primary ideals (see [19, 20, 11]).

Acknowledgments

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