

# CASTELNUOVO-MUMFORD REGULARITY AND RATLIFF-RUSH CLOSURE

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ABSTRACT. We establish strong relationships between the Castelnuovo-Mumford regularity and the Ratliff-Rush closure of an ideal. Our results have several interesting consequences on the computation of the Ratliff-Rush closure, the stability of the Ratliff-Rush filtration, the invariance of the reduction number, and the computation of the Castelnuovo-Mumford regularity of the Rees algebra and the fiber ring. In particular, we prove that the Castelnuovo-Mumford regularity of the Rees algebra and of the fiber ring are equal for large classes of monomial ideals in two variables, thereby verifying a conjecture of Eisenbud and Ulrich for these cases.

*Dedicated to Giuseppe Valla on the occasion of his seventieth anniversary*

## INTRODUCTION

The Castelnuovo-Mumford regularity is a homological invariant which controls the complexity of graded structures over a polynomial rings [7]. It is well known that the Castelnuovo-Mumford regularity plays an important role in computational Algebraic Geometry and Commutative Algebra [1, 36]. It is less known that the Castelnuovo-Mumford regularity can be also defined for graded structures over a commutative ring and that the Castelnuovo-Mumford regularity of the Rees algebra controls the complexity of an ideal; see e.g. [32, 33, 34]. The main aim of this paper is to explore this aspect of the Castelnuovo-Mumford regularity by establishing relationships between the Castelnuovo-Mumford regularity of the Rees algebra and the seemingly unrelated Ratliff-Rush closure of an ideal.

The motivation for our work originates from the following conjecture of Eisenbud and Ulrich [8, Conjecture 1.3].

**Conjecture.** Let  $A$  be a standard graded algebra over a field  $k$ . Let  $I$  be an  $\mathfrak{m}$ -primary graded ideal generated by forms of the same degree, where  $\mathfrak{m}$  denotes the maximal graded ideal of  $A$ . Then  $\text{reg } R(I) = \text{reg } F(I)$ , where  $R(I) = \bigoplus_{n \geq 0} I^n$  is the Rees algebra and  $F(I) = \bigoplus_{n \geq 0} I^n / \mathfrak{m}I^n$  is the fiber ring of  $I$ .

For an arbitrary graded ideal  $I$ , it is known that  $\text{reg } I^n$  is asymptotically a linear function [4, 19, 35]. However, very little is known on the stability index of  $\text{reg } I^n$ , i.e.

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the least number  $n$  where  $\text{reg } I^n$  becomes a linear function afterward. Eisenbud and Ulrich [8] showed that under the above assumption,  $\text{reg } I^n$  is related to the presentation of  $R(I)$  as a direct sum of modules over  $F(I)$ , which has led them to raise the above conjecture.

The conjecture of Eisenbud and Ulrich is not true if one does not put further assumption on the base ring  $A$ . We shall see that if the conjecture were true, then  $A$  must be a Buchsbaum ring. In fact, Ulrich communicated to the authors that the conjecture should be formulated for polynomial rings. To give an answer to this modified conjecture seems to be difficult because there were no tools, which allow us to compare  $\text{reg } R(I)$  and  $\text{reg } F(I)$ . Note that  $R(I)$  also has an  $\mathbb{N}$ -graded structure over  $k$ , which has a different Castelnuovo-Mumford regularity. This Castelnuovo-Mumford regularity was studied by Herzog, Popescu and Trung in [14].

We shall see in this paper that superficial sequences of  $I$  can be used as a common tool to characterize  $\text{reg } R(I)$  and  $\text{reg } F(I)$ . As a first consequence, we show that  $\text{reg } R(I) = \text{reg } F(I)$  provided  $\text{depth } G(I) \geq \dim A - 1$ , where  $G(I) := \bigoplus_{n \geq 0} I^n / I^{n+1}$  is the associated graded ring of  $I$ . Our main results are however the findings that  $\text{reg } R(I)$  and  $\text{reg } F(I)$  control the behavior of the Ratliff-Rush closure and the Ratliff-Rush filtration of  $I$ . If  $\dim A = 2$ , we can express  $\text{reg } R(I)$  and  $\text{reg } F(I)$  in terms of the stability of the Ratliff-Rush filtration. These relationships are of independent interests as we shall see below.

Let  $(A, \mathfrak{m})$  be an arbitrary local ring and  $I$  an arbitrary ideal of  $A$ . The Ratliff-Rush closure of  $I$  is defined as the ideal  $\tilde{I} := \bigcup_{n \geq 1} I^{n+1} : I^n$ . It is a refinement of the integral closure of  $I$  and  $\tilde{I} = I$  if  $I$  is integrally closed. If  $I$  is a regular ideal, i.e. if  $I$  contains non-zerodivisors,  $\tilde{I}$  is the largest ideal sharing the same higher powers with  $I$  [23]. In particular, the Ratliff-Rush filtration  $\{\tilde{I}^n\}$ , carries important information on the blowups, the associated graded ring of  $I$ , and the Hilbert function of an  $\mathfrak{m}$ -primary ideal  $I$  (see e.g. [11, 12, 16, 28]).

In general, the computation of  $\tilde{I}$  is hard because  $I^{n+1} : I^n = I^n : I^{n-1}$  does not imply  $I^{n+2} : I^{n+1} = I^{n+1} : I^n$  [25]. Let  $s(I)$  denote the least integer  $m \geq 0$  such that  $\tilde{I} = I^{m+1} : I^m$  for all  $n \geq m$ . We show that

$$s(I) \leq \max\{\text{reg } R(I) - 1, 0\}.$$

Since there are various bound for  $\text{reg } R(I)$  in terms of other well known invariants of  $I$  ([5], [6], [20], [21], [27], [29], [36]), the above bound provides us an *effective tool for the computation of  $\tilde{I}$*  because  $\tilde{I} = I^{c+1} : I^c$  if  $c$  is an upper bound for  $\text{reg } R(I)$ . As far as we know, an upper bound for  $s(I)$  has been given before only for  $\mathfrak{m}$ -primary ideals in Cohen-Macaulay rings by Elias [9].

A remarkable feature of the Ratliff-Rush filtration is that  $\tilde{I}^n = I^n$  for  $n \gg 0$  if  $I$  is a regular ideal [23]. Again, if  $\tilde{I}^n = I^n$ , it does not necessarily imply  $\tilde{I}^{n+1} = I^{n+1}$  [25]. Let  $s^*(I)$  denote the least integer  $m \geq 1$  such that  $\tilde{I}^n = I^n$  for all  $n \geq m$ . This invariant was studied to some extent in [22, 25]. We show that

$$s^*(I) \leq \max\{\text{reg } R(I), 1\}.$$

On the other hand, we can characterize  $\text{reg } R(I)$  in terms of  $s^*(I)$  if  $I$  is an  $\mathfrak{m}$ -primary ideal, which is not a parameter ideal, in a two-dimensional regular local ring.

In fact, we show that

$$\operatorname{reg} R(I) = \max\{r_J(I), s^*(I)\} = \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\},$$

where  $J$  is an arbitrary minimal reduction of  $I$  and  $r_J(I)$  denotes the reduction number of  $I$  with respect to  $J$ . The first formula has interesting consequences on the invariance of the reduction number, while the last formula gives *an effective tool for the computation of  $\operatorname{reg} R(I)$* . Rossi and Swanson in [25, Section 4] asked whether  $s^*(I) \leq r_J(I)$  holds in general. Using Huckaba's example on the non-invariance of the reduction number [17] we give a negative answer to this question.

Similarly, we show that if  $A$  is a polynomial ring in two variables and  $I$  is a homogeneous  $\mathfrak{m}$ -primary ideal generated by forms of the same degree  $d$ , which is not a parameter ideal, then

$$\operatorname{reg} F(I) = \min\{n \geq r_J(I) \mid (\widetilde{I}^n)_{nd} = (I^n)_{nd}\}$$

for any homogeneous minimal reduction  $J$  of  $I$ . Since  $nd$  is the initial degree of both ideals  $I^n$  and  $\widetilde{I}^n$ , this formula shows that  $\operatorname{reg} F(I)$  controls the difference between  $\widetilde{I}^n$  and  $I^n$  at the initial degree, while  $\operatorname{reg} R(I)$  controls the difference between  $\widetilde{I}^n$  and  $I^n$  at all degrees.

