PRIMARY IDEALS WITH GOOD ASSOCIATED GRADED RING

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Abstract

Let $(A,\mathcal{M})$ be a local Cohen-Macaulay ring of dimension d. Let I be an $\mathcal{M}-$primary ideal and let J be the ideal generated by a maximal superficial sequence for I. Under these assumptions Valabrega and Valla [VV] proved that the associated graded ring G of I is Cohen-Macaulay if and only if $I^n \cap J = JI^{n-1}$ for every integer n.

In this paper we consider the class of the $\mathcal{M}-$primary ideals I such that, for some positive integer k, we have $I^n \cap J = JI^{n-1}$ for $n \le k$ and $\lambda(I^{k+1}/JI^k) \le 1$. In this case G need not be Cohen-Macaulay. In Theorem 2.2. we show that G is Cohen-Macaulay unless the ideals we are considering are of maximal Cohen-Macaulay type. One can use the ideas of [RV2] and [RV3] to prove that, for the ideals we consider, the depth of G is at least $d-1$ and that its $h-$vector has no negative components. We characterize the possible Hilbert function of G.

Our approach gives a proof of an extended version of a conjecture of J. Sally (proved in [RV2] and independently in [W] in the case $I = \mathcal{M}$). Several results proved in [S2], [H], [RV1], [RV2],[RV3] are unified and generalized.

Introduction

Let $(A,\mathcal{M})$ be a local Cohen-Macaulay ring of dimension d and let I be an $\mathcal{M}-$primary ideal. We denote the associated graded ring of I by

$$G := gr_I(A) = \oplus_{n\ge 0} I^n/I^{n+1}$$

In the literature there are several papers which discuss the depth of the associated graded ring in terms of certain numerical invariants of I.

If J is an ideal generated by a maximal superficial sequence for I, then Valabrega and Valla in [VV] proved that G is Cohen-Macaulay if and only if $I^n \cap J = JI^{n-1}$ for every integer n. If this is the case one says that J is generated by a *standard* base (see [RoV]).

In this paper we consider the $\mathcal{M}-$ primary ideals such that, for some integer k, $I^n \cap J = JI^{n-1}$ for $n \leq k$. A primary ideal which satisfies this condition for some integer k is called $k-standard$.
It is clear that if I is $k-standard$ and $I^{k+1} = JI^k$ then, by Valabrega-Valla's condition, G is Cohen-Macaulay.

It is natural to consider the $k-standard$ ideals I such that $\lambda(I^{k+1}/JI^k) = 1$, where $\lambda(\quad)$ denote the length as $A-$module. In this case G is not necessarily Cohen-Macaulay.

For example, if we consider the $\mathcal{M}-$primary ideals I of multiplicity

$$e(I) = \lambda(I/I^2) + (1-d)\lambda(A/I) + 1$$

then $\lambda(I^2/JI) = 1$ and we are in the above situation for $k = 1$.

In this case, if $I = \mathcal{M}$, Sally proved that G is Cohen-Macaulay unless the local rings of maximal type (see [S2]). If this is the case, in [RV2] and, with a different method, in [W] it was proved that depth $G \geq d-1$.
The question about depth$G$ was still open for a general $\mathcal{M}-$primary ideal.
Corollary 2.3. and Corollary 3.8. give a complete answer to this problem.

By following the method used in [RV2], Huckaba proved that if $I^2 \cap J = JI$ and $\lambda(I^3/JI^2) \leq 1$, then depth $G \geq d-1$ (see [H]). In particular the author gives a positive answer to the Sally conjecture for an $\mathcal{M}-$primary ideal in the integrally closed case. In this case I is $2-standard$.

If $(A, \mathcal{M})$ is a Cohen-Macaulay local ring of embedding dimension N, multiplicity e and initial degree $t \geq 2$, it is well known that

$$e \geq \binom{N-d+t-1}{t-1}$$

and, if the equality holds, then G is Cohen-Macaulay.

If $e = \binom{N-d+t-1}{t-1} + 1$, then $I^n \cap J = JI^{n-1}$ for every $n \leq t$ and $\lambda(I^t/JI^{t-1}) = 1$.
In this case I is $t-standard$.
Once more, it is known that depth$G \geq d-1$ ([RV3]) and, if the the Cohen-Macaulay type of A is not maximal, then G is Cohen-Macaulay ([RV1]).

In this paper in order to extend the above facts, we consider the $\mathcal{M}-$ primary ideals which are $k-standard$ for some integer k. If $\lambda(I^{k+1}/JI^k) \leq 1$, we study the depth of the associated graded ring and the Hilbert function of I.

We denote by $\tau(I)$ the Cohen-Macaulay type of I, which is defined as

$$\tau(I) := \lambda(J : I/J)$$

One remarks that if $I = \mathcal{M}$, then $\tau(I)$ coincides with the Cohen-Macaulay type of A.

In Theorem 2.2. we give a condition in terms of $\tau(I)$ in order to get G Cohen-Macaulay. In any case we characterize the Hilbert function of I or equivalently the $h-$polynomial of I.

More generally in Theorem 3.2. we estimate the depth of G and we characterize the possible Hilbert functions of G.

