POINCARÉ SERIES OF MODULES OVER COMPRESSED GORENSTEIN LOCAL RINGS

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Abstract. Given two positive integers e and s we consider Gorenstein Artinian local rings R whose maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^s \neq 0 = \mathfrak{m}^{s+1}$ and $\operatorname{rank}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = e$. We say that R is a *compressed Gorenstein local ring* when it has maximal length among such rings. It is known that generic Gorenstein Artinian algebras are compressed. If $s \neq 3$, we prove that the Poincaré series of all finitely generated modules over a compressed Gorenstein local ring are rational, sharing a common denominator. A formula for the denominator is given. When s is even this formula depends only on the integers e and s. Note that for $s = 3$ examples of compressed Gorenstein local rings with transcendental Poincaré series exist, due to Bøgvad.

Introduction

Let $(R, \mathfrak{m}, k)$ be a local commutative ring with maximal ideal $\mathfrak{m}$ and residue field k. For a finitely generated R-module M we study the *Poincaré series*

$$\mathrm{P}_M^R(z) = \sum_{i \geqslant 0} \beta_i^R(M) z^i ,$$

where $\beta_i^R(M)$ are the *Betti numbers* of M defined as $\beta_i^R(M) = \operatorname{rank}_k \operatorname{Tor}_i^R(M, k)$. We would like to understand when this series is *rational*, meaning that it represents a rational function. This problem has a long history, fueled in part by the question attributed to Kaplansky and Serre of whether $\mathrm{P}_k^R(z)$ is rational.

Following Roos [23], we say that R is *good* if all finitely generated R-modules have rational Poincaré series, sharing a common denominator, and R is *bad* otherwise.

Significant classes of good rings are known. Complete intersections are among them by a result of Gulliksen [14]. For some of these classes, concrete information on the common denominator has been successfully used to further understand the asymptotic behavior of the sequences $\{\beta_i^R(M)\}_{i \geqslant 0}$ (see [29], [30]) and to prove results on vanishing of (co)homology (see [3], [20], [24], [25]).

Bad rings exist too, with examples including rings R with transcendental $\mathrm{P}_k^R(z)$, cf. Anick [1], and rings R admitting sequences of R-modules $\{M_n\}$ with $\mathrm{P}_{M_n}^R(z)$ rational, but sharing no common denominator, cf. [23].

Since both good and bad behavior exist, it is interesting to understand which one is most likely to occur. If we consider families of rings parametrized by points

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of some geometric object, then this problem can be made precise by asking the question: Is a generic ring in such a family good or bad?

Gorenstein Artinian algebras of fixed *embedding dimension* $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, and *socle degree* s, defined by $\mathfrak{m}^s \neq 0 = \mathfrak{m}^{s+1}$, provide a good setting for studying this question, as they can be parametrized by points in projective space via the Macaulay inverse system. When R is a generic algebra in this family, it is known, cf. Iarrobino [18], that its length is maximal.

In this paper we consider, more generally, local Artinian Gorenstein rings R of possibly mixed characteristic with maximal length

$$\lambda(R) = \sum_{i=0}^{s} \varepsilon_i \quad \text{where} \quad \varepsilon_i = \left\{ \binom{e-1+s-i}{e-1} \binom{e-1+i}{e-1} \right\},$$

and e and s are as above. For such rings one also has $\operatorname{rank}_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \varepsilon_i$ for all i with $0 \leq i \leq s$, and the associated graded ring $R^{\mathfrak{g}}$ with respect to the maximal ideal is Gorenstein. We call such rings *compressed Gorenstein local rings* of socle degree s and embedding dimension e.

We say that $\widehat{R} = Q/I$ is a *minimal Cohen presentation* of R if Q is a regular local ring and $I \subseteq \mathfrak{n}^2$, where $\mathfrak{n}$ is the maximal ideal of Q; such a presentation exists by Cohen's Structure Theorem.

Main Theorem. *Let e, s be integers such that $2 \leq s \neq 3$ and $e > 1$ and let $(R, \mathfrak{m}, k)$ be a compressed Gorenstein local ring of socle degree s and embedding dimension e. Let $R = Q/I$ be any minimal Cohen presentation of R.*

For every finitely generated R-module M there exists a polynomial $p_M(z) \in \mathbb{Z}[z]$ such that

$$\mathrm{P}_R^M(z) = \frac{p_M(z)}{d_R(z)},$$

where $p_k(z) = (1+z)^e$ and

$$d_R(z) = 1 - z\big(\mathrm{P}_R^Q(z) - 1\big) + z^{e+1}(1+z).$$

If s is even, then $d_R(z)$ depends only on the integers e and s as follows

$$d_R(z) = 1 + z^{e+2} + (-z)^{-\frac{s-2}{2}} \cdot \left((1+z)^e \cdot \big(\sum_{i \geqslant 0} (-1)^i \varepsilon_i \big) - 1 - z^{s+e} \right)$$

and in particular $\mathrm{P}_k^R(z) = \mathrm{P}_k^{R^{\mathfrak{g}}}(z)$.

When s is odd a formula for $\mathrm{P}_R^Q(z)$, and thus for $d_R(z)$, does not depend anymore only on e on s, without stronger hypotheses. The values of the (graded) Betti numbers of R over Q for generic Gorenstein Artinian graded algebras R of odd socle degree are conjectured by Boij [9].

In case $s = 2$ all Gorenstein local rings are compressed and our theorem recovers a result of Sjödin [28]. When $s = 3$ any ring R for which $R^{\mathfrak{g}}$ is Gorenstein is compressed. Bøgvad [8] constructed examples of compressed Gorenstein local rings with $s = 3$ and $e = 12$ which have transcendental Poincaré series, hence rationality results cannot be achieved without stronger hypotheses when $s = 3$. Nevertheless, we can use in this case results of Henriques and Şega [16], Conca, Rossi and Valla [10], Elias and Rossi [11], and we obtain that the following statement holds without any restrictions on e and s.

Corollary. *If e and s are positive integers, then a generic Gorenstein Artinian algebra of socle degree s and embedding dimension e is good.*

To prove the Main Theorem we construct a Golod homomorphism from a hypersurface ring onto R; a result of Levin then gives the desired conclusions on the rationality of the Poincaré series.

Throughout, we try to state the intermediary results with only the hypotheses that are needed for the proof. The division of the paper into sections is dictated by the progressive addition of hypotheses. Section 5 ties everything together and gives the proof of the Main Theorem, which becomes mainly a consequence of preceding work.

In the last section we pursue a suggestion of Jürgen Herzog and we note that $R/\mathfrak{m}^i$ is a Golod ring for R as in the Main Theorem and all i with $2 \le i \le s$. When s is even, a simple proof of this fact is available and yields a more direct proof of the formula for $d_R(z)$ in the theorem.

1. Golod homomorphisms: a criterion

Throughout the paper, the notation $(R, \mathfrak{m}, k)$ identifies R as a local Noetherian commutative ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. If $(R, \mathfrak{m}, k)$ is a local ring and M is a finitely generated R-module, we denote $\beta_i^R(M)$ its Betti numbers, defined as $\beta_i^R(M) = \operatorname{rank}_k \operatorname{Tor}_i^R(M,k)$. The *Poincare series* of M over R is the formal power series

$$\mathrm{P}_M^R(z) = \sum_{i \ge 0} \beta_i^R(M) z^i \,.$$

1.1. *Golod homomorphisms.* Let $\varkappa\colon (P, \mathfrak{p}, k) \to (R, \mathfrak{m}, k)$ be a surjective homomorphism of local rings. Let $\mathcal{D}$ be a minimal free resolution of k over P with a graded-commutative DG algebra structure; such a resolution always exists, see [15]. We set $\mathcal{A} = \mathcal{D} \otimes_P R$. If $x \in \mathcal{A}$, let $|x|$ denote the degree of x and set $\overline{x} = (-1)^{|x|+1}x$.

Let $\boldsymbol{h} = \{h_i\}_{i \ge 1}$ denote a homogeneous basis of the graded k-vector space $\mathrm{H}_{\ge 1}(\mathcal{A})$. According to Gulliksen [13], the homomorphism $\varkappa : P \to R$ is said to be *Golod* (or $\mathcal{A}$ admits a *trivial Massey operation*) if there is a function $\mu\colon \bigsqcup_{n=1}^{\infty} \boldsymbol{h}^n \to \mathcal{A}$ satisfying:

(1.1.1) $\qquad \mu(h)$ is a cycle in the homology class of h for each $h \in \boldsymbol{h}$;

(1.1.2) $$\partial\mu(h_1, \dots, h_n) = \sum_{i=1}^{n-1} \overline{\mu(h_1, \dots, h_i)}\mu(h_{i+1}, \dots, h_n) \quad \text{for each } m \ge 2\,;$$

(1.1.3) $\qquad \mu(\boldsymbol{h}^n) \subseteq \mathfrak{m}\mathcal{A} \quad$ for each $n \ge 1$.

The following criterion formalizes ideas from [22] and [2].

Lemma. *Let a be a positive integer such that*

(1) *The map $\operatorname{Tor}_i^P(R,k) \to \operatorname{Tor}_i^P(R/\mathfrak{m}^a, k)$ induced by the projection $R \to R/\mathfrak{m}^a$ is zero for all $i > 0$.*

(2) *The map $\operatorname{Tor}_i^P(\mathfrak{m}^{2a}, k) \to \operatorname{Tor}_i^P(\mathfrak{m}^a, k)$ induced by the inclusion $\mathfrak{m}^{2a} \hookrightarrow \mathfrak{m}^a$ is zero for all $i \ge 0$.*

Then $\varkappa$ is a Golod homomorphism. Furthermore, the Massey operation μ can be constructed so that $\mu(\mathcal{A}) \subseteq \mathfrak{m}^a\mathcal{A}$.