The above formulae can be used effectively to compare  $\operatorname{reg} R(I)$  and  $\operatorname{reg} F(I)$ . In fact, we will use them to prove that  $\operatorname{reg} R(I) = \operatorname{reg} F(I)$  for large classes of monomial ideals in two variables. To compute the Ratliff-Rush closure of monomial ideals generated by monomials of the same degree in two variables one may use a method developed by Crispin Quinonez in [3].

We note that Cortadellas and Zarzuela [2], and Jayanthan and Nanduri [10] gave some conditions for the equality  $\operatorname{reg} R(I) = \operatorname{reg} F(I)$  in the case  $I$  is an ideal in a local ring. However, their results are too specific to be recalled here.

The paper is divided into four sections. In Section 1 we present a characterization of  $\operatorname{reg} R(I)$  in terms of a superficial sequence of  $I$  and discuss its relationship to  $s(I)$ . In Section 2 we show that  $\operatorname{reg} R(I)$  is an upper bound for  $s^*(I)$  and that one can characterize  $\operatorname{reg} R(I)$  in terms of  $s^*(I)$  in the two-dimensional case. In Section 3 we give a characterization of  $\operatorname{reg} F(I)$  in terms of a superficial sequence of  $I$  and of the Ratliff-Rush filtration. In Section 4 we apply our results to monomial ideals in two variables.

## 1. REGULARITY OF THE REES ALGEBRA

Let  $(A, \mathfrak{m})$  be a local ring with  $\dim A > 0$  and  $I$  an ideal of  $A$ . Without loss of generality we may assume that the residue field of  $A$  is infinite.

An element  $x \in I$  is called *superficial* for  $I$  if there is an integer  $c$  such that

$$(I^{n+1} : x) \cap I^c = I^n$$

for all large  $n$ . A system of elements  $x_1, \dots, x_s$  in  $I$  is called a *superficial sequence* of  $I$  if  $x_i$  is a superficial element of  $I$  in  $A/(x_1, \dots, x_{i-1})$ ,  $i = 1, \dots, s$ . A superficial sequences can be constructed by means of the following notion.

A system of homogeneous elements  $z_1, \dots, z_s$  in the associated graded ring  $G(I)$  is called *filter-regular* if

$$[(z_1, \dots, z_{i-1}) : z_i]_n = (z_1, \dots, z_{i-1})_n,$$

for sufficiently large  $n$ ,  $i = 1, \dots, s$ . It is easy to see that  $z_1, \dots, z_s$  is filter-regular if and only if  $x_i \notin P$  for all associated primes  $P \not\supseteq G(I)_+$  of  $(z_1, \dots, z_{i-1})$ ,  $i = 1, \dots, s$  (see [32]).

For every element  $x \in I$  we denote by  $x^*$  the residue class of  $x$  in  $I/I^2 \subset G(I)$ .

**Lemma 1.1.** [33, Lemma 6.2]  $x_1, \dots, x_s$  is a superficial sequence of  $I$  if and only if  $x_1^*, \dots, x_s^*$  forms a filter-regular sequence of  $G(I)$ .

An ideal  $J \subseteq I$  is called a *reduction* of  $I$  if there exists an integer  $n$  such that  $I^{n+1} = JI^n$ . The least integer  $n$  with this property is called the *reduction number* of  $I$  with respect to  $J$ . We will denote it by  $r_J(I)$ . A reduction is minimal if it is minimal with respect to containment.

**Lemma 1.2.** Every minimal reduction  $J$  of  $I$  can be generated by a superficial sequence of  $I$ .

*Proof.* Let  $Q$  denote the ideal in  $G(I)$  generated by the elements  $x^*$ ,  $x \in J$ . Then  $Q$  is generated by  $(J + I^2)/I^2$ . Since  $I^{n+1} = JI^n$ ,  $Q_{n+1} = G(I)_{n+1}$ . Therefore,  $Q \not\subseteq P$  for any prime  $P \not\supseteq G(I)_+$ . Using prime avoidance we can find a filter-regular sequence  $z_1, \dots, z_s \in G(I)$  such that  $Q = (z_1, \dots, z_s)$ . Choose  $x_i \in J$  such that  $x_i^* = z_i$ ,  $i = 1, \dots, s$ . Then  $(x_1, \dots, x_s) + I^2 = J + I^2$ . Hence

$$(x_1, \dots, x_s)I^n + I^{n+2} = JI^n + I^{n+2} = I^{n+1}.$$

By Nakayama's Lemma, this implies  $I^{n+1} = (x_1, \dots, x_s)I^n$ . Therefore,  $(x_1, \dots, x_s)$  is a reduction of  $I$ . By the minimality of  $J$ , we must have  $J = (x_1, \dots, x_s)$ . The conclusion now follows from Lemma 1.1.  $\square$

The above notions can be used to characterize the Castelnuovo-Mumford regularity of  $R(I)$ . Recall that the Castelnuovo-Mumford regularity of a standard graded algebra  $R$  over a commutative ring is defined as follows. Let  $H_{R_+}^i(R)$  denote the  $i$ -th local cohomology module of  $R$  with respect to the graded ideal  $R_+$  of elements of positive degree and set  $a_i(R) = \max\{n \mid H_{R_+}^i(R)_n \neq 0\}$  with the convention  $a_i(R) = -\infty$  if  $H_{R_+}^i(R) = 0$ . Then one defines

$$\text{reg } R := \max\{a_i(R) + i \mid i \geq 0\}.$$

It is known that  $\text{reg } R(I)$  can be characterized in terms of a reduction  $J = (x_1, \dots, x_s)$  of  $I$  such that  $x_1^*, \dots, x_s^*$  form a filter-regular sequence in  $G(I)$  [33, Theorem 4.8]. By Lemma 1.1 and Lemma 1.2 we can replace a filter-regular sequence in  $G(I)$  by a superficial sequence of  $I$  as follows.

**Theorem 1.3.** Let  $x_1, \dots, x_s$  be a superficial sequence of  $I$  such that  $J = (x_1, \dots, x_s)$  is a reduction of  $I$ . Then

$$\begin{aligned} \text{reg } R(I) &= \text{reg } G(I) \\ &= \min \{n \geq r_J(I) \mid I^{n+1} \cap [(x_1, \dots, x_{i-1}) : x_i] = (x_1, \dots, x_{i-1})I^n, i = 1, \dots, s\}. \end{aligned}$$

Moreover, the equations in the above formula holds for all  $n \geq \text{reg } R(I)$ .

Theorem 1.3 will play a crucial role in our paper. It can be also used to compute  $\text{reg } F(I)$  as we shall see later.

**Corollary 1.4.** *Let  $x$  be a superficial element of  $I$ . Then  $I^{n+1} \cap (x) = xI^n$  for  $n \geq \operatorname{reg} R(I)$ .*

*Proof.* Using Lemma 1.1 we can extend  $x_1 := x$  to a superficial sequence  $x_1, \dots, x_s$  of  $I$  such that  $J = (x_1, \dots, x_s)$  is a reduction of  $I$ ,  $s \geq 2$ . By Theorem 1.3, we have  $I^{n+1} \cap [(x_1) : x_2] = x_1 I^n$  for  $n \geq \operatorname{reg} R(I)$ . Since

$$I^{n+1} \cap [(x_1) : x_2] \supseteq I^{n+1} \cap (x_1) \supseteq x_1 I^n,$$

we obtain  $I^{n+1} \cap (x_1) = x_1 I^n$  for  $n \geq \operatorname{reg} R(I)$ .  $\square$

Corollary 1.4 leads us to the following property of colon ideals.