After this paper was completed, I came to know that J. Elias in [E], A.Corso, C.Polini and M.Vaz Pinto in [CPV] obtained results which partially overlap the content of the last section of this paper.

1. Preliminaries

Let $(A, \mathcal{M})$ be a local ring of dimension d and let I be an $\mathcal{M}-$primary ideal. For all $n \geq 0$, I^n/I^{n+1} is a finite $A/I-$ module and we denote by $\lambda(I^n/I^{n+1})$ the corrispondent length.

The Hilbert function of I is by definition the Hilbert function of the associated graded ring of I which is the homogeneous A/I-module

$$G := gr_I(A) = \oplus_{n \geq 0} I^n/I^{n+1}.$$

Hence

$$H_I(n) = H_G(n) = \lambda(I^n/I^{n+1}).$$

The generating function of this numerical function is the power series

$$P_I(z) = \sum_{n \in \mathbb{N}} H_I(n) z^n$$

which is called the Hilbert Series of I. This series is rational and there exists a polynomial $h(z) \in \mathbb{Z}[z]$ such that

$$P_I(z) = \frac{h(z)}{(1-z)^d}$$

where $h(1) \geq 1$.

The polynomial $h(z) = h_0 + h_1 z + \cdots + h_s z^s$ is called the h-polynomial of I and the vector $(h_0, h_1, \ldots, h_s)$ the h-vector of I.

For every $i \geq 0$, the Hilbert coefficients of I can be characterized as follows

$$e_i(I) := \frac{h^{(i)}(1)}{i!}$$

where $h^{(i)}$ denotes the i-th derivative of $h(z)$. In particular $e_0(I) = h(1) := e(I)$ is the multiplicity of I (see [BH],Chap.4).

We recall that an element x in I is called *superficial* for I if there exists an integer $c > 0$ such that

$$(I^n : x) \cap I^c = I^{n-1}$$

for every $n > c$.

It is easy to see that a superficial element x is not in I^2 and that x is superficial for I if and only if $x^* := \overline{x} \in I/I^2$ does not belong to the relevant associated primes of G. Hence, if the residue field is infinite, superficial elements always exist. By passing, if needed, to the local ring $A[X]_{(\mathcal{M},X)}$ we may assume that the residue field is infinite.

Further if depth $A \geq 1$ every superficial element for I is also a regular element in A.

A sequence $x_1, \ldots, x_r$ in the local ring $(A, \mathcal{M})$ is called a *superficial sequence* for I, if x_1 is superficial for I and $\overline{x_i}$ is superficial for $I/(x_1, \ldots, x_{i-1})$ for $2 \leq i \leq r$.

If depth $A \geq r$, every superficial sequence $x_1, \ldots, x_r$ for I is also a regular sequence in A.

The following well known properties of the superficial sequences will be needed.

In particular if $J = (x_1, \ldots, x_r)$, $\overline{I} = I/J$ and $B = A/J$, then

- $e_i(I) = e_i(\overline{I})$ for $i = 0, \ldots, d - r$.
- $depth(gr_I(A)) \geq r \iff x_1^*, \ldots, x_r^*$ is a regular sequence in $gr_I(A) \iff$ $P_I(z) = \frac{P_{\overline{I}}(z)}{(1-z)^r} \iff I^j \cap J = JI^{j-1}$ for every $j \geq 1$.
- $depth(gr_I(A)) \geq r + 1 \iff depth(gr_{\overline{I}}(B)) \geq 1$.

This last property is the so called *Sally's machine* which is a very important trick to reduce dimension in questions relating to depth properties of $gr_I(A)$ (see Lemma 2.2 in [HM]).
Moreover if x is a superficial element for I, in [ERV] and [GR] it was proved :

- $e_d(I) = e_d(I/(x)) \iff depth(gr_I(A)) \geq 1$

If A is Cohen-Macaulay, we denote by $J = (x_1, \ldots, x_d)$ the ideal generated by a maximal superficial sequence and, for every $j \geq 0$, we denote by v_j the following integer

$$v_j := \lambda(I^{j+1}/JI^j)$$

We recall that, if A is a **one**-dimensional Cohen-Macaulay local ring we have $H_I(j) = e(I) - v_j$ and then

$$e_1(I) = \sum_{j=0}^{s-1} v_j$$

$$e_2(I) = \sum_{j=1}^{s-1} j v_j$$

where s is the reduction number of I, such that the least integer n such that $v_n = 0$.

Similar formulas can be found in the **two** dimensional case by using the following construction due to Ratliff and Rush (see [RR]).

Let $(A, \mathcal{M})$ be a Cohen-Macaulay local ring and I an $\mathcal{M}-$primary ideal. For every n we consider the chain of ideals

$$I^n \subseteq I^{n+1} : I \subseteq I^{n+2} : I^2 \subseteq \cdots \subseteq I^{n+k} : I^k \subseteq \cdots$$

This chain stabilizes at an ideal which was denoted by Ratliff and Rush as

$$\widetilde{I^n} := \bigcup_{k \geq 1} (I^{n+k} : I^k).$$

We define for every $n \geq 0$

$$\rho_n := \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}).$$

If dim$A$ $= 2$, the following formulas have been proved in [HM] Corollary 4.13 by using a homological approach and, with a different proof, in [GR] Corollary 1.10.

$$e_1(I) = \sum_{j \geq 0} \rho_j$$

and

$$e_2(I) = \sum_{j \geq 1} j\rho_j$$

2. k-standard $\mathcal{M}$-primary ideals

We recall that since we are assuming A to be Cohen-Macaulay of positive dimension d, we can always find a maximal superficial sequence $x_1, \ldots, x_d$ for I.