Proof. Let $\mathcal{D}$ be a minimal free resolution of k over P as above, and set $\mathcal{A} = \mathcal{D} \otimes_P R$. We need to construct a trivial Massey operation on $\mathrm{H}_{\geq 1}(\mathcal{A})$.

The map in (1) can be indentified with the map

$$\mathrm{H}_i(\mathcal{A}) \to \mathrm{H}_i(\mathcal{A}/\mathfrak{m}^a\mathcal{A})$$

induced by the projection $\mathcal{A} \to \mathcal{A}/\mathfrak{m}^a\mathcal{A}$. The map in (2) can be identified with the map

$$\mathrm{H}_i(\mathfrak{m}^{2a}\mathcal{A}) \to \mathrm{H}_i(\mathfrak{m}^a\mathcal{A})$$

induced by the inclusion of complexes $\mathfrak{m}^{2a}\mathcal{A} \hookrightarrow \mathfrak{m}^a\mathcal{A}$. The hypothesis (1) gives that every element in $\mathrm{H}_{\geq 1}(\mathcal{A})$ can be represented as $\mathrm{cls}(x)$ for some $x \in \mathfrak{m}^a\mathcal{A}$. Then if x_1 and x_2 are such representatives of two elements in $\mathrm{H}_{\geq 1}(\mathcal{A})$ we have $x_1x_2 \in \mathfrak{m}^{2a}\mathcal{A}$. The hypothesis (2) gives that x_1x_2 is a cycle in $\mathfrak{m}^a\mathcal{A}$.

Thus for any two elements $\mathrm{cls}(x_1)$ and $\mathrm{cls}(x_2)$ in $\mathrm{H}_{\geq 1}(\mathcal{A})$ one can pick representatives x_1, x_2 in $\mathfrak{m}^a\mathcal{A}$ such that x_1x_2 is the boundary of an element in $\mathfrak{m}^a\mathcal{A}$. This property allows for an inductive definition of the Massey operation μ. □

In the case of interest for this paper, condition (1) of Lemma 1.1 will be verified using Lemma 1.3 below.

1.2. *Maps induced by powers of the maximal ideal.* Let $(R, \mathfrak{m}, k)$ be a local ring and let $\widehat{R} = Q/I$ be a minimal Cohen presentation. We let R^{g}, respectively Q^{g}, denote the associated graded rings:

$$R^{\mathrm{g}} = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \quad \text{and} \quad Q^{\mathrm{g}} = \bigoplus_{i \geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}\,.$$

If M is an R-module, we denote

$$\nu_j^R(M)\colon\ \mathrm{Tor}_j^R(\mathfrak{m}M, k) \to \mathrm{Tor}_j^R(M, k) \tag{1.2.1}$$

the maps induced by the inclusion $\mathfrak{m}M \subseteq M$. Şega [26, Theorem 3.3] shows that

$$\nu_j^R(\mathfrak{m}^i) = 0 \quad \text{for all } j \geq 0 \text{ and all } i \geq \mathrm{reg}_{Q^{\mathrm{g}}}(\widehat{R}^{\mathrm{g}})\,,$$

where $\mathrm{reg}_{Q^{\mathrm{g}}}(-)$ denotes Castelnouvo-Mumford regularity over Q^{g}, which is a polynomial ring. It follows that the maps

$$\rho_j^R(\mathfrak{m}^i)\colon\ \mathrm{Tor}_j^R(R/\mathfrak{m}^{i+1}, k) \to \mathrm{Tor}_j^R(R/\mathfrak{m}^i, k)$$

induced by the projection $R/\mathfrak{m}^{i+1} \to R/\mathfrak{m}^i$ are zero for all $j > 0$ and all $i \geq \mathrm{reg}_{Q^{\mathrm{g}}}(\widehat{R}^{\mathrm{g}})$.

Lemma 1.3. *Let $(Q, \mathfrak{n}, k)$ be a regular local ring and $t \geq 2$. Let $I \subseteq \mathfrak{n}^t$ and set $R = Q/I$ and $\mathfrak{m} = \mathfrak{n}/I$. If $P = Q/(h)$ with $h \in I \smallsetminus \mathfrak{n}^{t+1}$, then the maps*

$$\rho_i^P(\mathfrak{m}^{t-1})\colon\ \mathrm{Tor}_i^P(R/\mathfrak{m}^t, k) \to \mathrm{Tor}_i^P(R/\mathfrak{m}^{t-1}, k)$$
$$\rho_i^Q(\mathfrak{m}^{t-1})\colon\ \mathrm{Tor}_i^Q(R/\mathfrak{m}^t, k) \to \mathrm{Tor}_i^Q(R/\mathfrak{m}^{t-1}, k)$$

induced by the projection $R/\mathfrak{m}^t \to R/\mathfrak{m}^{t-1}$ are zero for all $i > 0$.

Proof. Set $\mathfrak{p} = \mathfrak{n}/(h)$, the maximal ideal of P. Since $I \subseteq \mathfrak{n}^t$, we have that $R/\mathfrak{m}^t = P/\mathfrak{p}^t$ and $R/\mathfrak{m}^{t-1} = P/\mathfrak{p}^{t-1}$; these identifications are canonical. Thus, to show $\rho_i^P(\mathfrak{m}^{t-1}) = 0$ for all $i > 0$, it suffices to verify that the map

$$\rho_i^P(\mathfrak{p}^{t-1})\colon\ \mathrm{Tor}_i^P(P/\mathfrak{p}^t, k) \to \mathrm{Tor}_i^P(P/\mathfrak{p}^{t-1}, k)$$

is zero for all $i > 0$. This follows from 1.2, since $\mathrm{reg}_{Q^{\mathrm{g}}}(P^{\mathrm{g}}) = t - 1$.

The same argument yields the conclusion for $\rho_i^Q(\mathfrak{m}^{t-1})$. $\square$

2. A change of ring and homological computations

2.1. In this section $(Q, \mathfrak{n}, k)$ and $(P, \mathfrak{p}, k)$ denote local rings such that $P = Q/(h)$ for some $h \in \mathfrak{n}^t \smallsetminus \mathfrak{n}^{t+1}$ with $t \geq 2$ and M is a finitely generated P-module. For each i we denote φ_i^M the map

$$\varphi_i^M\colon\ \operatorname{Tor}_i^Q(M,k) \to \operatorname{Tor}_i^P(M,k) \tag{2.1.1}$$

induced by the canonical projection $Q \to P$. A free resolution of M over P can be constructed as in [4, Theorem 3.1.3] from a minimal free resolution of M over Q; this construction is due to Shamash [27]. If $h \in \mathfrak{n}\operatorname{ann}_Q(M)$, then this resolution is minimal, see [4, Remark 3.3.5], and it follows that φ_i^M is injective for all i. In particular, if $\mathfrak{n}^{t-1}M = 0$ then φ_i^M is injective for all i.

If Λ is graded vector space over k such that $\operatorname{rank}_k(\Lambda_i)$ is finite for each i and $\Lambda_i = 0$ for $i < r$, then the formal power series

$$HS_\Lambda(z) = \sum_{i \geqslant r} \operatorname{rank}_k(\Lambda_i) z^i$$

is called the *Hilbert series* of Λ.

2.2. We discuss now the relationship between $\mathrm{P}_M^Q(z)$ and $\mathrm{P}_M^P(z)$. As explained in the proof of [4, 3.2.2], the construction of Shamash leads to the definition of maps χ_i^M which fit into an exact sequence

$$\cdots \to \operatorname{Tor}_{i-1}^P(M,k) \to \operatorname{Tor}_i^Q(M,k) \xrightarrow{\varphi_i^M} \operatorname{Tor}_i^P(M,k) \xrightarrow{\chi_i^M} \operatorname{Tor}_{i-2}^P(M,k) \to$$
$$\to \operatorname{Tor}_{i-1}^Q(M,k) \xrightarrow{\varphi_{i-1}^M} \operatorname{Tor}_{i-1}^P(M,k) \to \dots$$

A rank count in this sequence gives:

$$\beta_i^P(M) - \beta_{i-2}^P(M) = \beta_i^Q(M) - \operatorname{rank}_k(\operatorname{Ker}\varphi_i^M) - \operatorname{rank}_k(\operatorname{Ker}\varphi_{i-1}^M) \quad \text{for all } i,$$

which yields :

$$\sum_{i \geqslant 0} \beta_i^P(M) z^i - z^2 \sum_{i \geqslant 0} \beta_{i-2}^P(M) z^{i-2} = \sum_{i \geqslant 0} \beta_i^P(M) z^i -$$
$$- \sum_{i \geqslant 0} \operatorname{rank}_k(\operatorname{Ker}\varphi_i^M) z^i - z \cdot \sum_{i \geqslant 0} \operatorname{rank}_k(\operatorname{Ker}\varphi_{i-1}^M) z^{i-1}$$

and hence

$$\mathrm{P}_M^P(z) = \frac{\mathrm{P}_M^Q(z) - (1+z) \cdot HS_{\operatorname{Ker}\varphi_*^M}(z)}{1 - z^2}\,. \tag{2.2.1}$$

Below, we use the notation introduced in (1.2.1).