**Proposition 1.5.** *Let  $I$  be a regular ideal. Then  $I^{n+1} : I = I^n$  for  $n \geq \operatorname{reg} R(I)$ .*

*Proof.* It is well-known that if  $I$  is a regular ideal, every superficial element of  $I$  is a non-zerodivisor (see e.g. [26, Lemma 1.2]). Therefore,  $0 : x = 0$  if  $x$  is a superficial element for  $I$ . By Corollary 1.4,  $I^{n+1} : x = I^n + (0 : x) = I^n$  for  $n \geq \operatorname{reg} R(I)$ . Since  $I^n \subseteq I^{n+1} : I \subseteq I^{n+1} : x$ , this implies  $I^{n+1} : I = I^n$  for  $n \geq \operatorname{reg} R(I)$ .  $\square$

**Remark 1.6.** Corollary 1.4 can be also deduced from [33, Lemma 4.4 (i)], whose proof shows that  $I^{n+1} \cap (x) = xI^n$  for  $n \geq \max\{a_0(G(I)), a_1(G(I)) + 1\}$ . If  $I$  is a regular ideal,  $a_0(G(I)) < a_1(G(I))$  by [15, Theorem 5.2]. Therefore,  $I^{n+1} : I = I^n$  for  $n \geq a_1(G(I)) + 1$ .

Recall that the Ratliff-Rush closure of  $I$  is defined by

$$\tilde{I} := \bigcup_{n \geq 1} I^{n+1} : I^n.$$

Let  $s(I)$  denotes the least integer  $m \geq 0$  such that  $I^{n+1} : I^n = \tilde{I}$  for all  $n \geq m$ .

**Theorem 1.7.** *Let  $I$  be a regular ideal. Then  $s(I) \leq \max\{\operatorname{reg} R(I) - 1, 0\}$ .*

*Proof.* Applying Proposition 1.5 we have

$$I^{n+1} : I^n = (I^{n+1} : I) : I^{n-1} = I^n : I^{n-1}$$

for  $n \geq \operatorname{reg} R(I)$ . Thus,  $I^{n+1} : I^n$  is the same ideal for all  $n \geq \operatorname{reg} R(I) - 1$  and equals  $\tilde{I}$ .  $\square$

There are plenty examples with  $s(I) = 0$  and  $\operatorname{reg} R(I)$  arbitrarily large. For instance, if  $I = \mathfrak{m}$  then  $s(\mathfrak{m}) = 0$ . Since  $\operatorname{reg} R(\mathfrak{m}) \geq r_J(\mathfrak{m})$  for any minimal reduction  $J$  of  $\mathfrak{m}$ , one can easily construct local rings such that  $\operatorname{reg} R(\mathfrak{m})$  is arbitrarily large.

By Theorem 1.7, if we know an upper bound  $c$  of  $\operatorname{reg} R(I)$ , we can easily compute  $\tilde{I}$  because  $\tilde{I} = I^{c+1} : I^c$ . There are several upper bounds for  $\operatorname{reg} R(I)$  in terms of other invariants [5, 6, 20, 21, 27, 29, 36].

**Corollary 1.8.** *Let  $I$  be an  $\mathfrak{m}$ -primary ideal in a Cohen-Macaulay ring  $A$  and  $d = \dim A$ . Let  $e(I)$  be the the multiplicity  $e(I)$  of  $I$ . Then*

- (i)  $s(I) \leq e(I) - 1$  if  $d = 1$ ,
- (ii)  $s(I) \leq e(I)^{2(d-1)-1} [e(I) - 1]^{(d-1)!}$  if  $d \geq 2$ .

*Proof.* It was proved in [27, Corollary 3.4] that the right-hand sides of the bounds in (i) and (ii) are upper bounds for  $\operatorname{reg} R(I)$  in the case  $I = \mathfrak{m}$ . However, it can be checked that the proof also holds for an arbitrary  $\mathfrak{m}$ -primary ideal  $I$  (cf. [20, Corollary 4.4], where a more compact but weaker bound for  $\operatorname{reg} R(I)$  is given). Therefore, the assertion follows from Theorem 1.7.  $\square$

A similar upper bound for  $s(I)$  was already given for this case by Elias [9, Theorem 2.1(ii)], which is worse than Corollary 1.8. This bounds are deduced from a bound for  $s(I)$  in terms of the postulation numbers of  $I$  and of ideals of the form  $I/(x)$ , where  $x$  is an element of a given superficial sequence of  $I$  generating a minimal reduction of  $I$  [9, Theorem 2.1(i)].

## 2. RATLIFF-RUSH FILTRATION

Let  $(A, \mathfrak{m})$  be a local ring with  $\dim A > 0$  and  $I$  an ideal of  $A$ . One calls the sequence of ideals  $\widetilde{I}^n$ ,  $n \geq 1$ , the *Ratliff-Rush filtration* with respect to  $I$ . It is well known that for  $n \geq 1$ ,

$$\widetilde{I}^n = \bigcup_{t \geq 0} I^{n+t} : I^t.$$

To compute  $\widetilde{I}^n$ , we need to know an upper bound for the least number  $t$  with  $\widetilde{I}^n = I^{n+t} : I^t$ . The following result gives an upper bound for such  $t$  in terms of  $\operatorname{reg} R(I)$ . Moreover, we also obtain an upper bound for the stability index  $s^*(I)$ , which is the least integer  $m \geq 1$  such that  $\widetilde{I}^n = I^n$  for  $n \geq m$ .

**Proposition 2.1.** *Let  $I$  be a regular ideal. Then*

- (i)  $\widetilde{I}^n = I^{n+t} : I^t$  for  $t \geq \operatorname{reg} R(I) - n$ ,
- (ii)  $s^*(I) \leq \max\{\operatorname{reg} R(I), 1\}$ .

*Proof.* By Proposition 1.5 we have  $I^{n+1} : I = I^n$  for  $n \geq \operatorname{reg} R(I)$ . Therefore,

$$I^{n+t+1} : I^{t+1} = (I^{n+t+1} : I) : I^t = I^{n+t} : I^t$$

for  $t \geq \operatorname{reg} R(I) - n$ , which proves (i). If  $n \geq \operatorname{reg} R(I)$ , we can put  $t = 0$  in (i). Hence  $\widetilde{I}^n = I^n : I^0 = I^n$ , which proves (ii).  $\square$

As pointed out in Remark 1.6, we can replace  $\operatorname{reg} R(I)$  by  $a_1(G(I))+1$  in Proposition 2.1. So we can recover the bound  $s^*(I) \leq \max\{a_1(G(I))+1, 1\}$  given by Puthenpurakal in [22, Theorem 4.3]. Note that Puthenpurakal considers the least integer  $m \geq 0$  such that  $\widetilde{I}^n = I^n$  for  $n \geq m$ , whereas we require  $m \geq 1$  because one always has  $\widetilde{I}^0 = I^0 = A$ . Proposition 2.1(ii) can be also deduced from this bound because  $a_1(G(I)) + 1 \leq \operatorname{reg} G(I) = \operatorname{reg} R(I)$  by Theorem 1.3.

Recall that a system of elements  $x_1, \dots, x_r$  in  $A$  is a *d-sequence* if the following two conditions are satisfied:

- (i)  $x_i$  is not contained in the ideal generated by the rest of the system,  $i = 1, \dots, r$ ,
- (ii)  $(x_1, \dots, x_i) : x_{i+1}x_k = (x_1, \dots, x_i) : x_{i+1}$  for all  $i = 0, \dots, r-1$  and  $k = i+1, \dots, r$ .

This notion was introduced by Huneke in [18]. Examples of *d-sequences* are abundant such as the maximal minors of an  $r \times (r+1)$  generic matrix and systems of parameters in Buchsbaum rings.

It was known that  $I$  is generated by a  $d$ -sequence if and only if  $\text{reg } R(I) = 0$  [33, Corollary 5.7]. By Proposition 2.1(ii),  $\text{reg } R(I) = 0$  implies  $s^*(I) = 1$ . This means  $\widetilde{I}^n = I^n$  for all  $n \geq 1$ . So we obtain the following consequence.

**Corollary 2.2.** *Let  $I$  be a regular ideal generated by a  $d$ -sequence. Then  $\widetilde{I}^n = I^n$  for all  $n \geq 1$ .*

It is also known that  $\widetilde{I}^n = I^n$  for all  $n \geq 1$  if and only if  $G(I)$  contains a nonzerodivisor of positive degree [12, (1.2)]. This fact can be used to construct examples with  $s^*(I) = 1$  and  $\text{reg } R(I)$  arbitrarily large.