Definition 2.1. *Let $(A, \mathcal{M})$ be a Cohen-Macaulay local ring of dimension $d > 0$, k a positive integer and I an $\mathcal{M}-$primary ideal. We say that I is $k - standard$ if there exists an ideal J generated by a maximal superficial sequence such that*

$$I^n \cap J = I^{n-1} J \quad \forall \ \ n \leq k$$

In the following if I is $k - standard$, we denote by J an ideal generated by a maximal superficial sequence such that $I^n \cap J = I^{n-1} J \quad \forall \ \ n \leq k$.

Notice that :

- If I is $k - standard$ with $k > 1$, then I is $(k-1) - standard$.
- Every $\mathcal{M}-$primary ideal I is $1 - standard$.
- If I is integrally closed, then I is $2 - standard$.

Let $(A, \mathcal{M})$ be a Cohen-Macaulay local ring with embedding dimension N.

• If the initial degree $indeg(A) := \min \left\{ j \ : dim_k(\mathcal{M}^j/\mathcal{M}^{j+1}) < \binom{N+j-1}{j} \right\} \geq t$, then $\mathcal{M}$ is $t - standard$ (see [RV3], Proposition 2.2)

• If A is a stretched ring with Cohen-Macaulay type $\tau < N-d$ and multiplicity $e = N-d+3$, then $\mathcal{M}$ is $3-standard$ (see [RV4]).

It is clear that if I is $k-standard$ and $I^{k+1} = JI^k$, then $I^n \cap J = JI^{n-1}$ for every integer n and so G is Cohen- Macaulay.
If $\lambda(I^{k+1}/JI^k) = 1$, the following examples point out the different situations.

• $A = k[[t^4, t^5, t^{11}]]$, $I = \mathcal{M}$ and $J = (t^4)$, $\lambda(I^2/JI) = 1$ and depth $G = 0$.

• $A = k[[x, y, z, u]]$, $I = (x^2, y^2, z^2, u^2, xy+zu)$ and $J = (x^2, y^2, z^2, u^2)$, $\lambda(I^2/JI) = 1$ and depth $G = 3$.

• $A = k[[t^7, t^{10}, t^{12}, t^{15}, t^{18}]]$, $I = \mathcal{M}$ and $J = (t^7)$, then I is $2-standard$, $\lambda(I^3/JI^2) = 1$ and in this case G is Cohen-Macaulay.

• $A = k[[t^7, t^8, t^{12}, t^{13}, t^{18}]]$, $I = \mathcal{M}$ and $J = (t^7)$, I is $2-standard$, $\lambda(I^3/JI^2) = 1$ and depth$G = 0$.

Let $(A, \mathcal{M})$ be a Cohen-Macaulay local ring of dimension d, embedding dimension N and multiplicity e.
Sally proved that if $e = N-d+2$ (or equivalently $\lambda(\mathcal{M}^2/J\mathcal{M}) = 1$) the exceptions to the Cohen-Macaulayness of G come from the Cohen-Macaulay local rings of maximal Cohen-Macaulay type (see [S2]).

In order to extend this fact, we define

$$\tau(I) := \lambda(J : I/J)$$

and we call it the Cohen-Macaulay type of I.

Theorem 2.2. *Let $(A, \mathcal{M})$ be a d-dimensional Cohen-Macaulay local ring, k a positive integer and I a $k-standard$ $\mathcal{M}-$primary ideal.*
Suppose $\lambda(I^{k+1}/JI^k) \leq 1$ and $\tau(I) < \lambda(I^k/JI^{k-1}) - 1$, then
1) G is Cohen-Macaulay
2) the $h-$polynomial of I is

$$h(z) = \sum_{n=0}^{k} \lambda(I^n/(I^{n+1} + JI^{n-1}))z^n + c\, z^{k+1}$$

where $c = \lambda(I^{k+1}/JI^k)$.

Proof. We have

$$JI^k \subseteq I^{k+1} \cap J \subseteq I^{k+1}$$

If $JI^k \neq I^{k+1} \cap J$, then $I^{k+1} \subseteq J$ and so $I^k \subseteq J : I$. It follows $\tau(I) \geq \lambda(I^k + J/J) = \lambda(I^k/JI^{k-1})$. Then

$$JI^k = I^{k+1} \cap J$$

If one proves

$$I^{k+2} = JI^{k+1}$$

then G is Cohen-Macaulay since $I^n \cap J = JI^{n-1}$ for every integer n.

We may suppose $\lambda(I^{k+1}/JI^k) = 1$. Then there exist $a \in I$ and $b \in I^k$ such that

$$I^{k+1} = JI^k + (ab)$$

with $ab\mathcal{M} \subseteq JI^k$. In particular

$$I^{k+2} = JI^{k+1} + aI^{k+1}$$

We have to prove $aI^{k+1} \subseteq JI^{k+1}$.