Lemma 2.3. *If $\mathfrak{n}^t M = 0$, then the following statements are equivalent:*

(1) $\nu_i^P(M) = 0$ *for all i;*
(2) $\operatorname{rank}_k(\operatorname{Im}\nu_i^Q(M)) = \operatorname{rank}_k(\operatorname{Ker}\varphi_i^M)$ *for all i.*

Proof. Consider the exact sequence

$$0 \to \mathfrak{n}M \to M \to M/\mathfrak{n}M \to 0$$

and the one induced in homology:

$$\ldots \to \operatorname{Tor}_i^Q(\mathfrak{n}M, k) \xrightarrow{\nu_i^Q(M)} \operatorname{Tor}_i^Q(M, k) \longrightarrow \operatorname{Tor}_i^Q(M/\mathfrak{n}M, k) \longrightarrow$$
$$\longrightarrow \operatorname{Tor}_{i-1}^Q(\mathfrak{n}M, k) \xrightarrow{\nu_{i-1}^Q(M)} \operatorname{Tor}_{i-1}^Q(M, k) \to \ldots$$

A rank count (similar to the one spelled out in 2.2) in the last sequence gives the formula:

$$z\,\mathrm{P}^Q_{\mathfrak{n}M}(z) + \mathrm{P}^Q_M(z) - \mathrm{P}^Q_{M/\mathfrak{n}M}(z) = (1+z)\cdot HS_{\operatorname{Im}\nu_*^Q(M)}(z)\,. \tag{2.3.1}$$

Proceed similarly over the ring P and obtain:

$$z\,\mathrm{P}^P_{\mathfrak{n}M}(z) + \mathrm{P}^P_M(z) - \mathrm{P}^P_{M/\mathfrak{n}M}(z) = (1+z)\cdot HS_{\operatorname{Im}\nu_*^P(M)}(z)\,. \tag{2.3.2}$$

Since $\mathfrak{n}^{t-1}(\mathfrak{n}M) = \mathfrak{n}^t M = 0 = \mathfrak{n}^{t-1}(M/\mathfrak{n}M)$, 2.1 gives that $\varphi_i^{\mathfrak{n}M}$ and $\varphi_i^{M/\mathfrak{n}M}$ are injective for all i. Using (2.2.1), we have thus:

$$\mathrm{P}^P_{M/\mathfrak{n}M}(z) = \frac{\mathrm{P}^Q_{M/\mathfrak{n}M}(z)}{1-z^2} \quad \text{and} \quad \mathrm{P}^P_{\mathfrak{n}M}(z) = \frac{\mathrm{P}^Q_{\mathfrak{n}M}(z)}{1-z^2}\,. \tag{2.3.3}$$

Combining equations (2.3.1)-(2.3.3) and (2.2.1), we obtain:

$$HS_{\operatorname{Im}\nu_*^P(M)}(z) = \frac{HS_{\operatorname{Im}\nu_*^Q(M)}(z) - HS_{\operatorname{Ker}\varphi_*^M}(z)}{1-z^2}\,.$$

We have thus $\nu_i^P(M) = 0$ for all i if and only if $HS_{\operatorname{Im}\nu_*^Q(M)}(z) = HS_{\operatorname{Ker}\varphi_*^M}(z)$. □

Lemma 2.4. *Let N be a submodule of M and let i be an integer such that the map*

$$\operatorname{Tor}_i^Q(N, k) \to \operatorname{Tor}_i^Q(M, k)$$

induced by the inclusion $N \hookrightarrow M$ is zero. If $\varphi_i^{M/N}$ is injective, then φ_i^M is injective.

Proof. Starting with the exact sequence

$$0 \to N \to M \to M/N \to 0$$

one obtains a commutative diagram:

$$\begin{tikzcd}
\operatorname{Tor}_i^Q(N,k) \arrow[r, "0"] \arrow[d, "\varphi_i^N"] & \operatorname{Tor}_i^Q(M,k) \arrow[r] \arrow[d, "\varphi_i^M"] & \operatorname{Tor}_i^Q(M/N,k) \arrow[d, "\varphi_i^{M/N}"] \\
\operatorname{Tor}_i^P(N,k) \arrow[r] & \operatorname{Tor}_i^P(M,k) \arrow[r] & \operatorname{Tor}_i^P(M/N,k)
\end{tikzcd}$$

We know that righmost map in the top row is injective, since the preceding map in the top sequence is zero. Also, $\varphi_i^{M/N}$ is injective by hypothesis. The rightmost commutative diagram then gives that φ_i^M is injective. □

In what follows, we let K^Q denote the Koszul complex on a minimal system of generators of $\mathfrak{n}$. If M is a Q-module, we set $K^M = K^Q \otimes_Q M$. Note that K^Q is a DG algebra and K^M has an induced DG module structure over K^Q, in the sense of [4, §1]. We denote by $Z_i(K^M)$ the cycles of K^M of homological degree i. The notation ∂ identifies the differential of any of the complexes that we work with.

Proposition 2.5. *Assume $(Q,\mathfrak{n},k)$ is regular and $h \in \mathfrak{n}^2$. Set $e = \dim(Q)$ and $P = Q/(h)$. Let $G \in K_1^Q$ such that $\partial G = h$. If M is a P-module such that*

$$Z_e(K^M) = GZ_{e-1}(K^M)$$

then $\varphi_e^M = 0$.

Proof. Let g denote the image of G in K_1^P. Note that K^Q is a minimal free resolution of k over the regular local ring Q. Since $P = Q/(h)$ with $h \in \mathfrak{n}^2$, a minimal free resolution $\mathcal{D}$ of k over P has underlying graded algebra

$$\mathcal{D}_* = K_*^P \otimes_P \mathsf{\Gamma}_*^P\big(PY\big)$$

and differential induced from that of K^P, together with the relation $\partial Y = g$, and extended according to DGΓ algebra axioms. This construction goes back to Cartan; for the description in terms of DGΓ algebra structures we refer to [15] or [4].

Set $D^M = \mathcal{D} \otimes_P M$. We view K^P as a subcomplex of $\mathcal{D}$ and K^M as a subcomplex of $\mathcal{D}^M$. Computing homology over Q by means of K^Q and homology over P by means of $\mathcal{D}$, one can identify the map $\varphi_e^M\colon \operatorname{Tor}_e^Q(M,k) \to \operatorname{Tor}_e^P(M,k)$ with the canonical map

$$\mathrm{H}_e(K^M) \to \mathrm{H}_e(\mathcal{D}^M)$$

induced by the inclusion $K^M \subseteq \mathcal{D}^M$. Note that K^M has a canonical structure of DG module over K^P and $\mathcal{D}^M$ has a canonical structure of DG module over $\mathcal{D}$.

We have $\mathrm{H}_e(K^M) = Z_e(K^M)$. Let $z \in Z_e(K^M)$. By the hypothesis, we have $z = Gz'$ with $z' \in Z_{e-1}(K^M)$. We can view z' as a cycle in $\mathcal{D}^M$. Using the DG module axioms for $\mathcal{D}^M$, we have:

$$z = Gz' = gz' = \partial(Y)z' = \partial(Yz') - Y\partial(z') = \partial(Yz')$$

Hence z becomes a boundary in $\mathcal{D}^M$. □

3. A Golod homomorphism

In this section, the assumptions are as follows:

3.1. Let s,t be positive integers with

$$2 \le 2t-2 \le s \le 3t-4\,. \tag{3.1.1}$$

We set $r = s+1-t$.

Let $(Q,\mathfrak{n},k)$ $(P,\mathfrak{p},k)$, $(R,\mathfrak{m},k)$ be three local rings such that:

(1) Q is a regular local ring of dimension e.
(2) $R = Q/I$ with $I \subseteq \mathfrak{n}^t$, and $P = Q/(h)$ with $h \in I \cap \mathfrak{n}^t \smallsetminus \mathfrak{n}^{t+1}$.
(3) $\mathfrak{m}^{s+1} = 0 \neq \mathfrak{m}^s$.

Let $\varkappa\colon P \to R$ be the canonical projection.

The inequalities in (3.1.1) give $2t-2 \ge r+1 \ge r \ge t-1$, hence we have inclusions

$$\mathfrak{m}^{2t-2} \subseteq \mathfrak{m}^{r+1} \subseteq \mathfrak{m}^r \subseteq \mathfrak{m}^{t-1}\,. \tag{3.1.2}$$

Let M be an R-module. Note that M inherits both a Q-module and a P-module structure. We use the notation φ_i^M as introduced in (2.1.1) and $\nu_i^Q(M)$, $\nu_i^P(M)$ as in (1.2.1). We denote $\operatorname{Soc}(M)$ the *socle* of M, defined as

$$\operatorname{Soc}(M) = \{x \in M \mid \mathfrak{m}x = 0\}\,.$$

Since Q is regular of dimension e, we compute $\operatorname{Tor}_e^Q(M,k)$ as $\mathrm{H}_e(K^M)$, hence $\operatorname{Tor}_e^Q(M,k)$ can be canonically identified with $\operatorname{Soc}(M)$.

Lemma 1.1 finds an application as follows:

Proposition 3.2. *If $\nu_i^P(\mathfrak{m}^r)=0$ for all $i\geq 0$, then $\varkappa$ is Golod.*

Proof. We verify the hypotheses of Lemma 1.1 with $a=t-1$.

The condition (1) of Lemma 1.1 follows immediately from Lemma 1.3. The inclusions in (3.1.2) show that the map $\operatorname{Tor}_i^P(\mathfrak{m}^{2a},k)\to\operatorname{Tor}_i^R(\mathfrak{m}^a,k)$ induced by the inclusion $\mathfrak{m}^{2a}\subseteq\mathfrak{m}^a$ factors through $\nu_i^P(\mathfrak{m}^r)$ and it is thus zero for all i, hence condition (2) holds. $\square$

Theorem 3.3. *In addition to the assumptions in* 3.1*, assume the following:*

(a) $\operatorname{Soc}(R)\subseteq\mathfrak{m}^{r+1}$*;*
(b) $\varphi_e^{\mathfrak{m}^r}=0$*;*
(c) $\nu_i^Q(\mathfrak{m}^r)=0$ *for all* $i<e$*.*

Then $\nu_i^P(\mathfrak{m}^r)=0$ for all $i\geq 0$, and consequently $\varkappa$ is Golod.