**Example 2.3.** Let  $R = k[X]/P$ , where  $k[X]$  is a polynomial ring and  $P$  is a homogeneous prime generated by forms of degree  $d$ . Let  $A$  be the localization of  $R$  at its maximal graded ideal. Then  $G(\mathfrak{m}) \cong R$ . Since  $\text{depth } R > 0$ ,  $s^*(\mathfrak{m}) = 1$ . By Theorem 1.3,  $\text{reg } R(\mathfrak{m}) = \text{reg } G(\mathfrak{m}) = \text{reg } R$ . Since  $\text{reg } R + 1$  is an upper bound for the degree of the generators of  $P$  [7],  $\text{reg } R(\mathfrak{m}) \geq d - 1$ .

Despite the possible large difference between  $\text{reg } R(I)$  and  $s^*(I)$  we can characterize  $\text{reg } R(I)$  in terms of  $s^*(I)$  in the following case.

Recall that  $A$  is called a *Buchsbaum ring* if every system of parameters  $x_1, \dots, x_r$  of  $A$  is a *weak sequence*, i.e.

$$(x_1, \dots, x_{i-1}) : x_i = (x_1, \dots, x_{i-1}) : \mathfrak{m}$$

for  $i = 1, \dots, r$ .

Huneke showed that  $A$  is a Buchsbaum ring if and only if every system of parameters forms a  $d$ -sequence [18, Proposition 1.7]. Therefore,  $\text{reg } R(I) = 0$  and  $s^*(I) = 1$  if  $I$  is a parameter ideal in a Buchsbaum ring. If  $I$  is not a parameter ideal, we have the following formula for  $\text{reg } R(I)$ , which is not known even in the case  $A$  is a regular local ring.

**Theorem 2.4.** *Let  $A$  be a two-dimensional Buchsbaum ring with  $\text{depth } A > 0$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal, which is not a parameter ideal. Let  $J$  be a minimal reduction of  $I$ . Then*

$$\text{reg } R(I) = \max\{r_J(I), s^*(I)\} = \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\}.$$

*Proof.* By Theorem 1.3,  $\text{reg } R(I) \geq r_J(I)$ . Since  $I$  is not a parameter ideal,  $r_J(I) \geq 1$ . Hence,  $\text{reg } R(I) \geq 1$ . By Proposition 2.1 (ii), this implies  $\text{reg } R(I) \geq s^*(I)$ . Thus,  $\text{reg } R(I) \geq \max\{r_J(I), s^*(I)\}$ . Since

$$\max\{r_J(I), s^*(I)\} \geq \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\},$$

it suffices to show that  $\text{reg } R(I) \leq \min\{n \geq r_J(I) \mid \widetilde{I}^n = I^n\}$ .

By Lemma 1.2, there is a superficial sequence  $x, y$  of  $I$  such that  $J = (x, y)$ . Since  $\text{depth } A > 0$ ,  $I$  is a regular ideal. Hence  $0 : x = 0$ . By Theorem 1.3, this implies

$$\text{reg } R(I) = \min\{n \geq r_J(I) \mid I^{n+1} \cap [(x) : y] = xI^n\}.$$

We will show that  $I^{n+1} \cap [(x) : y] = I^{n+1} \cap (x)$  for  $n \geq r_J(I)$ . Let  $f$  be an arbitrary element of  $I^{n+1} \cap [(x) : y]$ . Since  $I^{n+1} = (x, y)I^n$ , there are elements  $g, h \in I^n$  such that  $f = gx + hy$ . Since  $fy \in (x)$ ,  $h \in (x) : y^2$ . Since  $x, y$  is a  $d$ -sequence,  $(x) : y^2 = (x) : y$ . Hence  $hy \in y[(x) : y^2] \subseteq y[(x) : y] \subseteq (x)$ . This implies  $f \in (x)$ .

So we can conclude that  $I^{n+1} \cap [(x) : y] \subseteq I^{n+1} \cap (x)$ . Since the converse inclusion is obvious,  $I^{n+1} \cap [(x) : y] = I^{n+1} \cap (x)$ . Therefore,

$$\begin{aligned} \operatorname{reg} R(I) &= \min\{n \geq r_J(I) \mid I^{n+1} \cap (x) = xI^n\} \\ &= \min\{n \geq r_J(I) \mid I^{n+1} : x = I^n\}. \end{aligned}$$

We have  $I^n \subseteq I^{n+1} : x \subseteq \widetilde{I^{n+1}} : x$ . By [26, Lemma 3.1 (5)],  $\widetilde{I^{n+1}} : x = \widetilde{I^n}$ . Therefore,  $I^{n+1} : x = I^n$  if  $\widetilde{I^n} = I^n$ . So we can conclude that

$$\operatorname{reg} R(I) \leq \min\{n \geq r_J(I) \mid \widetilde{I^n} = I^n\}.$$

□

The formula  $\operatorname{reg} R(I) = \min\{n \geq r_J(I) \mid \widetilde{I^n} = I^n\}$  provides an effective tool for the computation of  $\operatorname{reg} R(I)$  because we only need to check the condition  $\widetilde{I^n} = I^n$  successively for  $n \geq r_J(I)$ . Moreover, comparing to Theorem 1.3, we do not need a superficial sequence which generates a reduction of  $I$ . To find such a sequence is not easy in general.

**Remark 2.5.** The proof of Theorem 2.4 leads us to raise the question of whether  $\operatorname{reg} R(I) = \min\{n \geq r_J(I) \mid I^{n+1} : I = I^n\}$ . We could not find any counter-example to this formula.

Rossi and Swanson [25, Section 4] asked whether  $s^*(I) \leq r_J(I)$  always holds. Under the assumption of Theorem 1.3, a positive answer would imply  $r_J(I) = \operatorname{reg} R(I)$  for all minimal reductions  $J$  of  $I$ . However, Huckaba [17, Example 3.1] showed that  $r_J(I)$  depends on the choice of  $J$ . So we obtain a negative answer to the above question.

On the other hand, we can give the following simple condition for the invariance of the reduction numbers.

**Corollary 2.6.** *Let  $A$  be a two-dimensional Buchsbaum ring with  $\operatorname{depth} A > 0$  and  $I$  an  $\mathfrak{m}$ -primary ideal. If there exists a minimal reduction  $J$  of  $I$  such that  $s^*(I) < r_J(I)$ , then the reduction numbers of all minimal reductions of  $I$  equal  $\operatorname{reg} R(I)$ .*

*Proof.* Since  $r_J(I) > 1$ ,  $I$  is not a parameter ideal. By Theorem 2.4, the assumption implies  $\operatorname{reg} R(I) = r_J(I)$ . If there is a minimal reduction  $J'$  of  $I$  with  $r_{J'}(I) \neq r_J(I)$ , from the formula  $\operatorname{reg} R(I) = \max\{r_{J'}(I), s^*(I)\}$  we can deduce that  $\operatorname{reg} R(I) = s^*(I)$ , a contradiction. □

One can easily find examples, where  $r_J(I)$  is arbitrarily larger than  $s^*(I)$ . For instance, let  $A$  be a local ring as in Example 2.3. Then  $s^*(\mathfrak{m}) = 1$  while  $\operatorname{reg} R(\mathfrak{m})$  can be arbitrarily large. If we choose  $A$  to be a two-dimensional Buchsbaum ring, then  $r_J(\mathfrak{m}) = \operatorname{reg} R(\mathfrak{m})$  by Theorem 2.4.

**Remark 2.7.** Let  $br(I)$  denote the big reduction number of  $I$  which is defined by

$$br(I) := \max\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

If  $br(I) \geq s^*(I)$ , then  $\operatorname{reg} R(I) = br(I)$  by Theorem 2.4. Under the assumption of Theorem 2.4, we could not find any example with  $br(I) < s^*(I)$ . So we raise the question whether  $\operatorname{reg} R(I) = br(I)$  always holds in this case.