First we remark that

$$I^k = (b) + (J : \ a) \cap I^k$$

If $x \in I^k$, then $xa \in I^{k+1} = JI^k + (ab)$. Hence there exists $c \in A$ such that $(x - cb)a \in JI^k$, in particular $x - cb \in (J : a) \cap I^k$ and the equality follows. We have

$$JI^{k-1} \subseteq (J : I) \cap I^k \subseteq (J : a) \cap I^k \subseteq I^k$$

Since $b\mathcal{M} \subseteq (J : a)$, we have $\lambda(I^k/(J : a) \cap I^k) = 1$. Moreover $\lambda((J : I) \cap I^k/JI^{k-1}) = \lambda((J : I) \cap I^k/J \cap I^k) = \lambda((J : I) \cap I^k + J/J) \leq \tau(I) \leq \lambda(I^k/JI^{k-1}) - 2$, then

$$(J : I) \cap I^k \neq (J : a) \cap I^k.$$

We find an element $p \in (J : a) \cap I^k$, $p \notin J : I$. We can write

$$I^{k+1} = JI^k + pI$$

and so

$$aI^{k+1} \subseteq JI^{k+1} + apI \subseteq JI^{k+1},$$

as required.
In particular the $h-$polynomial of I concides with that of its artinian reduction. We only remark that $\lambda(I^n + J/I^{n+1} + J) = \lambda(I^n/I^{n+1} + JI^{n-1})$ if $n \leq k$, moreover $I^{k+2} \subseteq J$ and so $\lambda(I^{k+1} + J/I^{k+2} + J) = \lambda(I^{k+1}/JI^k) = c$.

• Actually one remarks that in the above Theorem we get the same result with the weaker assumption $\lambda((J : I) \cap I^k/JI^{k-1}) < \lambda(I^k/JI^{k-1}) - 1$.
Notice that if $A = k[[x, y, z, u]]$, $I = (x^2, y^2, z^2, u^2, xy + zu)$ and $J = (x^2, y^2, z^2, u^2)$, then $\lambda((J : I) \cap I/J) = 5$, while $\tau(I) = 10$.

We recall that if $(A, \mathcal{M})$ is a $d-$dimensional Cohen-Macaulay ring and I is an $\mathcal{M}-$primary ideal, then

$$e(I) \geq \lambda(I/I^2) + (1 - d)\lambda(A/I)$$

(see [V], Lemma 1). If the equality holds, then $I^2 = JI$ for every ideal J generated by a maximal superficial sequence and in particular G is Cohen-Macaulay.
If $e(I) = \lambda(I/I^2) + (1 - d)\lambda(A/I) + 1$, then $\lambda(I^2/JI) = 1$ and G is not necessarily Cohen-Macaulay.
If we apply Theorem 2.2. with $k = 1$, we get the following result.

Corollary 2.3. *With the above notations, if* $e(I) = \lambda(I/I^2) + (1-d)\lambda(A/I) + 1$ *and* $\tau(I) < e(I) - \lambda(A/I) - 1$, *then*
1) G is Cohen-Macaulay
2) the h−polynomial of I is

$$h(z) = \lambda(A/I) \ + \ (\lambda(I/I^2) \ - \ d\lambda(A/I))z \ + \ z^2.$$

If $I = \mathcal{M}$, the above result was proved by Sally in [S2].

Let $(A, \mathcal{M})$ be a Cohen-Macaulay ring of dimension d, embedding dimension N and multiplicity e. If $\text{indeg}(A) \geq t$, then $\mathcal{M}$ is $t-$ *standard* .
Moreover if $t \geq 2$, then (see [RV1])

$$e \geq \binom{N-d+t-1}{t-1}.$$

If the equality holds, then $\mathcal{M}^t = J\mathcal{M}^{t-1}$ for every ideal J generated by a maximal superficial sequence and in particular G is Cohen-Macaulay.
Suppose $e = \binom{N-d+t-1}{t-1} + 1$, then $\lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) = 1$. If we apply Theorem 2.2. with $I = \mathcal{M}$ and $k = t-1$, we recover the following result which was proved in [RV1]

Corollary 2.4. *With the above assumptions, if* $e = \binom{N-d+t-1}{t-1} + 1$ *and* $\tau(\mathcal{M}) < \binom{N-d+t-2}{t-1}$, *then*
1) G is Cohen-Macaulay
2) the h−polynomial of I is

$$h(z) = \sum_{i=0}^{t-1} \binom{N-d+i-1}{i} z^i \ + \ z^t.$$

We now remark that, if A is Gorenstein and $e = N-d+2$, then G is Gorenstein (see [S2]). If A is Gorenstein and $e = N-d+3$, then G is Cohen-Macaulay (see [S3] or for an extended version [RV4]).

We may conclude that, if A is Gorenstein and $e \leq N-d+3$, then G is Cohen-Macaulay.
If $e \geq N-d+4$ and A is Gorenstein, then G is not necessarily Cohen-Macaulay. In fact, if we consider the Gorenstein ring $A = k[[t^6, t^7, t^{15}]]$ of multiplicity $e = 6 = N-d+4$, then depth $G = 0$ (see [S3]) and the $h-$ polynomial is $h(z) = 1 + 2z + z^2 + z^3 + z^5$. Notice that $\lambda(\mathcal{M}^3/J\mathcal{M}^2) = 2$.