Furthermore, for every finitely generated R–module M there exists a polynomial $p_M(z)\in\mathbb{Z}[z]$ with

$$\mathrm{P}_M^R(z)d_R(z)=p_M(z)$$

and such that $p_k(z)=(1+z)^e$, where

$$d_R(z)=1-z(\mathrm{P}_R^Q(z)-1)+az^{e+1}(1+z)$$

with $a=\operatorname{rank}_k\operatorname{Soc}(\mathfrak{m}^r)$.

The proof of the Theorem, mainly a consequence of Proposition 3.2, is given at the end of the section. Some preliminaries are needed in order to establish the formula for $d_R(z)$.

Lemma 3.4. *Under the hypotheses of Theorem* 3.3*, the following hold:*

(1) $\varphi_i^{\mathfrak{m}^{t-1}}$ *is injective for all* $i<e$*;*
(2) $HS_{\operatorname{Ker}\varphi_*^R}(z)=z+az^e$*.*

Proof. (1) The hypothesis that $\nu_i^Q(\mathfrak{m}^r)=0$ for all $i<e$ and the sequence of inclusions in (3.1.2) show that the map

$$\operatorname{Tor}_i^Q(\mathfrak{m}^{2t-2},k)\to\operatorname{Tor}_i^Q(\mathfrak{m}^{t-1},k)$$

induced by the inclusion $\mathfrak{m}^{2t-2}\subseteq\mathfrak{m}^{t-1}$ is zero for all $i<e$. In Lemma 2.4 take $M=\mathfrak{m}^{t-1}$ and $N=\mathfrak{m}^{2t-2}$ and note that $\varphi_i^{M/N}$ is injective for all i because $\mathfrak{n}^{t-1}(M/N)=0$, and hence 2.1 applies. Thus Lemma 2.4 gives that $\varphi_i^{\mathfrak{m}^{t-1}}$ is injective for all $i<e$.

(2) We compute now $\operatorname{Ker}(\varphi_i^R)$ for all i. Clearly, we have $\operatorname{Ker}(\varphi_i^R)=0$ for $i>e$ and $\operatorname{Ker}(\varphi_0^R)=0$, since φ_0^R is an isomorphism.

Claim 1. $\varphi_e^R=0$. Indeed, the hypothesis $\operatorname{Soc}(R)\subseteq\mathfrak{m}^{r+1}$ gives $\operatorname{Soc}(\mathfrak{m}^r)=\operatorname{Soc}(R)$, hence $\operatorname{Tor}_e^Q(\mathfrak{m}^r,k)\cong\operatorname{Tor}_e^Q(R,k)$. We have thus a commutative diagram:

$$\begin{tikzcd}
\operatorname{Tor}_e^Q(\mathfrak{m}^r,k) \arrow[r, "\cong"] \arrow[d, "\varphi_e^{\mathfrak{m}^r}"] & \operatorname{Tor}_e^Q(R,k) \arrow[d, "\varphi_e^R"] \\
\operatorname{Tor}_e^P(\mathfrak{m}^r,k) \arrow[r] & \operatorname{Tor}_e^P(R,k)
\end{tikzcd}$$

which shows that $\varphi_e^R=0$ because $\varphi_e^{\mathfrak{m}^r}=0$.

Claim 2. $\operatorname{Ker}(\varphi_i^R) = 0$ for all i with $1 < i < e$.

Set $\overline{R} = R/\mathfrak{m}^{t-1}$. Recall from Lemma 1.3 that the maps

$$\rho_i^P(\mathfrak{m}^{t-1})\colon \operatorname{Tor}_i^P(R/\mathfrak{m}^t, k) \to \operatorname{Tor}_i^P(\overline{R}, k)$$
$$\rho_i^Q(\mathfrak{m}^{t-1})\colon \operatorname{Tor}_i^Q(R/\mathfrak{m}^t, k) \to \operatorname{Tor}_i^Q(\overline{R}, k)$$

are zero for all $i > 0$. Consequently, the maps

$$\operatorname{Tor}_i^P(R, k) \to \operatorname{Tor}_i^P(\overline{R}, k) \quad \text{and} \quad \operatorname{Tor}_i^Q(R, k) \to \operatorname{Tor}_i^Q(\overline{R}, k)$$

induced by the canonical projection $R \to \overline{R}$ are zero.

Note that $\mathfrak{n}^{t-1}\overline{R} = 0$, hence $\varphi_i^{\overline{R}}$ is injective for all i, as discussed in 2.1. Consider the exact sequence

$$0 \to \mathfrak{m}^{t-1} \to R \to \overline{R} \to 0$$

and write the long exact sequences induced by applying $-\otimes_P k$, respectively $-\otimes_Q k$, and also the change of rings sequences as in 2.2. The maps χ_i introduced in 2.2 can also be understood as given by the action of the *Eisenbud operator*, in the particular case of a hypersurface, see [4, Construction 9.1.1]. Eisenbud operators are natural in the module arguments and they commute with connecting maps induced by short exact sequences, cf. [4, Proposition 9.1.3]. Together with the result of (1) that the maps $\varphi_i^{\mathfrak{m}^{t-1}}$ are injective for all $i < e$, these facts yield for each i with $1 < i < e$ the commutative diagram below, with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \operatorname{Tor}_{i+1}^Q(\overline{R}, k) \arrow[r] \arrow[d, "\varphi_i^{\overline{R}}"] & \operatorname{Tor}_i^Q(\mathfrak{m}^{t-1}, k) \arrow[r] \arrow[d, "\varphi_i^{\mathfrak{m}^{t-1}}"] & \operatorname{Tor}_i^Q(R, k) \arrow[r] \arrow[d, "\varphi_i^R"] & 0 \\
0 \arrow[r] & \operatorname{Tor}_{i+1}^P(\overline{R}, k) \arrow[r] \arrow[d, "\chi_{i+1}^{\overline{R}}"] & \operatorname{Tor}_i^P(\mathfrak{m}^{t-1}, k) \arrow[r] \arrow[d, "\chi_i^{\mathfrak{m}^{t-1}}"] & \operatorname{Tor}_i^P(R, k) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Tor}_{i-1}^P(\overline{R}, k) \arrow[r] \arrow[d] & \operatorname{Tor}_{i-2}^P(\mathfrak{m}^{t-1}, k) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

A use of the snake lemma in this diagram shows that φ_i^R is injective.

Claim 3. $\operatorname{rank}_k \operatorname{Ker}(\varphi_1^R) = 1$. Indeed, the map φ_1^R can be identified with the canonical projection $I/\mathfrak{n}I \to I/(\mathfrak{n}I, h)$. Since $I \subseteq \mathfrak{n}^t$ and $h \notin \mathfrak{n}^{t+1}$, we conclude $h \notin \mathfrak{n}I$, hence the kernel has rank 1. □

We can now prove the theorem.

Proof of Theorem 3.3. The hypothesis that $\operatorname{Soc}(R) \subseteq \mathfrak{m}^{r+1}$ yields that $\operatorname{Soc}(\mathfrak{m}^{r+1}) = \operatorname{Soc}(\mathfrak{m}^r)$, hence $\nu_e^Q(\mathfrak{m}^r)$ is an isomorphism.

By taking $M = \mathfrak{m}^r$ and $N = \mathfrak{m}^{r+1}$ in Lemma 2.4 and noting that $\varphi_i^{\mathfrak{m}^r/\mathfrak{m}^{r+1}}$ is injective in view of 2.1, we conclude that $\varphi_i^{\mathfrak{m}^r}$ is injective for all $i < e$.

Note that $\mathfrak{n}^t(\mathfrak{m}^r) = \mathfrak{m}^{s+1} = 0$. Condition (2) of Lemma 2.3 then holds with $M = \mathfrak{m}^r$. Indeed, $\operatorname{Ker}(\varphi_i^{\mathfrak{m}^r}) = 0 = \operatorname{Im} \nu_i^Q(\mathfrak{m}^r)$ for all $i < e$ and $\operatorname{Ker}(\varphi_e^{\mathfrak{m}^r}) = \operatorname{Soc}(\mathfrak{m}^r) = \operatorname{Im} \nu_e^Q(\mathfrak{m}^r)$. Lemma 2.3 gives thus $\nu_i^P(\mathfrak{m}^r) = 0$ for all i, and Proposition 3.2 then gives that $\varkappa$ is Golod.

Thus R is a homomorphic image of a hypersurface ring via a Golod homomorphism. Results of Levin cf. [5, Proposition 5.18] give then the remainder of the conclusion, except for the formula for $d_R(z)$.