In the following we will give an alternative formula for  $\operatorname{reg} R(I)$ , which involves only the Ratliff-Rush closure of a power of  $I$ . This is based on the following observation.



**Lemma 2.8.** *Let  $A$  be a two-dimensional Buchsbaum ring with depth  $A > 0$  and  $I$  an  $\mathfrak{m}$ -primary ideal. Let  $x, y$  be a superficial sequence of  $I$  such that  $J = (x, y)$  is a minimal reduction of  $I$ . Set  $r = r_J(I)$ . For  $n \geq r$ , we have*

$$I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x).$$

*Proof.* The case  $n = r$  is trivial. For  $n \geq r + 1$ , let  $f$  be an arbitrary element of  $I^{n+1} : x$ . Since  $I^{n+1} = (x, y)I^n$ , there are elements  $g, h \in I^n$  such that  $xf = xg + yh$ . From this it follows that  $h \in I^n \cap [(x) : y]$ . As showed in the proof of Theorem 2.4,

$$I^n \cap [(x) : y] = I^n \cap (x) = x(I^n : x).$$

Hence  $h = xh'$  for some element  $h' \in I^n : x$ . Thus,  $xf = xg + xyh'$ . Since  $x$  is a non-zerodivisor,  $f = g + yh' \in I^n + y(I^n : x)$ . So we have  $I^{n+1} : x \subseteq I^n + y(I^n : x)$ . Since the inverse inclusion is obvious, we can conclude that

$$I^{n+1} : x = I^n + y(I^n : x).$$

Applying this formula to  $I^n : x$  and so on, we get  $I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x)$ .  $\square$

**Theorem 2.9.** *Let  $A$  be a two-dimensional Buchsbaum ring with depth  $A > 0$  and  $I$  an  $\mathfrak{m}$ -primary ideal. Let  $J$  be a minimal reduction of  $I$ . Set  $r = r_J(I)$ . Then*

$$\text{reg } R(I) = \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}.$$

*Proof.* If  $r = 0$ ,  $I$  is a parameter ideal. Since  $A$  is Buchsbaum,  $I$  is generated by a  $d$ -sequence. Hence  $\text{reg } R(I) = 0$  by [33, Corollary 5.7]. In this case, the above formula is trivial. Therefore, we may assume that  $I$  is not a parameter ideal.

Let  $J = (x, y)$ , where  $x, y$  is a superficial sequence of  $I$ . By the proof of Theorem 2.4, we have

$$\text{reg } R(I) = \min\{n \geq r \mid I^{n+1} : x = I^n\}.$$

By Lemma 2.8,

$$I^{n+1} : x = I^n + y^{n-r}(I^{r+1} : x) \subseteq I^n + I^{n-r}(\widetilde{I}^{r+1} : x).$$

By [26, Lemma 3.1 (5)],  $\widetilde{I}^{r+1} : x = \widetilde{I}^r$ . If  $\widetilde{I}^r = I^n : I^{n-r}$ , we have

$$I^n + I^{n-r}(\widetilde{I}^{r+1} : x) = I^n + I^{n-r}(I^n : I^{n-r}) = I^n \subseteq I^{n+1} : x.$$

So we can conclude  $I^{n+1} : x = I^n$ . Hence,

$$\text{reg } R(I) \geq \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}.$$

To show the converse inequality we observe that  $\widetilde{I}^n : x^{n-r} = \widetilde{I}^r$  [26, Lemma 3.1 (5)]. Since  $\widetilde{I}^r \subseteq \widetilde{I}^n : I^{n-r} \subseteq \widetilde{I}^n : x^{n-r}$ , this implies  $\widetilde{I}^r = \widetilde{I}^n : I^{n-r}$ . If  $\widetilde{I}^n = I^n$ , then  $\widetilde{I}^r = I^n : I^{n-r}$ . Therefore, using Theorem 2.4, we have

$$\text{reg } R(I) = \min\{n \geq r \mid \widetilde{I}^n = I^n\} \leq \min\{n \geq r \mid \widetilde{I}^r = I^n : I^{n-r}\}.$$

$\square$

### 3. REGULARITY OF THE FIBER RING

Throughout this section let  $A$  be a finitely generated standard graded algebra over a field  $k$  with  $\dim A > 0$ . Let  $I$  be an ideal generated by homogeneous elements of the same degree  $d$ ,  $d \geq 1$ . Under this assumption, the fiber ring  $F(I)$  is isomorphic to the algebra  $k[I_d]$  generated by the elements of the component  $I_d$  of  $I$ .

In the following we will identify  $F(I)$  with  $k[I_d]$ . Let  $\mathfrak{n}$  denote the maximal graded ideal of  $F(I)$ . Since  $F(I)$  is a standard graded algebra over  $k$ ,  $F(I) \cong G(\mathfrak{n})$ . By Theorem 1.3,

$$\operatorname{reg} F(I) = \operatorname{reg} G(\mathfrak{n}) = \operatorname{reg} R(\mathfrak{n}),$$

and we can use a minimal reduction of  $\mathfrak{n}$  to compute  $\operatorname{reg} F(I)$ .

In general, there is a natural correspondence between minimal reductions of  $I$  and  $\mathfrak{n}$ . In our setting, this correspondence can be described as follows.

**Lemma 3.1.** *Let  $J$  be a homogeneous reduction of  $I$ . Let  $\mathfrak{q}$  be the ideal generated by  $J_d$  in  $F(I)$ . Then  $\mathfrak{q}$  is a reduction of  $\mathfrak{n}$  with  $r_{\mathfrak{q}}(\mathfrak{n}) = r_J(I)$ .*

*Proof.* By our assumption, the ideals  $\mathfrak{n}$  and  $I$  or  $\mathfrak{q}$  and  $J$  have the the same generating set. From this it follows that  $\mathfrak{n}^{n+1} = \mathfrak{q}\mathfrak{n}^n$  if and only if  $I^{n+1} = JI^n$ . Hence the conclusion is immediate.  $\square$

**Theorem 3.2.** *Let  $x_1, \dots, x_s$  be a superficial sequence of homogeneous elements of  $I$  such that  $J = (x_1, \dots, x_s)$  is a reduction of  $I$ . Then*

$$\begin{aligned} \operatorname{reg} F(I) &= \min \{t \geq r_J(I) \mid \\ &((I^{n+1}) \cap [(x_1, \dots, x_{i-1}) : x_i])_{(n+1)d} = ((x_1, \dots, x_{i-1})I^n)_{(n+1)d} \text{ for } n \geq t, i = 1, \dots, s\}. \end{aligned}$$

*Proof.* By Theorem 1.3 we have

$$I^{n+1} \cap [(x_1, \dots, x_{i-1}) : x_i] = (x_1, \dots, x_{i-1})I^n$$

for all  $n \geq R(I)$ ,  $i = 1, \dots, s$ . Therefore,

$$(I^{n+1} \cap [(x_1, \dots, x_{i-1}) : x_i])_{(n+1)d} = ((x_1, \dots, x_{i-1})I^n)_{(n+1)d}$$

for all  $n \geq \operatorname{reg} R(I)$ . Since  $\mathfrak{n}^{t+1} = \bigoplus_{n \geq t} (I^{n+1})_{(n+1)d}$ , this implies

$$\mathfrak{n}^{t+1} \cap [(x_1, \dots, x_{i-1})F(I) : x_i] = (x_1, \dots, x_{i-1})\mathfrak{n}^t$$

for  $t \geq \operatorname{reg} R(I)$ ,  $i = 1, \dots, s$ . Note that  $\operatorname{reg} R(I) \geq r_J(I) = r_{\mathfrak{q}}(\mathfrak{n})$  by Theorem 1.3 and Lemma 3.1. Then the above equations implies that  $x_1^*, \dots, x_s^*$  form a filter-regular sequence of  $F(I) = G(\mathfrak{n})$  [33, Theorem 4.7]. Hence,  $x_1, \dots, x_s$  is a superficial sequence of  $\mathfrak{n}$  by Lemma 1.1. Now, we can apply Theorem 1.3 to the reduction  $\mathfrak{q} = (x_1, \dots, x_s)F(I)$  of  $\mathfrak{n}$ , and obtain

$$\begin{aligned} \operatorname{reg} F(I) &= \\ &\min \{t \geq r_J(I) \mid \mathfrak{n}^{t+1} \cap [(x_1, \dots, x_{i-1})F(I) : x_i] = (x_1, \dots, x_{i-1})\mathfrak{n}^t, i = 1, \dots, s\}. \end{aligned}$$

This formula can be rewritten as in the statement of the theorem.  $\square$

The relations in Theorem 3.2 are only the initial parts of the relations in Theorem 1.3. As an immediate consequence, we get the inequality  $\operatorname{reg} R(I) \geq \operatorname{reg} F(I)$ , which was already observed by Eisenbud and Ulrich in [8, Section 1].