We can prove the following result

Corollary 2.5. *Let* $(A, \mathcal{M})$ *be a Gorenstein local ring of dimension d, embedding dimension N and multiplicity* $e \geq N-d+4$.
If $\lambda(\mathcal{M}^3/J\mathcal{M}^2) \leq 1$, *then G is Cohen-Macaulay and the h−polynomial is*

$$h(z) = 1 \ + \ (N-d)z \ + \ (e-N+d-2)z^2 \ + \ z^3.$$

Proof. We may apply Theorem 2.2. with $I = \mathcal{M}$ and $k = 2$. It is enough remark that $\lambda(\mathcal{M}^2/J\mathcal{M}) = e - N + d - 1 \geq 3$ and so $\tau(\mathcal{M}) = 1 < \lambda(\mathcal{M}^2/J\mathcal{M}) - 1$.

3. Depth of the associated graded ring

In this section we prove that if I a $k-$ *standard* $\mathcal{M}-$primary ideal and we assume $\lambda(I^{k+1}/JI^k) \leq 1$, then the associated graded ring has depth at least $d-1$. We need some more results on the $k-$ *standard* $\mathcal{M}-$primary ideals.

Proposition 3.1. *Let $(A, \mathcal{M})$ be a two-dimensional Cohen-Macaulay local ring, k a positive integer and I a $k-$ standard $\mathcal{M}-$primary ideal.*
For all $n = 1, ..., k$ we have

1) $I^n : x = I^{n-1}$ for every superficial element x in J.

2) $\rho_{n-1} - v_{n-1} = \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}} + I^n)$.

Proof. Since $I^n \cap J = JI^{n-1}$, the first assertion is an easy fact. As for the second one, we have
$$J\widetilde{I^{n-1}} \subseteq J\widetilde{I^{n-1}} + I^n \subseteq \widetilde{I^n}$$
hence
$$\lambda(\widetilde{I^n}/J\widetilde{I^{n-1}} + I^n) = \rho_{n-1} - \lambda(J\widetilde{I^{n-1}} + I^n/J\widetilde{I^{n-1}}) = \rho_{n-1} - \lambda(I^n/J\widetilde{I^{n-1}} \cap I^n).$$
Since
$$JI^{n-1} \subseteq J\widetilde{I^{n-1}} \cap I^n \subseteq I^n \cap J = JI^{n-1},$$
we get $JI^{n-1} = J\widetilde{I^{n-1}} \cap I^n$ and the conclusion follows.

Theorem 3.2. *Let $(A, \mathcal{M})$ be a d-dimensional Cohen-Macaulay local ring, k a positive integer and I a $k-$ standard $\mathcal{M}-$primary ideal.*
If we suppose $\lambda(I^{k+1}/JI^k) \leq 1$, then
1) depth$(G) \geq d-1$
2) the $h-$polynomial of I is
$$h(z) = \sum_{n=0}^{k-1} \lambda(I^n/I^{n+1} + JI^{n-1})z^n \ + \ (\lambda(I^k/JI^{k-1}) - c\ )z^k \ + \ c\ z^s$$
where $c = \lambda(I^{k+1}/JI^k)$ and $s \geq k+1$.
G is Cohen-Macaulay if and only if $s = k+1$.

First we recall that in the following we may assume $(A, \mathcal{M})$ be a d-dimensional Cohen-Macaulay local ring, k a positive integer and I a $k-$ *standard* $\mathcal{M}-$primary ideal with $\lambda(I^{k+1}/JI^k) = 1$.

With this assumption, we need the following easy facts.

Lemma 3.3. *With the above assumptions, then the following conditions hold:*

1) $\mathcal{M}I^{j+1} \subseteq JI^j$ *for every* $j \geq k$

2) There exist $w \in I$ *and* $z \in I^k$ *such that* $I^{j+1} = JI^j + (zw^{j+1-k})$ *for every* $j \geq k$

3) $\lambda(I^{j+1}/JI^j) \leq 1$ *for every* $j \geq k$

Proof. One has

$$I^{k+1} \supseteq \mathcal{M}I^{k+1} + JI^k \supseteq JI^k$$

and also, if $I^{k+1} = \mathcal{M}I^{k+1} + JI^k$, by Nakayama $I^{k+1} = JI^k$. Hence

$$\mathcal{M}I^{k+1} \subseteq JI^k$$

and the first assertion follows by multiplication by I.

We pass to the second assertion.
Since $\lambda(I^{k+1}/JI^k) = 1$, there exists an element $w \in I$ such that $wI^k \nsubseteq JI^k$. Then there exists $z \in I^k$ such that

$$I^{k+1} = JI^k + (zw).$$

Now $I^{k+2} \subseteq JI^{k+1} + zwI \subseteq JI^{k+1} + wI^{k+1} = JI^{k+1} + (zw^2)$ and so on.
The third assertion follows from 1) and 2).

We prove now a result in the one-dimensional case which will be used later. The notations are those we introduced in the first section. We only recall that s denotes the reduction number of I, such that the least integer n for which $v_n = \lambda(I^{n+1}/JI^n) = 0$.