Note that
$$d_R(z) = (1+z)^e \cdot (\mathrm{P}_k^R(z))^{-1}\,.$$
We need thus to find a formula for $\mathrm{P}_k^R(z)$. Since $\varkappa$ is Golod, the trivial Massey operation can be used to describe the minimal free resolution of k over R, see [13, Prop. 1], showing that the following formula holds:

$$\mathrm{P}_k^R(z) = \frac{\mathrm{P}_k^P(z)}{1 - z(\mathrm{P}_R^P(z)-1)}\,. \tag{3.4.1}$$

Using the conclusion (2) of Lemma 3.4 in formula (2.2.1), we have

$$\mathrm{P}_R^P(z) = \frac{\mathrm{P}_R^Q(z) - (1+z)(z+az^e)}{1-z^2}\,. \tag{3.4.2}$$

Since P is a hypersurface, we also have:

$$\mathrm{P}_k^P(z) = \frac{(1+z)^e}{1-z^2}\,. \tag{3.4.3}$$

Plugging (3.4.2) and (3.4.3) into (3.4.1) and simplifying, we obtain:

$$\mathrm{P}_k^R(z) = \frac{(1+z)^e}{1 - z(\mathrm{P}_R^Q(z)-1) + az^e + az^{e+1}}\,. \tag{3.4.4}$$

This formula gives the desired expression for $\mathrm{P}_k^R(z)$, and thus for $d_R(z)$. □

4. Compressed Gorenstein local rings

Let $(R,\mathfrak{m},k)$ be a local ring. The *Hilbert function* h_R of R is defined by
$$h_R(i) = \mathrm{rank}_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) \quad \text{for all } i \geq 0\,.$$
The number $h_R(1)$ is the *embedding dimension* of R, denoted $\mathrm{edim}(R) = e$. Assume from now that R is Artinian.

4.1. Let $R = Q/I$ be a minimal Cohen presentation, with $(Q,\mathfrak{n},k)$ a regular local ring with $\dim(Q) = e$. If $t \geq 1$, then the following conditions are equivalent:

(1) $I \subseteq \mathfrak{n}^t$;
(2) $h_R(t-1) = h_Q(t-1)$;
(3) $h_R(i) = h_Q(i)$ for all i with $0 \leq i \leq t-1$.

Since $h_Q(i) = \binom{e-i+1}{e-1}$ for all $i \geq 0$, these conditions are independent on the choice of the minimal Cohen presentation, and so is the number

$$v(R) = \max\{i \colon I \subseteq \mathfrak{n}^i\}\,. \tag{4.1.1}$$

One has thus $v(R) \geq t$ if and only if $I \subseteq \mathfrak{n}^t$, and $v(R) = t$ if and only if $I \subseteq \mathfrak{n}^t$ and $I \not\subseteq \mathfrak{n}^{t+1}$.

We say that an element $g \in R$ has valuation i in R and we write $v_R(g) = i$ if $g \in \mathfrak{m}^i \smallsetminus \mathfrak{m}^{i+1}$. If g is an element of valuation j of R, we denote by g^* the homogeneous element of degree j in R^{g} given by $\overline{g} \in \mathfrak{m}^j/\mathfrak{m}^{j+1}$. Note that if a and b are elements of R with $v_R(a) = i$ and $v_R(b) = j$, then we have
$$a^*b^* = (ab)^* \iff a^*b^* \neq 0 \iff v_R(ab) = i+j$$

We recall that $R^{\mathrm{g}} = Q^{\mathrm{g}}/I^*$, where I^* is the homogeneous ideal generated by all the elements g^* for $g \in I$. Note that $v(R) = v(R^{\mathrm{g}})$. This number can be thought of as the *initial degree* of I^*, since we have equalities:

$$v(R) = \inf\{v_R(h) \colon h \in I\} = \inf\{\deg(h^*) \colon h^* \in I^*\} = v(R^{\mathrm{g}})$$

We let $\lceil - \rceil$ denote the ceiling function, that is, $\lceil x \rceil$ is the smallest integer not less than x. If M is an R-module, we denote by $\lambda(M)$ its length and we set $M^\vee = \operatorname{Hom}_R(M, R)$.

The following result is known when the ring R is equicharacteristic, and it is thus an Artinian k-algebra, see [17], [18]. The characteristic assumption is unnecessary, and we provide a complete proof in order to make this point.

Proposition 4.2. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local Artinian ring of socle degree s and embedding dimension $e > 1$. We set $t = \left\lceil \dfrac{s+1}{2} \right\rceil$ and*

$$\varepsilon_i = \min\left\{ \binom{e-1+s-i}{e-1}, \binom{e-1+i}{e-1} \right\} \quad \textit{for all } i \textit{ with } 0 \le i \le s.$$

Then $\lambda(R) \le \sum_{i=0}^{e} \varepsilon_i$ and the following conditions are equivalent:

(1) $\lambda(R) = \sum_{i=0}^{e} \varepsilon_i$;
(2) $v(R) \ge t$ *and* $\operatorname{ann}(\mathfrak{m}^t) = \mathfrak{m}^{s+1-t}$;
(3) $h_R(i) = \varepsilon_i$ *for all i with* $0 \le i \le s$.

These conditions imply the following ones:

(a) $v(R) = t$;
(b) $\operatorname{ann}(\mathfrak{m}^i) = \mathfrak{m}^{s+1-i}$ *for all i with* $0 \le i \le s+1$;
(c) R^{g} *is Gorenstein.*

Definition. Let R be as in Proposition 4.2. We say that R is a *compressed Gorenstein ring* of socle degree s and embedding dimension e if R has maximal length, that is, $\lambda(R) = \sum_{i=0}^{e} \varepsilon_i$.

Proof of Proposition 4.2. Set $\varepsilon_i = 0$ when $i < 0$. Let $R = Q/I$ be a minimal Cohen presentation. We have $h_Q(j) = \varepsilon_j$ for all $j \le t-1$. Since $h_R(j) \le h_Q(j)$ for all $j \ge 0$ we have thus

$$h_R(j) \le \varepsilon_j \quad \text{for all } j \le t-1\,. \tag{4.2.1}$$

For each i with $0 \le i \le s+1$ we have $\mathfrak{m}^{s+1-i} \subseteq \operatorname{ann}(\mathfrak{m}^i)$, hence

$$\lambda(R/\mathfrak{m}^i) = \lambda\big((R/\mathfrak{m}^i)^\vee\big) = \lambda(\operatorname{ann}(\mathfrak{m}^i)) \ge \lambda(\mathfrak{m}^{s+1-i})\,. \tag{4.2.2}$$

The first equality above is due to the fact that R is Gorenstein. Next, note that

$$\lambda(R/\mathfrak{m}^i) = \sum_{j=0}^{i-1} h_R(j) \quad \text{and} \quad \lambda(\mathfrak{m}^{s+1-i}) = \sum_{j=s+1-i}^{s} h_R(j) = \lambda(R) - \sum_{j=0}^{s-i} h_R(j)$$

The inequality (4.2.2) becomes thus:

$$\lambda(R) \le \sum_{j=0}^{i-1} h_R(j) + \sum_{j=0}^{s-i} h_R(j) \tag{4.2.3}$$

When we take $i = t$ in (4.2.3) and we use (4.2.1) we obtain:

$$\text{(4.2.4)}\quad \lambda(R) \le \sum_{j=0}^{t-1} h_R(j) + \sum_{j=0}^{s-t} h_R(j) \le \sum_{j=0}^{t-1} \varepsilon_j + \sum_{j=0}^{s-t} \varepsilon_j = \sum_{j=0}^{t-1} \varepsilon_j + \sum_{j=t}^{s} \varepsilon_{s-j} = \sum_{j=0}^{s} \varepsilon_j$$

For the second inequality, note that $s-t \le t-1$, hence $h_R(j) \le \varepsilon_j$ for all $j \le s-t$ by (4.2.1), and for the last equality, note that $\varepsilon_j = \varepsilon_{s-j}$ for all j with $0 \le j \le s$.

(1) $\Longleftrightarrow$ (2): Equalities hold in (4.2.4) if and only if equalities hold in (4.2.1) for all $j \le t-1$ and in (4.2.2) for $i = t$. Equalities hold in (4.2.1) for all $j \le t-1$ if and only if $h_R(i) = h_Q(i)$ for all $i \le t-1$, and this is equivalent to $v(R) \ge t$ by 4.1. Equalities hold in (4.2.2) for $i = t$ if and only if $\lambda(\operatorname{ann}(\mathfrak{m}^t)) = \lambda(\mathfrak{m}^{s+1-t})$, and this is equivalent to $\operatorname{ann}(\mathfrak{m}^t) = \mathfrak{m}^{s+1-t}$.

(2) $\Longrightarrow$ (3): Assume (2) holds. Since $v(R) \ge t$, we know that $h_R(i) = \varepsilon_i$ for all $i \le t-1$ and $I \subseteq \mathfrak{n}^t$. We prove by induction on n :

$$\text{(4.2.5)}\qquad \operatorname{ann}(\mathfrak{m}^n) = \mathfrak{m}^{s+1-n} \quad \text{for all } n \text{ with } t \le n \le s+1.$$

The basis for the induction is the case $n = t$, which holds by assumption. Assume now that the statement holds for $n \ge t$. We prove that $\operatorname{ann}(\mathfrak{m}^{n+1}) = \mathfrak{m}^{s-n}$. Let $x \in R$ with $x\mathfrak{m}^{n+1} = 0$. Then $x\mathfrak{m} \subseteq \operatorname{ann}(\mathfrak{m}^n) = \mathfrak{m}^{s+1-n}$. In particular, if X denotes the preimage of x in Q, then:

$$X\mathfrak{n} \subseteq \mathfrak{n}^{s+1-n} + I \subseteq \mathfrak{n}^{s+1-n}$$

where the inclusion is due to the fact that $I \subseteq \mathfrak{n}^t \subseteq \mathfrak{n}^{s+1-n}$, since we have inequalities $t \ge s+1-t \ge s+1-n$. Since Q is regular, it follows that $X \in \mathfrak{n}^{s-n}$ and hence $x \in \mathfrak{m}^{s-n}$. This shows that $\operatorname{ann}(\mathfrak{m}^{n+1}) = \mathfrak{m}^{s-n}$ and it finishes the induction.