Since  $I$  is generated by elements of the same degree,  $R(I)$  is a bigraded algebra over  $k$ . Eisenbud and Ulrich used the bigraded minimal free resolution to define  $\text{reg } R(I)$ . That definition coincides with the definition of  $\text{reg } R(I)$  by means of local cohomology modules [24].

Let  $\mathfrak{m}$  be the maximal graded ideal of  $A$ . Eisenbud and Ulrich conjectured that if  $I$  is an  $\mathfrak{m}$ -primary ideal, then  $\text{reg } R(I) = \text{reg } F(I)$  [8, Conjecture 1.3]. This conjecture is not true if  $A$  is a non-Buchsbaum ring. This follows from the following observation. Note that  $A$  is called a Buchsbaum ring if  $A_{\mathfrak{m}}$  is a Buchsbaum ring.

**Proposition 3.3.** *Assume that  $\text{reg } R(Q) = \text{reg } F(Q)$  for every parameter ideal  $Q$  generated by forms of the same degree in graded quotient rings of  $A$ . Then  $A$  is a Buchsbaum ring.*

*Proof.* It is well known that every system of parameters is analytically independent. From this it follows that  $F(Q)$  is isomorphic to a polynomial ring over  $k$ . Hence  $\text{reg } R(Q) = \text{reg } F(Q) = 0$ . By [33, Corollary 5.7], this implies that  $Q$  is generated by a  $d$ -sequence. In particular, every system of parameters of  $A$ , which consists of forms of the same degree, is a  $d$ -sequence.

Let  $x_1, \dots, x_s$  be a homogeneous system of parameters of degree 2 of  $A$ ,  $s = \dim A$ . Applying the above fact to the quotient ring  $A/(x_1, \dots, x_i)$ ,  $i < s$ , we can deduce that every homogeneous system of parameters  $x_1, \dots, x_i, y_1, \dots, y_{s-i}$  of  $A$ , where  $y_1, \dots, y_{s-i}$  are linear forms, is a  $d$ -sequence. By [30, Corollary 2.6], a local ring  $(B, \mathfrak{n})$  is Buchsbaum if there exists a system of parameters  $x'_1, \dots, x'_s$  in  $\mathfrak{n}^2$ ,  $s = \dim B$ , and a generating set  $S$  for  $\mathfrak{n}$  such that  $x'_1, \dots, x'_i, y'_1, \dots, y'_{s-i}$  is a  $d$ -sequence for every family  $y'_1, \dots, y'_{s-i}$  of  $s - i$  elements of  $S$ ,  $i = 1, \dots, s$  (the term absolutely superficial sequence was used there for  $d$ -sequence). From this it follows that  $A$  is a Buchsbaum ring.  $\square$

The following example shows that  $\text{reg } R(I)$  can be arbitrarily larger than  $\text{reg } F(I)$  even when  $I$  is a parameter ideal in an one-dimensional non-Buchsbaum ring.

**Example 3.4.** Let  $A = k[x, y]/(x^t, xy^{t-1})$ ,  $t \geq 2$ . Then  $A$  is a non-Buchsbaum ring for  $t \geq 3$ . Let  $I = yA$ . It is clear that  $\text{reg } F(I) = 0$ . Using Theorem 1.3, we have  $\text{reg } R(I) = \min\{n \geq 0 \mid y^{n+1}A \cap (0 : yA) = 0\} = t - 2$ .

We can prove the conjecture of Eisenbud and Ulrich in the following case.

**Theorem 3.5.** *Let  $A$  be a Buchsbaum ring with  $\dim A \geq 1$ , Let  $I$  be an  $\mathfrak{m}$ -primary ideal such that  $\text{depth } G(I) \geq \dim A - 1$ . Then*

$$\text{reg } R(I) = \text{reg } F(I) = r_J(I)$$

for every minimal reduction  $J$  of  $I$ .

*Proof.* By [32, Theorem 1.2], the assumption implies that  $\text{reg } R(I) = r_J(I)$ . By Theorem 3.2,  $r_J(I) \leq \text{reg } F(I) \leq \text{reg } R(I)$ . So we must have  $\text{reg } R(I) = \text{reg } F(I) = r_J(I)$ .  $\square$

The condition  $\text{depth } G(I) \geq \dim A - 1$  is satisfied if  $\dim A = 1$ . Therefore, we have the following consequence.

**Corollary 3.6.** *Let  $A$  be an one-dimensional Buchsbaum ring. Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Then  $\text{reg } R(I) = \text{reg } F(I)$ .*

In the following we will show that there is a strong relationship between  $\text{reg } F(I)$  and the initial behavior of the ideals  $\widetilde{I}^n$ .

Let  $s_{\text{in}}^*(I)$  denote the least integer  $m$  such that  $(\widetilde{I}^n)_{nd} = (I^n)_{nd}$  for all  $n \geq m$ . Note that  $nd$  is the initial degree of  $\widetilde{I}^n$  and  $I^n$ .

**Lemma 3.7.** *Let  $I$  be a regular ideal. Then  $s_{\text{in}}^*(I) \leq \max\{\text{reg } F(I), 1\}$ .*

*Proof.* We have to show that  $(\widetilde{I}^n)_{nd} = (I^n)_{nd}$  for  $n \geq \text{reg } F(I)$ . Since  $\widetilde{I}^n = \cup_{t \geq 0} I^{n+t} : I^t$ , it suffices to show that  $(I^{n+t} : I^t)_{nd} \subseteq (I^n)_{nd}$  for  $n \geq \text{reg } F(I)$ .

Let  $x \in I_d$  be a superficial element of  $\mathfrak{n}$ . Since  $\mathfrak{n}$  is a regular ideal,  $x$  is a non-zerodivisor. By Corollary 1.4,  $\mathfrak{n}^{n+1} \cap (x) = x\mathfrak{n}^n$  for  $n \geq \text{reg } F(I)$ . This implies  $\mathfrak{n}^{n+1} : x = \mathfrak{n}^n$ . From this it follows that  $\mathfrak{n}^{n+t} : x^t = \mathfrak{n}^n$  for  $n \geq \text{reg } F(I)$ . Note that  $(\mathfrak{n}^n)_d = (I^n)_{nd}$ . Then

$$(I^{n+t} : I^t)_{nd} \subseteq (I^{n+t} : x^t)_{nd} = (\mathfrak{n}^{n+t} : x^t)_d = (\mathfrak{n}^n)_d = (I^n)_{nd}$$

for  $n \geq \text{reg } F(I)$ , which implies the conclusion.  $\square$

**Theorem 3.8.** *Let  $A$  be a two-dimensional Buchsbaum ring with  $\text{depth } A > 0$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal, which is not a parameter ideal. Let  $J$  be a homogeneous minimal reduction of  $I$ . Then*

$$\text{reg } F(I) = \max\{r_J(I), s_{\text{in}}^*(I)\} = \min\{n \geq r_J(I) \mid (\widetilde{I}^n)_{nd} = (I^n)_{nd}\}.$$

*Proof.* By Theorem 3.2 we have  $\text{reg } F(I) \geq r_J(I)$ . Since  $I$  is not a parameter ideal,  $\mathfrak{n}$  is not generated by two elements. From this it follows that the defining equations of  $F(I)$  have degree  $> 1$ . Hence  $\text{reg } F(I) > 0$  [7]. By Lemma 3.7, this implies  $\text{reg } F(I) \geq s_{\text{in}}^*(I)$ . Thus,

$$\text{reg } F(I) \geq \max\{r_J(I), s_{\text{in}}^*(I)\} \geq \min\{n \geq r_J(I) \mid (\widetilde{I}^n)_{nd} = (I^n)_{nd}\}.$$

Therefore, it suffices to show that  $\text{reg } F(I) \leq \min\{n \geq r_J(I) \mid (\widetilde{I}^n)_{nd} = (I^n)_{nd}\}$ .