Lemma 3.4. *Let* $(A, \mathcal{M})$ *be a Cohen-Macaulay local ring of dimension one, with the above assumptions we have*

$$e_1(I) = \sum_{n=0}^{k-1} v_n + s - k$$

and the $h-$*polynomial of* I *is*

$$h(z) = \sum_{n=0}^{k-1} \lambda(I^n/I^{n+1} + JI^{n-1})z^n \ + \ (\lambda(I^k/JI^{k-1}) - 1)z^k \ + \ z^s$$

Proof. We recall that $e_1(I) = \sum_{j=0}^{s-1} v_j$ and since the $h-$vector is the first difference of the Hilbert function, we have $h(z) = \lambda(A/I) + \sum_{n=1}^{s}(v_{n-1} \ - \ v_n)z^n$.
If $n \leq k-1$,

$$v_{n-1} \ - \ v_n = \lambda((I^{n+1} + J)/J) - \lambda((I^n + J)/J) =$$

$$= \lambda((I^n + J)/(I^{n+1} + J)) = \lambda(I^n/(I^{n+1} + JI^{n-1})).$$

It is enough to remark that, by Lemma 3.3.3), $v_k = \cdots = v_{s-1} = 1$.

We come back now to the general case of the Theorem.

First of all we have $s = k+1$ if and only if $h-$vector of I concides with that of its artinian reduction. Hence the last assertion on the Cohen-Macaulyness of G is clear.

If $d \geq 2$, we let
$$B := A/(x_1, \ldots, x_{d-2})$$
and
$$C := A/(x_1, \ldots, x_{d-1})$$
where $x_1, \ldots, x_{d-1}$ is a superficial sequence in I.
We have $dim(C) = 1$, $dim(B) = 2$; and all the assumptions of the Theorem 3.2. are stable modulo a superficial sequence, that is if I is a $k - standard$ primary ideal, also $I/(x_1, \ldots, x_{d-2})$ and $I/(x_1, \ldots, x_{d-1})$ are $k - standard$ primary ideals and the assumption on the lenght is preserved.
Hence if we can prove that the depth of the associated graded ring of $I/(x_1, \ldots, x_{d-2})$ is positive then, by using Sally's machine, we get $depth(G) \geq 1 + d - 2 = d - 1$.

Henceforth we may assume $dim(A) = 2$, I a $k - standard$ $\mathcal{M}-$primary ideal with $\lambda(I^{k+1}/JI^k) = 1$ and, by the above remark, we need to prove that $depth(G) \geq 1$.

As before, we let $J = (x, y)$ be the ideal generated by a superficial sequence and $R := A/xA$ the one-dimensional Cohen-Macaulay local ring.
As before we denote by s the reduction exponent of $\overline{I} = I/(x)$.
If $s \leq k$, then the associated graded ring of $\overline{I}$ is Cohen-Macaulay and, against by Sally's machine, we may conclude that G is Cohen-Macaulay.
We will suppose $s \geq k+1$ and then $\lambda(\overline{I}^{j+1}/y\overline{I}^j) = 1$ for $j = k, ..., s-1$.

In order to prove the Theorem, we will need the following crucial results. We do not insert here the proofs since they have been proved in a similar version in [RV2] and [RV3].

Proposition 3.5. *With the above notations the following conditions hold.*

1) $\lambda(I^{j+1}/JI^j) = 1$ *for every* $j = k, \ldots, s-1$.

2) $I^{j+1} : x = I^j$ *for every superficial element* x *in* J *and* $j = 0, \ldots, s-1$.

3) $v_j = 1$ *for every* $j = k, \ldots, s-1$.

4) $e_1(I) = \sum_{j=0}^{k-1} \lambda(I^{j+1}/JI^j) + s - k$

5) $depth(G) > 0 \iff I^{s+1} = JI^s$.

Proposition 3.6. *Let* $(A, \mathcal{M})$ *be a local ring,* I *a* $\mathcal{M}-$*primary ideal,* p *a positive integer and* $J \subseteq I$ *an ideal of* A. *For every integer* $n = 1, \ldots, p$ *suppose we are given ideals*
$$I_n = (a_{1n}, \ldots, a_{\nu_n n}) \subseteq \widetilde{I^n}.$$
Assume $\widetilde{I^{p+1}} = J\widetilde{I^p}$ *and for* $n = 0, \ldots, p-1$,

$$\widetilde{I^{n+1}} = J\widetilde{I^n} + I_{n+1} + I^{n+1}$$

Let $w \in I$ and set $\nu := \sum_{i=1}^{p} \nu_i$. Then there exists an element $\sigma \in JI^{\nu-1}$ such that for every $n = 1, \dots, p$ and $i = 1, \dots, \nu_n$

$$w^{\nu} a_{in} \equiv \sigma \ a_{in} \ mod \ I^{\nu+n}.$$

We can finish now the proof of the theorem.