We prove now $h_R(i) = \varepsilon_i$ for all $t \le i \le s+1$. Note that

$$(R/\mathfrak{m}^i)^\vee \cong \operatorname{ann}(\mathfrak{m}^i) \quad \text{and} \quad (R/\mathfrak{m}^{i+1})^\vee \cong \operatorname{ann}(\mathfrak{m}^{i+1})\,.$$

hence, using duality, we have

$$h_R(i) = \operatorname{rank}_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \operatorname{rank}_k\left(\operatorname{ann}(\mathfrak{m}^{i+1})/\operatorname{ann}(\mathfrak{m}^i)\right).$$

By (4.2.5), we have then

$$h_R(i) = \operatorname{rank}_k\left(\operatorname{ann}(\mathfrak{m}^{i+1})/\operatorname{ann}(\mathfrak{m}^i)\right) = \operatorname{rank}_k(\mathfrak{m}^{s-i}/\mathfrak{m}^{s+1-i}) = h_R(s-i)$$

for all i with $t \le i \le s+1$. Note that $s-i \le 2t-1-i \le t-1$ for all such i, hence we further have:

$$h_R(i) = h_R(s-i) \le \varepsilon_{s-i} = \varepsilon_i\,.$$

Assume now that (1)-(3) hold. To prove (a), note that we already know $v(R) \ge t$. To show that equality holds, use 4.1, noting that

$$h_R(t) = \varepsilon_t < h_Q(t)\,.$$

For part (b) note that the equalities $h_R(j) = \varepsilon_j$ for all j force equalities in (4.2.3), and thus in (4.2.2), for all i with $0 \le i \le s$.

We prove part (c) as a consequence of (b). To show that $R^{\mathfrak{g}}$ is Gorenstein, it is enough to prove that $\operatorname{Soc}(R^{\mathfrak{g}}) \subseteq (\mathfrak{m}^{\mathfrak{g}})^s$. Assume $x^* \in \operatorname{Soc}(R^{\mathfrak{g}})$ for some $x \in R$ with $v_R(x) = i \le s-1$. Then $x\mathfrak{m} \subseteq \mathfrak{m}^{i+2}$, hence $x\mathfrak{m}^{s-i} \subseteq \mathfrak{m}^{s+1} = 0$ and thus $x \in \operatorname{ann}(\mathfrak{m}^{s-i}) = \mathfrak{m}^{i+1}$ from (b), a contradiction. $\square$

4.3. For the remainder of the section, we assume that $(R, \mathfrak{m}, k)$ is a compressed Gorenstein local ring of socle degree s and embedding dimension $e > 1$.

Let $R = Q/I$ be a minimal Cohen presentation where $(Q, \mathfrak{n}, k)$ is a regular local ring. We set $t = v(R)$. By Proposition 4.2, we know $t = \lceil (s+1)/2 \rceil$. Also, we set $r = s+1-t$. Note that $r = t-1$ when $s = 2t-2$ is even and $r = t$ when $s = 2t-1$ is odd.

Lemma 4.4. *The map*

$$\nu_i^Q(\mathfrak{m}^r)\colon \operatorname{Tor}_i^Q(\mathfrak{m}^{r+1}, k) \to \operatorname{Tor}_i^Q(\mathfrak{m}^r, k)$$

is zero for all $i < e$ and is bijective for $i = e$.

Proof. Since R is Gorenstein, $\operatorname{Soc}(\mathfrak{m}^r) = \operatorname{Soc}(\mathfrak{m}^{r+1}) = \mathfrak{m}^s$, hence $\nu_e^Q(\mathfrak{m}^r)$ is bijective. Proposition 4.2 gives $\operatorname{ann}(\mathfrak{m}^t) = \mathfrak{m}^r$ and $\operatorname{ann}(\mathfrak{m}^{t-1}) = \mathfrak{m}^{r+1}$, hence

$$(R/\mathfrak{m}^t)^\vee \cong \operatorname{ann}(\mathfrak{m}^t) = \mathfrak{m}^r \quad \text{and} \quad (R/\mathfrak{m}^{t-1})^\vee \cong \operatorname{ann}(\mathfrak{m}^{t-1}) = \mathfrak{m}^{r+1}\,.$$

Proposition 4.2 and 4.1 give that $I \subseteq \mathfrak{n}^t$. In particular, $R/\mathfrak{m}^j = Q/\mathfrak{n}^j$ for all $j \le t$. We have thus canonical isomorphisms

$$(Q/\mathfrak{n}^t)^\vee \cong \mathfrak{m}^r \quad \text{and} \quad (Q/\mathfrak{n}^{t-1})^\vee \cong \mathfrak{m}^{r+1}\,. \tag{4.4.1}$$

Since R is Gorenstein, note that $M^\vee \cong \operatorname{Ext}_Q^e(M, Q)$ for any finitely generated R-module M. Consequently, if

$$F: \qquad F_e \to \cdots \to F_i \to F_{i-1} \to \cdots \to F_0$$

is a minimal free resolution of M over Q, then

$$F^*: \qquad (F_0)^* \to \ldots (F_{i-1})^* \to (F_i)^* \to \cdots \to (F_e)^*$$

is a minimal free resolution of $M^\vee$ over Q, where $(F_i)^* = \operatorname{Hom}_Q(F_i, Q)$. In particular, we have canonical identifications:

$$\operatorname{Tor}_i^Q(M^\vee, k) = \mathrm{H}_i(F^* \otimes_Q k) = \mathrm{H}_{e-i}(F \otimes_Q k) = \operatorname{Tor}_{e-i}^Q(M, k)\,.$$

In view of (4.4.1), we have $\nu_i(\mathfrak{m}^r) = 0$ for $i < e$ if and only if

$$\rho_{e-i}^Q(\mathfrak{n}^{t-1})\colon \operatorname{Tor}_{e-i}^Q(Q/\mathfrak{n}^t, k) \to \operatorname{Tor}_{e-i}^Q(Q/\mathfrak{n}^{t-1}, k)$$

is zero for all $i < e$. Since Q is regular, we can apply 1.2 to get this conclusion. $\square$

Below, we adopt the following notational convention: if X is an element of Q, then x denotes its image in R.

Lemma 4.5. *With the notation in* 4.3, *assume k is infinite. Let $h \in I$ with $v_R(h) = t$. There exists then a minimal system of generators $X_1, ..., X_e$ of $\mathfrak{n}$ such that*

(1) $(h) = (X_1^t + C)$ *with* $C \in (X_2, ..., X_e)$.
(2) $\mathfrak{m}^s = x_1^{t-1} \operatorname{ann}_R(x_2, ..., x_e)$.

Proof. Since Q is regular, Q^{g} is a polynomial ring over k. Say $Q^{\mathrm{g}} = k[Z_1, \ldots, Z_e]$ where $Z_1, \ldots, Z_e$ are variables of degree 1. Let $h \in I \cap \mathfrak{n}^t \smallsetminus \mathfrak{n}^{t+1}$, hence $\deg(h^*) = t$ in Q^{g}. Then after a generic change of coordinates in Q^{g} we may assume that $h^* = Z_1^t + A$ where $A \in (Z_2, \ldots, Z_e)_t$. Now let $X_1, \ldots, X_e$ be a minimal system of generators of $\mathfrak{n}$ such that $X_i^* = Z_i$. We have then

$$h = X_1^t + L \quad \text{with} \quad L \in (X_2, \ldots, X_e) + \mathfrak{n}^{t+1}\,.$$

Further, we can write $L = C' + X_1^t B$ with $C' \in (X_2, \dots, X_e)$ and $B \in (X_1)$, hence

$$h = X_1^t(1 + B) + C' \in I \tag{4.5.1}$$

where $1 + B$ is an unit in Q and (1) follows taking $C = (1 + B)^{-1}C'$. In particular we deduce

$$x_1^t \in (x_2, \dots, x_e)\,. \tag{4.5.2}$$

Set $\mathfrak{q} = \operatorname{ann}(x_2, \dots, x_n)$ and recall that $r = s + 1 - t$. Now we prove the following claims:

Claim 1. $\mathfrak{q} \subseteq \mathfrak{m}^r$.

Since R is Gorenstein, the statement is equivalent to $\operatorname{ann}(\mathfrak{m}^r) \subseteq \operatorname{ann}(\mathfrak{q})$. Noting that $\operatorname{ann}(\mathfrak{q}) = (x_2, \dots, x_n)$ we need thus to show that

$$\operatorname{ann}(\mathfrak{m}^r) \subseteq (x_2, \dots, x_e)\,.$$

Note that $\operatorname{ann}(\mathfrak{m}^r) = \mathfrak{m}^t$ by Proposition 4.2. Using (4.5.2), we have

$$\mathfrak{m}^t = (x_2, \dots, x_e)\mathfrak{m}^{t-1} + (x_1^t) \subseteq (x_2, \dots, x_e)\,.$$

Claim 2. $\mathfrak{q} \not\subseteq \mathfrak{m}^{r+1}$.

Since R is Gorenstein, the statement is equivalent to $\operatorname{ann}(\mathfrak{m}^{r+1}) \not\subseteq \operatorname{ann}(\mathfrak{q})$. Since $\operatorname{ann}(\mathfrak{m}^{r+1}) = \mathfrak{m}^{t-1}$ by Proposition 4.2, we need to show

$$\mathfrak{m}^{t-1} \not\subseteq (x_2, \dots, x_e)\,.$$

Assume that the inclusion holds. Then in the ring Q this implies

$$\mathfrak{n}^{t-1} \subseteq (X_2, \dots, X_e) + I \subseteq (X_2, \dots, X_e) + \mathfrak{n}^t\,.$$

The second inclusion is due to the fact that $I \subseteq \mathfrak{n}^t$. Using Nakayama, it follows that $\mathfrak{n}^{t-1} \subseteq (X_2, \dots, X_e)$. This is a contradiction, since $\dim Q = e$.