Since  $I = (I_d)$  is a regular ideal,  $I_d$  contains a non-zerodivisor. From this it follows that  $\text{depth } F(I) = \text{depth } k[I_d] > 0$ . Let  $\mathfrak{n}$  be the maximal graded ideal of  $F(I)$  and  $\mathfrak{q}$  the ideal generated by  $J_d$  in  $F(I)$ . Then  $\mathfrak{q}$  is a minimal reduction of  $\mathfrak{n}$  with  $r_{\mathfrak{q}}(\mathfrak{n}) = r_J(I)$  by Lemma 3.1. By Lemma 1.2 we may assume that  $\mathfrak{q} = (x, y)$ , where  $x, y$  is a superficial sequence for  $\mathfrak{n}$ . By the proof of Theorem 1.3 to  $\mathfrak{n}$  we have

$$\text{reg } F(I) = \min\{n \geq r_J(I) \mid \mathfrak{n}^{n+1} : x = \mathfrak{n}^n\}.$$

Thus, we only need to show that if  $(\widetilde{I}^n)_{nd} = (I^n)_{nd}$  for  $n \geq r_J(I)$ , then  $\mathfrak{n}^{n+1} : x = \mathfrak{n}^n$ . Note that  $\mathfrak{n}^{n+1} : x = \bigoplus_{t \geq n} (I^{t+1} : x)_{td}$  and  $\mathfrak{n}^n = \bigoplus_{t \geq n} (I^t)_{td}$ .

Without restriction we may also assume that  $x, y$  is a superficial sequence for  $I$  and  $J = (x, y)$ . By Lemma 2.8, we have  $I^{t+1} : x = I^t + y^{t-n}(I^{n+1} : x)$  for  $t \geq n \geq r_J(I)$ . Since  $I^{n+1} : x \subseteq \widetilde{I}^{n+1} : x = \widetilde{I}^n$  [26, Lemma 3.1(5)],  $I^{t+1} : x \subseteq I^t + y^{t-n}\widetilde{I}^n$ . If  $(\widetilde{I}^n)_{nd} = (I^n)_{nd}$ , then

$$(I^{t+1} : x)_{td} \subseteq (I^t)_{td} + y^{t-n}(\widetilde{I}^n)_{nd} \subseteq (I^t)_{td} + y^{t-n}(I^n)_{nd} = (I^t)_{td} \subseteq (I^{t+1} : x)_{td}.$$

From this it follows that  $(I^{t+1} : x)_{td} = (I^t)_{td}$  for all  $t > n$ . Hence,  $\mathfrak{n}^{n+1} : x = \mathfrak{n}^n$ , as required.  $\square$

**Corollary 3.9.** *Let  $A$  be a two-dimensional Buchsbaum ring with  $\text{depth } A > 0$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Let  $J$  be a homogeneous minimal reduction of  $I$ . Then  $\text{reg } R(I) = \text{reg } F(I)$  if and only if  $\widetilde{I}^n = I^n$  for the least integer  $n \geq r_J(I)$  such that  $(\widetilde{I}^n)_{nd} = (I^n)_{nd}$ .*

*Proof.* If  $I$  is a parameter ideal, we have  $\text{reg } R(I) = \text{reg } F(I) = 0$  by [33, Corollary 5.7] and  $\widetilde{I}^n = I^n$  for  $n \geq 1$  by Corollary 2.2. If  $I$  is not a parameter ideal, the conclusion follows from Theorem 2.4 and Theorem 3.8.  $\square$

#### 4. MONOMIAL IDEALS IN TWO VARIABLES

In this section we will use the relationship between Castelnuovo-Mumford regularity and Ratliff-Rush closure to investigate the conjecture of Eisenbud and Ulrich for monomial ideals in two variables.

Let  $A = k[x, y]$  be a polynomial ring over a field  $k$  and  $\mathfrak{m} = (x, y)$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal generated by monomials of degree  $d$ ,  $d \geq 1$ . Then  $I$  contains  $x^d, y^d$ , and  $J = (x^d, y^d)$  is a minimal reduction of  $I$ . It is well-known [28] and easy to see that

$$\widetilde{I}^n = \bigcup_{t \geq 0} I^{n+t} : (x^{td}, y^{td}).$$

**Lemma 4.1.** *Let  $I = (x^d, y^d) + (x^{d-i}y^i \mid a \leq i \leq b)$ , where  $a \leq b < d$  are given positive integers. Then  $\widetilde{I}^n = I^n$  for all  $n \geq 1$ .*

*Proof.* Let  $x^i y^j$  be an arbitrary monomial of  $\widetilde{I}^n$ . Then  $x^{i+td} y^j \in I^{n+t}$  for some  $t \geq 1$ . Since  $I^{n+t}$  is generated by monomials of degree  $(n+t)d$ ,  $x^{i+td} y^j$  is divisible by a monomial  $x^{(n+t)d-c} y^c \in I^{n+t}$ . The divisibility implies  $i+td \geq (n+t)d-c$  and  $j \geq c$ .

If  $j < na$ , then

$$(n+t)d-c \geq (n+t)d-j > (n+t)d+na = td+n(d-a).$$

Let  $M = \{x^d, x^{d-a}y^a, x^{d-a-1}y^{a+1}, \dots, x^{d-b}y^b, y^d\}$  be the set of the monomial generators of  $I$ . Then  $x^{(n+t)d-c} y^c$  is a product of  $n+t$  monomials of  $M$ . Let  $s$  be the number of copies of  $x^d$  among these  $n+t$  monomials of  $M$ . If  $s < t$ , we would have  $(n+d)d-c \leq sd + (n+t-s)(d-a)$  because the exponent of  $x$  in each monomial in  $M \setminus \{x^d\}$  is less or equal  $d-a$ . Since

$$sd + (n+t-s)(d-a) = td + n(d-a) - (t-s)a < td + n(d-a),$$

we would get  $(n+d)d-c < td + n(d-a)$ , a contradiction. Therefore, we must have  $s \geq t$ . From this it follows that  $x^{nd-c} y^c = x^{(n+t)d-c} y^c / x^{td}$  is a product of  $n$  monomials in  $M$ . Hence  $x^{nd-c} y^c \in I^n$ . Since  $x^i y^j$  is divisible by  $x^{nd-c} y^c$ ,  $x^i y^j \in I^n$ .

By symmetry, if  $i < n(d-b)$ , we can also show that  $x^i y^j \in I^n$ .

Now, we may assume that  $i \geq n(d-b)$  and  $j \geq na$ . Let  $Q$  denote the ideal generated by the monomials  $x^{d-j} y^j$ ,  $a \leq j \leq b$ . It is clear that  $Q^n$  is generated by the monomials  $x^{nd-j} y^j$ ,  $na \leq j \leq nb$ . If  $j < nb$ ,  $x^i y^j$  is divisible by  $x^{nd-j} y^j$  because  $i \geq nd-c \geq nd-j$ . Therefore,  $x^i y^j \in Q^n$ . Since  $Q \subset I$ , we obtain  $x^i y^j \in I^n$ . If  $j \geq nb$ , then  $x^i y^j$  is divisible by  $x^{n(d-b)} y^{nb} = (x^{d-b} y^b)^n \in I^n$ . Thus, we always have  $x^i y^j \in I^n$ . Therefore, we can conclude that  $\widetilde{I}^n = I^n$ .  $\square$

Lemma 4.1 gives a large class of monomial ideals in two variables for which the conjecture of Eisenbud and Ulrich holds.