Theorem 3.7. *Let $(A, \mathcal{M})$ be a Cohen-Macaulay local ring of dimension two, k a positive integer and I a $k-$standard $\mathcal{M}-$primary ideal such that $\lambda(I^{k+1}/JI^k) = 1$. Then*

1) $depth(G) \geq 1$

2) the $h-$polynomial of I is

$$h(z) = \sum_{n=0}^{k-1} \lambda(I^n/I^{n+1} + JI^{n-1})z^n \ + \ (\lambda(I^k/JI^{k-1}) - 1)z^k \ + \ z^s.$$

for some integer $s \geq k+1$

Proof. For every $n \geq 1$ we have $\lambda(\widetilde{I^n}/J\widetilde{I^{n-1}} + I^n) \leq \rho_{n-1} = \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}})$ and equality holds if and only if $I^n \subseteq J\widetilde{I^{n-1}}$. Further we can find elements $a_{1n}, \dots, a_{\nu_n n} \in \widetilde{I^n}$ such that their residue classes modulo $J\widetilde{I^{n-1}} + I^n$ form a minimal system of generators of the module $\widetilde{I^n}/J\widetilde{I^{n-1}} + I^n$. It is clear that we have

$$\nu_n \leq \lambda(\widetilde{I^n}/J\widetilde{I^{n-1}} + I^n) \leq \rho_{n-1}$$

and if $I^n \not\subseteq J\widetilde{I^{n-1}}$, then $\nu_n < \rho_{n-1}$.
If $n \leq k-1$, by Proposition 3.1., we can be more precise, namely

$$\nu_n \leq \rho_{n-1} - v_{n-1}.$$

If we let $I_n := (a_{1n}, \dots, a_{\nu_n n})$, then $I_n \subseteq \widetilde{I^n}$ and $\widetilde{I^n} = J\widetilde{I^{n-1}} + I^n + I_n$. Since $\widetilde{I} = I + I_1$ we get $\widetilde{I^2} = J\widetilde{I} + I^2 + I_2 = JI + I^2 + I_2$.
Going on in this way, we obtain for every $r \geq 1$

$$\widetilde{I^r} = \sum_{j=1}^{r} J^{r-j} I_j + I^r.$$

Now we recall that by Proposition 3.1. , for every $j \leq k-1$, we have $\rho_j \geq v_j = \lambda(I^{j+1}/JI^j)$, hence by Proposition 3.5.

$e_1(I) = \sum_{j=0}^{k-1} v_j + s - k = \sum_{j\geq 0} \rho_j = \sum_{j=0}^{k-1} \rho_j + \sum_{j\geq k} \rho_j \geq \sum_{j=0}^{k-1} v_j + \sum_{j\geq k} \rho_j$
It follows that

$$\sum_{j\geq k} \rho_j \leq s - k.$$

We distinguish two cases:

i) $\rho_k = \cdots = \rho_{s-1} = 1$.

With this assumption the above inequality turns out to be an equality and this implies $\rho_j = v_j$ for every $j \leq s-1$ and $\rho_j = 0$ for $j \geq s$.

Since by Proposition 3.5. we have $I^{j+1} : x = I^j$ for every $j \leq s-1$, we get $\lambda(\overline{I}^{j+1}/y\overline{I}^j) = \lambda(I^{j+1}/JI^j) = \rho_j$ for every $j \leq s-1$. In particular

$$\lambda(\overline{I}^{j+1}/y\overline{I}^j) = \rho_j$$

for every $j \geq 0$.

It follows

$$e_2(I) = \sum_{j\geq 0} j\rho_j = \sum_{j\geq 0} j\lambda(\overline{I}^{j+1}/y\overline{I}^j) = e_2(\overline{I})$$

and hence $\operatorname{depth}(G) \geq 1$.

ii) There exists an integer j such that $k \leq j \leq s-1$, $\rho_j \neq 1$.

In this case we let p the least integer $n \leq s-1$ such that $\rho_n = 0$ and we also let c be the least integer n such that $I^{n+1} \subseteq J\widetilde{I^n}$.
We remark that we have $k \leq c \leq p \leq s-1$. In fact $I^k \nsubseteq JI^{k-1}$ otherwise by the good property of I, we get $I^k = JI^{k-1}$, which contradicts the assumption.
Further, since $\rho_p = 0$, $I^{p+1} \subseteq \widetilde{I^{p+1}} = J\widetilde{I^{k-1}}$.

By the true definition of c with the above notations, we have for every $j = k, \dots, c$

$$\nu_j < \rho_{j-1}$$

and for every $j \leq k \quad \nu_j \leq \rho_{j-1} - v_{j-1}$.

By Lemma 3.3., for every $j \geq k$ we can find $w \in I$ and $z \in I^k$ such that $I^{j+1} = JI^j + (zw^{j+1-k})$.
We get for every $n = 1, \dots, p-1$

$$wI_n \subseteq \widetilde{I^{n+1}} = \sum_{j=1}^{n+1} J^{n+1-j} I_j + I^{n+1}$$

and

$$wI_p \subseteq \widetilde{I^{p+1}} = J\widetilde{I^p} = \sum_{j=1}^{p} J^{p+1-j} I_j + JI^p \subseteq \sum_{j=1}^{p} J^{p+1-j} I_j + I^{p+1}.$$