Claim 3. $x_1^{t-1}\mathfrak{q} \neq 0$.

Assume that $x_1^{t-1}\mathfrak{q} = 0$. This means that $x_1^{t-1} \in \operatorname{ann}(\mathfrak{q}) = (x_2, \dots, x_e)$ and, in particular, it implies $\mathfrak{m}^{t-1} \subseteq (x_2, \dots, x_e)$. This cannot happen, see Claim 2.

Since $\mathfrak{q} \subseteq \mathfrak{m}^r$, we have $x_1^{t-1}\mathfrak{q} \subseteq \mathfrak{m}^s$. Since $x_1^{t-1}\mathfrak{q} \neq 0$ and R is Gorenstein, it follows that $\mathfrak{m}^s = x_1^{t-1}\mathfrak{q}$. $\square$

Proposition 4.6. *With the notation in* 4.3*, let $h \in I \cap \mathfrak{n}^t \smallsetminus \mathfrak{n}^{t+1}$ and let $(P, \mathfrak{p}, k)$ be the local ring defined by $P = Q/(h)$. Then the map*

$$\varphi_e^{\mathfrak{m}^r} \colon \operatorname{Tor}_e^Q(\mathfrak{m}^r, k) \to \operatorname{Tor}_e^P(\mathfrak{m}^r, k)$$

is zero.

Proof. After a flat change of ring(s), we may assume that k is infinite. We show that the hypotheses of Proposition 2.5 are satisfied and then we apply this result. Let $X_1, \dots, X_e$ be a minimal system of generators of $\mathfrak{n}$ as in Lemma 4.5. We may assume then $h = X_1^t + C$ for some $C \in (X_2, \dots, X_e)$ and

$$\operatorname{Soc}(\mathfrak{q}) = \operatorname{Soc}(R) = \mathfrak{m}^s = x_1^{t-1}\mathfrak{q} \quad \text{where} \quad \mathfrak{q} = \operatorname{ann}_R(x_2, ..., x_e)\,.$$

In Proposition 2.5, take K^Q to be the Koszul complex on $X_1, \dots, X_e$ and $M = \mathfrak{q}$.

Let $T_1, \dots, T_e$ be exterior algebra variables such that $K^R = \bigwedge_*^R\big(RT_1 \oplus \cdots \oplus RT_e\big)$, with $\partial T_i = x_i$. Since $h = X_1^t + C$ we can pick $G \in K_1^Q$ with $\partial(G) = h$ such that the image g of G in K^R satisfies $g = x_1^{t-1}T_1 + \alpha_2 T_2 + \cdots + \alpha_e T_e$ with $\alpha_2, \dots, \alpha_e \in R$.

Identify $K^{\mathfrak{q}}$ with $\mathfrak{q}K^R$. We need to verify the equality

$$Z_e(\mathfrak{q}K^R) = gZ_{e-1}(\mathfrak{q}K^R).$$

Indeed, we have:

$$\begin{aligned} Z_e(\mathfrak{q}K^R) &= \operatorname{Soc}(\mathfrak{q})T_1 \dots T_e = (x_1^{t-1}\mathfrak{q})T_1 \dots T_e = \mathfrak{q}(x_1^{t-1}T_1)(T_2, \dots, T_e) = \\ &= \mathfrak{q}(g - \alpha_2 T_2 - \dots \alpha_e T_e)T_2 \dots T_e = \mathfrak{q}gT_2 \dots T_e = g(\mathfrak{q}T_2 \dots T_e) \end{aligned}$$

If $q \in \mathfrak{q}$, then $qx_i = 0$ for all $i \neq 1$ and the Leibnitz rule gives that $\partial(qT_2 \dots T_e) = 0$. Thus $\mathfrak{q}T_2 \dots T_e \subseteq Z_{e-1}(\mathfrak{q}K^R)$.

Hence the hypotheses of Proposition 2.5 are satisfied, and we conclude $\varphi_e^{\mathfrak{q}} = 0$.

Recall that $\mathfrak{q} \subseteq \mathfrak{m}^r$ by Claim 1 of the proof of Lemma 4.5. An argument simliar to the one in the proof of Lemma 3.4(2) shows that $\varphi_e^{\mathfrak{q}} = 0$ implies $\varphi_e^{\mathfrak{m}^r} = 0$. $\square$

5. The main results

We are now ready to state our main result.

Theorem 5.1. *Let e, s be integers such that $2 \leq s \neq 3$ and $e > 1$ and let $(R, \mathfrak{m}, k)$ be a compressed Gorenstein local ring of socle degree s and embedding dimension e. Let $R = Q/I$ be any minimal Cohen presentation of R. Let $h \in I$ with $v_R(h) = v(R)$ and let $(P, \mathfrak{p}, k)$ be the local hypersurface ring defined by $P = Q/(h)$.*

The canonical map $\varkappa \colon P \to R$ is then Golod. Furthermore, for every finitely generated R–module M there exists a polynomial $p_M(z) \in \mathbb{Z}[z]$ with

$$\mathrm{P}_M^R(z) d_R(z) = p_M(z)$$

and such that $p_k(z) = (1+z)^e$, where

$$d_R(z) = 1 - z(\mathrm{P}_R^Q(z) - 1) + z^{e+1}(1+z)\,. \tag{5.1.1}$$

If s is even, then $d_R(z)$ depends only on the integers e and s as follows

$$d_R(z) = 1 + z^{e+2} + (-z)^{-\frac{s-2}{2}} \cdot \Big((1+z)^e \cdot \big(HS_{R^{\mathfrak{g}}}(-z) - 1 - z^{s+e} \Big) \tag{5.1.2}$$

and in particular $\mathrm{P}_k^R(z) = \mathrm{P}_k^{R^{\mathfrak{g}}}(z)$.

The proof is given at the end of the section.

Remark 5.2. The Poincaré series of Gorenstein local rings of embedding dimension $e \leq 4$ have been classified in [6]. According to this classification, the compressed Gorenstein local rings with $e = 4$ are of type GGO. This is because the type GGO is the only one in the list for which $d_R(z)$ has degree 6.

Remark 5.3. When $s = 2$, the formula (5.1.2) becomes

$$d_R(z) = HS_{R^{\mathfrak{g}}}(-z) \cdot (1+z)^e\,.$$

The result of the Theorem is subsumed, in this case, by the results of Sjödin [28], who proves the formulas for Poincaré series, and Avramov et. al. [2], who prove the conclusion about the Golod homomorphism.

Remark 5.4. Our methods cannot recover the results of [16] for $s = 3$. Furthermore, Bøgvad's examples [8] show that not all compressed Gorenstein local rings with $s = 3$ have rational Poincaré series.

5.5. *Generic Gorenstein Artinian algebras.* Let D denote the divided powers algebra over k on e variables of degree 1 and let $Q = k[[x_1, \dots, x_e]]$ be the power series ring in e variables. The ring Q acts then on D by "differentiation", as described for example in [18]. F. S. Macaulay proved that there exists a one-to-one correspondence between ideals $I \subseteq Q = k[[x_1, \dots, x_e]]$ such that $R = Q/I$ is an Artinian Gorenstein local algebra and cyclic R-submodules of D, see [18, Lemma 1.2]. Thus Gorenstein Artinian k-algebras of embedding dimension e and socle degree s can be parametrized by points in a certain projective space. There exists then a nonempty open (and thus dense) subset of this projective space such that the corresponding algebras have maximal length/Hilbert function, see [18, Theorem I]. In other words, a *generic* Gorenstein Artinian k-algebra of embedding dimension e and socle degree s is compressed.

Corollary 5.6. *Let e and s be any positive integers. Let R be a generic Gorenstein Artinian k-algebra of embedding dimension e and socle degree s. Then for every finitely generated R-module there exists a polynomial $p_M(z) \in \mathbb{Z}[z]$ with*

$$\mathrm{P}^R_M(z) d_R(z) = p_M(z)$$

and such that $p_k(z) = (1+z)^e$.

Proof. If $e \le 2$, then R is a complete intersection, and the statement is due to Gulliksen, see [14, Theorem 4.1]; in this case, $d_R(z) = (1-z^2)^e$. Assume now $e > 3$, and in particular $s \ge 2$. By 5.5, R is compressed. If $s \ne 3$, the result follows from Theorem 5.1. If $s = 3$, then Conca, Rossi and Valla [10, Claim 6.5] prove that a generic Gorenstein graded algebra R has an element x of degree 1 such that $\operatorname{ann}(x)$ is principal, and work of Henriques and Şega [16, Corollary 4.5] shows that such algebras satisfy the desired conclusion. A result of Elias and Rossi [11, Theorem 3.3] shows that the graded hypothesis can be removed and this completes the proof. □

Remark 5.7. When s is odd, the compressed hypothesis in the theorem is not enough to grant that $\mathrm{P}^Q_R(z)$, and thus $d_R(z)$, depends only on e and s. However, for graded rings, formulas for the Betti numbers of R over Q depending only on e and s are conjectured when R is generic, see [9]. Thus a formula for $d_R(z)$ is available in the cases when the conjecture is known to hold; see [21] for some cases.

We now prepare to give a proof of the theorem.

The Poincaré series of a finitely generated graded module over a standard graded algebra can be defined the same way as in the local case. If $N = \oplus N_i$ is a graded R^{g}-module and j is an integer, then the notation $N(-j)$ stands for the graded module whose ith graded component is N_{i+j}.