**Theorem 4.2.** *Let  $I = (x^d, y^d) + (x^{d-i}y^i \mid a \leq i \leq b)$ , where  $a \leq b < d$  are given positive integers. Then*

$$\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$$

for any homogeneous minimal reduction  $J$  of  $I$ .

*Proof.* By Lemma 4.1, we have  $s^*(I) = 1$ . Since  $s^*(I) \geq s_{\text{in}}^*(I) \geq 1$ , we also have  $s_{\text{in}}^*(I) = 1$ . Applying Theorem 2.4 and Theorem 3.8, we obtain  $\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$ .  $\square$

Theorem 4.2 contains the case  $I$  is an  $\mathfrak{m}$ -primary ideal generated by three monomials.

**Corollary 4.3.** *Assume that  $I = (x^d, x^{d-a}y^a, y^d)$ ,  $1 \leq a < d$ . Then*

$$\operatorname{reg} R(I) = \operatorname{reg} F(I) = d/(a, d) - 1.$$

*Proof.* This is the case  $a = b$  of Theorem 4.2. Therefore,  $\operatorname{reg} R(I) = \operatorname{reg} F(I) = r_J(I)$ . For  $J = (x^d, y^d)$ , it is easy to check that  $r_J(I) = d/(a, d) - 1$ .  $\square$

Now we will present another large class of monomial ideals in two variables for which the conjecture of Eisenbud and Ulrich holds.

**Theorem 4.4.** *Let  $I$  be an ideal in  $k[x, y]$  which is generated by monomials of degree  $d \geq 2$ . Assume that  $x^d, x^{d-1}y, y^d \in I$ . Then  $\operatorname{reg} R(I) = \operatorname{reg} F(I)$ .*

*Proof.* Let  $n$  be the least integer  $n \geq r_J(I)$  such that  $(\widetilde{I^n})_{nd} = (I^n)_{nd}$ . By Corollary 3.9, we only need to show that  $\widetilde{I^n} = (I^n)$ .

Let  $x^i y^j$  be an arbitrary monomial of  $\widetilde{I^n}$ . Then  $x^i y^{j+td} \in I^{n+t}$  for some  $t \geq 1$ . Since  $I^{n+t}$  is generated by monomials of degree  $(n+t)d$ , there exists a monomial  $x^a y^{(n+t)d-a} \in I^{n+t}$  such that  $x^i y^{j+td}$  is divisible by  $x^a y^{(n+t)d-a}$ . By the divisibility, we have  $i \geq a$  and  $j + td \geq (n+t)d - a$ .

If  $i \geq nd$ , then  $x^i y^j$  is divisible by  $x^{nd} \in I^n$ .

If  $i < nd$ , then  $a < nd$ . We have

$$(x^a y^{nd-a})x^{(nd-a)d} = (x^{d-1}y)^{nd-a} x^{nd} \in I^{nd-a+n}$$

$$(x^a y^{nd-a})y^{td} = x^a y^{(n+t)d-a} \in I^{n+t}.$$

Set  $s = \max\{(nd-a), t\}$ . Then  $(x^a y^{nd-a})x^{sd}, (x^a y^{nd-a})y^{sd} \in I^{n+s}$ . Hence  $x^a y^{nd-a} \in I^{n+s} : (x^{sd}, y^{sd}) \subseteq \widetilde{I^n}$ . Since  $(\widetilde{I^n})_{nd} \in (I^n)_{nd}$ ,  $x^a y^{nd-a} \in I^n$ . Since  $i \geq a$  and  $j \geq nd - a$ ,  $x^i y^j$  is divisible by  $x^a y^{nd-a}$ . Thus,  $x^i y^j \in I^n$ .

So we can conclude that  $\widetilde{I^n} = (I^n)$ .  $\square$

Actually, the proof of the above cases is obtained by translating everything in terms of lattice points in  $\mathbb{N}^2$ . For every set  $Q \subseteq A$  we consider the set  $\operatorname{Exp}(Q)$  of the exponent vectors of the monomials in  $Q$ .

Let  $E = \operatorname{Exp}(I_d)$ . Then  $\operatorname{Exp}((I^n)_{nd}) = nE$  for all  $n \geq 0$ , where  $nE$  is the Minkowski sum of  $n$  copies of  $E$ . Let

$$F_n = \{\mathbf{a} \in \mathbb{N}^2 \mid \mathbf{a} + t\mathbf{e}_1, \mathbf{a} + t\mathbf{e}_2 \in (n+t)E \text{ for some } t \geq 0\},$$

where  $\mathbf{e}_1 = (d, 0)$  and  $\mathbf{e}_2 = (0, d)$ . Then  $F_n = \text{Exp}(\widetilde{(I^n)}_{nd})$ . Hence  $\widetilde{(I^n)}_{nd} = (I^n)_{nd}$  if and only if  $F_n = nE$ . Similarly, if we let

$$F_n^* = \{\mathbf{a} \in \mathbb{N}^2 \mid \mathbf{a} + t\mathbf{e}_1, \mathbf{a} + t\mathbf{e}_2 \in (n+t)E + \mathbb{N}^2 \text{ for some } t \geq 0\},$$

then  $\text{Exp}(\widetilde{(I^n)}_{nd}) = F_n^*$ . Hence  $\widetilde{(I^n)}_{nd} = (I^n)_{nd}$  if and only if  $F_n^* = nE + \mathbb{N}^2$ .

**Remark 4.5.** Let  $S \subseteq \mathbb{N}^2$  denote the additive monoid generated by  $E$ . Let  $k[S]$  denote the semigroup ring  $k[S]$  of  $S$  over  $k$ . Then  $k[S] \cong F(I)$ . Let  $S^* = \cup_{n \geq 0} F_n^*$ . Then  $S^*$  is also an additive monoid. By [31, Lemma 1.1],  $k[S]$  is Cohen-Macaulay or Buchsbaum if and only if  $S^* = S$  or  $S^* + (S \setminus \{0\}) \subseteq S$ . In particular,

$$H_{\mathfrak{n}}^1(k[S]) \cong k[S^*]/k[S],$$

where  $\mathfrak{n}$  denote the maximal graded ideal of  $k[S]$ .

Now we will give an example such that  $\text{reg } R(I) = s^*(I) = r_J(I)$  for  $J = (x^d, y^d)$ , but  $s^*(I) > r_{J'}(I)$  for another minimal reduction  $J'$  of  $I$ . This example shows that  $r_J(I)$  is not always the minimal reduction number of  $I$  for  $J = (x^d, y^d)$ .

**Example 4.6.** Let  $I = (x^7, x^6y, x^2y^5, y^7)$ . First, we will show that  $\text{reg } R(I) = r_J(I)$  for  $J = (x^7, y^7)$ . By Theorem 4.4 we know that  $\text{reg } R(I) = \text{reg } F(I)$ . Since  $\text{reg } F(I) \geq r_J(I)$  by (2), it suffices to show that  $\text{reg } F(I) = r_J(I)$ . It is easy to check that  $r_J(I) = 4$  and  $(I^4)_{28} = (I^4)_{28}$ . By Theorem 3.8, this implies  $\text{reg } F(I) = 4$ . On the other hand, it is shown in [17, Example 3.1] that  $r_{J'}(I) \leq 3$  for  $J' = (x^7, x^6y + y^7)$ . By Theorem 2.4, this implies  $\text{reg } R(I) = s^*(I) = r_J(I)$ .

The following example shows that we may have  $\text{reg } R(I) = s^*(I) > r_J(I)$ , where  $J = (x^d, y^d)$ . We do not know whether  $br(I) = r_J(I)$  in this example. If  $br(I) = r_J(I)$ , this will give a negative answer to the question of Remark 2.7.

**Example 4.7.** Let  $I = (x^{17-i}y^i \mid i = 0, 1, 3, 5, 13, 14, 16, 17)$ . By [13, Example 3.2], we have  $\text{reg } F(I) = 4 > r_J(I) = 3$ , where  $J = (x^{17}, y^{17})$ . By Theorem 4.4 we have  $\text{reg } R(I) = \text{reg } F(I) = 4$ . Hence, Theorem 2.4 implies  $\text{reg } R(I) = s^*(I) > r_J(I)$ .

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