By applying the above Proposition with $\nu = \sum_{i=1}^{p} \nu_i$, we can find an element $\sigma \in JI^{\nu-1}$, such that for every $i = 1, \dots, \nu_n$

$$w^\nu a_{in} \equiv \sigma \; a_{in} \; mod \; I^{\nu+n}.$$

On the other hand, since $I^{c+1} \subseteq J\widetilde{I^c} = \sum_{j=1}^{c} J^{c+1-j} I_j + JI^c$, we can write

$$zw^{c+1-k} = \sum_{j=1}^{c} \sum_{i=1}^{\nu_j} m_{ij} a_{ij} + b$$

where $m_{ij} \in J^{c+1-j}$, $b \in JI^c$. Using these facts, exactly as in the proof of Theorem 2.5. [RV2] we get

$$zw^{\nu+c+1-k} \in JI^{\nu+c}$$

which, by Lemma 3.3.(2), implies

$$I^{\nu+c+1} = JI^{\nu+c}.$$

We finally remark that

$$\nu + c = \sum_{i=1}^{p} \nu_i + c \leq \sum_{i=1}^{k} (\rho_{i-1} - v_{i-1}) + \sum_{i=k+1}^{c} (\rho_{i-1} - 1) + \sum_{i=k+1}^{p} \rho_{i-1} + c \leq$$

$$\leq \sum_{i \geq 0} \rho_i - (c-k) + c - \sum_{i=0}^{k-1} v_i = e_1(I) + k - (e_1(I) - s + k) = s$$

Hence

$$I^{s+1} = JI^s$$

and then depth $G \geq 1$ by Proposition 3.5.
In particular $I^n : x = I^{n-1}$ for every integer n, and the $h-$polynomial of I is the $h-$polynomial of $I/(x)$. We conclude by using Lemma 3.4.

It is well known that, if $(A, \mathcal{M})$ is a $d-$dimensional Cohen-Macaulay ring and I is a $\mathcal{M}-$primary ideal, then $e(I) \geq \lambda(I/I^2) + (1-d)\lambda(A/I)$ (see [V], Lemma 1). If the equality holds, then G is Cohen-Macaulay.
Suppose $e(I) = \lambda(I/I^2) + (1-d)\lambda(A/I) + 1$, then $\lambda(I^2/JI) = 1$.
In the case $I = \mathcal{M}$, then depth $G \geq d-1$. The problem was open in the $\mathcal{M}-$primary case. If we apply Theorem 3.2. with $k = 1$, we get the following result.

Corollary 3.8. *Let $(A, \mathcal{M})$ be a $d-$dimensional Cohen-Macaulay ring and I an $\mathcal{M}-$primary ideal.*
If $e(I) = \lambda(I/I^2) + (1-d)\lambda(A/I) + 1$, then
1) depth$(G) \geq d-1$
2) $h(z) = \lambda(A/I) + (\lambda(I/I^2) - d\lambda(A/I))z + z^s$ for some $s \geq 2$

If we apply Theorem 3.2. with $k = 2$, we obtain the following result proved by Huckaba [H], Theorem 2.6. We also characterize the Hilbert function.

Corollary 3.9. *Let $(A, \mathcal{M})$ be a $d-$dimensional Cohen-Macaulay ring and I an $\mathcal{M}-$primary ideal. Assume that J is an ideal generated by a maximal superficial sequence for I such that $I^2 \cap J = JI$ and $\lambda(I^3/JI^2) \leq 1$.*
Then
1) depth$(G) \geq d-1$
2) The $h-$polynomial of I is

$$h(z) = \lambda(A/I) + (\lambda(I/I^2) - d\lambda(A/I))z + (\lambda(I^2/JI) - c)z^2 + c\, z^s$$

where $c = \lambda(I^3/JI^2)$ *and* $s \geq 3$

Let $(A, \mathcal{M})$ be a Cohen-Macaulay ring of dimension d, embedding dimension N and multiplicity e. If t a positive integer and indeg$(A) \geq t$, then $\mathcal{M}$ is $t-$ *standard.* As we have already said, if $t \geq 2$, then

$$e \geq \binom{N-d+t-1}{t-1}.$$

If the equality holds, then G is Cohen-Macaulay.
If $e = \binom{N-d+t-1}{t-1} + 1$, then $\lambda(\mathcal{M}^t/J\mathcal{M}^{t-1}) = 1$. In [RV3], Theorem 3.1., it was proved that depth$G \geq d-1$.
This result can be recovered if we apply Theorem 3.2. with $k = t-1$.

Corollary 3.10. *With the above notations, if* $e = \binom{N-d+t-1}{t-1} + 1$, *then*
1) depth$(G) \geq d-1$
2) the $h-$*polynomial is*

$$h(z) = \sum_{i=0}^{t-1} \binom{N-d+i-1}{i} z^i \ + \ z^s$$

for some $s \geq t$.

If $e = N-d+3$ and the Cohen-Macaulay type is $\tau(\mathcal{M}) < N-d$, in [RV4] it was proved that $\lambda(\mathcal{M}^3/J\mathcal{M}^2) \leq 1$ for every ideal J generated by a maximal superficial sequence, then:

Corollary 3.11. *With the above notations, if* $e = N-d+3$ *and* $\tau(\mathcal{M}) < N-d$, *then depth* $(G) \geq d-1$.

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