Lemma 5.8. *Under the hypotheses of the Theorem 5.1, if s is even, then*

$$\mathrm{P}^Q_R(z) = 1 + z^e + (-z)^{-\frac{s-2}{2}} \cdot \left(HS_{R^{\mathrm{g}}}(-z) \cdot (1+z)^e - 1 - z^{s+e} \right) .$$

Proof. Since s is even, Proposition 4.2 gives $s = 2t-2$ with $t = v(R)$. By Proposition 4.2(c), R^{g} is Gorenstein as well. Thus R^{g} is a compressed Gorenstein graded algebra of socle degree $2t-2$ and embedding dimension e.

Set $A = Q^{\mathrm{g}}$ and $\beta_i = \beta_i^A(R^{\mathrm{g}})$. According to [18, 4.7], a graded minimal free resolution of R^{g} over A is as follows:

$$(5.8.1) \quad 0 \to A(-e-s) \to A^{\beta_{e-1}}(-e-t+2) \to \cdots \to A^{\beta_1}(-t) \to A \to R^{\mathrm{g}} \to 0$$

where the ith term of the free resolution is $A^{\beta_i}(-i-t+1)$ for $0<i<e$.

A Hilbert series computation in (5.8.1) gives the following:

$$HS_{R^{\mathrm{g}}}(z)=HS_A(z)(1-\beta_1z^t+\beta_2z^{t+1}+\cdots+(-1)^{e-1}\beta_{e-1}z^{t+e-2}+(-1)^ez^{s+e})\,.$$

We further obtain

$$\begin{aligned}
\mathrm{P}^A_{R^{\mathrm{g}}}(-z)&=1-\beta_1z+\beta_2z^2+\cdots+(-1)^e\beta_ez^e\\
&=z^{-t+1}\left(1-\beta_1z^t+\beta_2z^{t+1}+\cdots+(-1)^{e-1}\beta_{e-1}z^{t+e-2}+(-1)^ez^{s+e}\right)+\\
&\qquad+1+(-z)^e-z^{-t+1}\cdot(1+(-1)^ez^{s+e})\\
&=z^{-t+1}\cdot HS_{R^{\mathrm{g}}}(z)\cdot(1-z)^e+1+(-z)^e-z^{-t+1}\cdot(1+(-1)^ez^{s+e})\\
&=1+(-z)^e+z^{-t+1}\cdot\left(HS_{R^{\mathrm{g}}}(z)\cdot(1-z)^e-1-(-1)^ez^{s+e}\right)
\end{aligned}$$

where the second equality uses the fact that $\beta_e=1$, and the third equality uses the equation above and the fact that $HS_A(z)=(1-z)^{-e}$. We have thus

$$\mathrm{P}^A_{R^{\mathrm{g}}}(z)=1+z^e+(-z)^{-t+1}\cdot\left(HS_{R^{\mathrm{g}}}(-z)\cdot(1+z)^e-1-z^{s+e}\right)\,.$$

Finally, the fact that the graded resolution of R^{g} over A is pure gives $\mathrm{P}^{Q^{\mathrm{g}}}_{R^{\mathrm{g}}}(z)=\mathrm{P}^Q_R(z)$, see Fröberg [12, Theorem 1]. $\square$

Proof of Theorem 5.1. Set $t=v(R)$. Recall from Proposition 4.2 that $t=\lceil(s+1)/2\rceil$. We want to apply Theorem 3.3. First, we need to check that the inequalities of (3.1.1) are satisfied. Since $s=2t-1$ or $s=2t-2$, these inequalities are consequences of the hypothesis $2\le s\ne 3$. Next, we check the hypotheses (a)–(c): Part (a) follows from the fact that R is Gorenstein. Part (b) follows from Proposition 4.6. Part (c) follows from Lemma 4.4.

All conclusions of the theorem follow then from Theorem 3.3, except for formula (5.1.2). When s is even, (5.1.2) is obtained by plugging the formula of Lemma 5.8 into (5.1.1). The formulas in Proposition 4.2(3) show that $HS_{R^{\mathrm{g}}}(z)$ depends only on e and s. To see that $\mathrm{P}^R_k(z)=\mathrm{P}^{R^{\mathrm{g}}}_k(z)$, note that R^{g} satisfies the same hypotheses as R, as already noted in the proof of Lemma 5.8. $\square$

6. Factoring out the socle

The development described below comes from a suggestion of Jürgen Herzog regarding a different way of computing a formula for $\mathrm{P}^R_k(z)$ in Theorem 5.1.

Let $(R,\mathfrak{m},k)$ be a local ring and $\widehat{R}=Q/I$ a minimal Cohen presentation. The ring R is said to be *Golod* if the induced map $Q\to\widehat{R}$ is Golod. This definition is independent of the choice of the presentation; see more details in [4]. It is known that R is Golod if and only if the formula:

$$\mathrm{P}^R_k(z)=\frac{(1+z)^e}{1-z(\mathrm{P}^Q_R(z)-1)}$$

holds, where e is the embedding dimension of R.

Lemma 6.1. *Let R be a Gorenstein local ring of embedding dimension e. The following equality then holds:*

$$\mathrm{P}^Q_{R/\operatorname{Soc}(R)}(z)=\mathrm{P}^Q_R(z)+z(1+z)^e-z^e(1+z)$$

Proof. Note that the map

$$\nu_i\colon \operatorname{Tor}_i^Q(\operatorname{Soc}(R),k) \to \operatorname{Tor}_i^Q(R,k)$$

induced by the inclusion $\operatorname{Soc}(R) \subseteq R$ is zero for all $i < e$. This follows by an argument similar to that in the proof of Lemma 4.4, using duality; we skip the details. Also, note that ν_e is bijective because $\operatorname{Soc}(\operatorname{Soc}(R)) = \operatorname{Soc}(R)$.

Set $S = R/\operatorname{Soc}(R)$. Consider now the exact sequence

$$0 \to \operatorname{Soc}(R) \to R \to S \to 0$$

and the long sequence induced in homology

$$\cdots \to \operatorname{Tor}_i^Q(R,k) \to \operatorname{Tor}_i^Q(S,k) \to \operatorname{Tor}_{i-1}^Q(\operatorname{Soc}(R),k) \to \dots$$

Note that $\operatorname{Soc}(R) \cong k$. One obtains equalities:

$$\beta_i^Q(S) = \beta_i^Q(R) + \beta_{i-1}^Q(k)$$

for all $i < e$. We also have:

$$\beta_e^Q(S) = \beta_{e-1}(k) \quad \text{and} \quad \beta_e^Q(R) = \beta_e(k) = 1$$

and the desired formula follows from here, since $\mathrm{P}_k^Q(z) = (1+z)^e$. □

Proposition 6.2. *Let R be a Gorenstein local ring of embedding dimension $e > 1$. The following statements are equivalent:*

(1) $\mathrm{P}_k^R(z) = \dfrac{(1+z)^e}{1 - z(\mathrm{P}_R^Q(z) - 1) + z^{e+1}(z+1)}$.

(2) *$R/\operatorname{Soc}(R)$ is Golod.*

Proof. Set $S = R/\operatorname{Soc}(R)$. By [7, Theorem 2], we know that the canonical homomorphism $R \to S$ is Golod and the following formula holds:

$$\mathrm{P}_k^S(z) = \frac{\mathrm{P}_k^R(z)}{1 - z^2\,\mathrm{P}_k^R(z)}$$

Rearranging this formula, we have:

$$\mathrm{P}_k^R(z) = \frac{\mathrm{P}_k^S(z)}{1 + z^2\,\mathrm{P}_k^S(z)} \tag{6.2.1}$$

Assume that S is Golod, hence

$$\mathrm{P}_k^S(z) = \frac{(1+z)^e}{1 - z(\mathrm{P}_S^Q(z) - 1)} = \frac{(1+z)^e}{1 - z(\mathrm{P}_R^Q(z) - 1) - z^2(1+z)^e + z^{e+1}(1+z)}$$

where we used Lemma 6.1 in the second equality. Plugging this into (6.2.1) and simplifying, one obtains the formula in (1).

Proceed similarly for the converse. □

In view of Theorem 5.1, it follows that $R/\operatorname{Soc}(R)$ is Golod for R satisfying the hypotheses of the theorem. A more straightforward explanation can be given when R has even socle degree, see the proof below.

Proposition 6.3. *If R is a Gorenstein compressed local ring of socle degree s with $2 \le s \ne 3$, then $R/\mathfrak{m}^i$ is a Golod ring for all i with $2 \le i \le s$.*

Proof. When $e = 1$, the result follows, for example, from [26, Proposition 6.10]. Assume $e > 1$. Let $t = v(R)$, so that $s = 2t - 1$ or $s = 2t - 2$.

Assume $t \leq i \leq 2t - 2$. Set $J = I + \mathfrak{n}^i$. Then $R/\mathfrak{m}^i = Q/J$ and

$$\mathfrak{n}^{2t-2} \subseteq \mathfrak{n}^i \subseteq J \subseteq \mathfrak{n}^t\,.$$

It is a known fact that Q/J is Golod when $\mathfrak{n}^{2t-2} \subseteq J \subseteq \mathfrak{n}^t$. In fact, note that Lemma 1.1 can be applied for the map $Q \to Q/J$ with $a = t - 1$ and $b = 2t - 2$. Indeed, to check the hypothesis (1), see Lemma 1.3. The hypothesis (2) is trivially satisfied, since $\mathfrak{m}^{2t-2}/J = 0$.

When $i \leq t$, then $R/\mathfrak{m}^i = Q/\mathfrak{n}^i$ and the ring $Q/\mathfrak{n}^i$ is Golod for all $2 \leq i$, since Q is regular.

The only case not covered yet is when $i = s = 2t - 1$. Theorem 5.1 and Proposition 6.2 give the conclusion in this case. □

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