# ON THE RATE OF POINTS IN PROJECTIVE SPACES

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Abstract. The rate of a standard graded $K$-algebra $R$ is a measure of the growth of the shifts in a minimal free resolution of $K$ as an $R$-module. It is known that $\text{rate}(R) = 1$ if and only if $R$ is Koszul and that $\text{rate}(R) \geq m(I) - 1$ where $m(I)$ denotes the highest degree of a generator of the defining ideal $I$ of $R$.

We show that the rate of the coordinate ring of certain sets of points $X$ of the projective space $\mathbf{P^n}$ is equal to $m(I) - 1$. This extends a theorem of Kempf. We study also the rate of algebras defined by a space of forms of some fixed degree $d$ and of small codimension.

## Introduction

Let $K$ be a field. A graded commutative Noetherian $K$-algebra $R = \bigoplus_{i \in \mathbf{N}} R_i$ is said to be standard if $R_0 = K$ and $R$ is generated (as a $K$-algebra) by elements of degree 1. Denote by $\mathcal{M}$ the homogeneous maximal ideal $\bigoplus_{i \geq 1} R_i$ of $R$. The minimal free resolution

$$\mathbf{F} : \cdots \to \mathbf{F_i} \to \mathbf{F_{i-1}} \to \cdots \to \mathbf{F_0} \to \mathbf{K} \to \mathbf{0}$$

of $K = R/\mathcal{M}$ as an $R$-module plays an important role in the study of the homological properties of $R$. For instance one knows that $\mathbf{F}$ is finite if and only if any finitely generated $R$-module $M$ has a finite free resolution as an $R$-module and, by the Auslander-Buchsbaum-Serre theorem, this is equivalent to the fact that $R$ is a polynomial ring. There are several invariants attached to $\mathbf{F}$. One is the Poincaré series of $R$ which is defined as the formal power series whose $i$-th coefficient is the rank of the free module $F_i$. Another important invariant is the Backelin rate which measures the growth of the shifts in $\mathbf{F}$. It is defined as follows. For any finitely generated graded $R-$module $M$

and for every integer $i \geq 0$, one sets

$$t_i^R(M) = \max\{j\colon \operatorname{Tor}_i^R(M,K)_j \neq 0\}$$

if $\operatorname{Tor}_i^R(M,K) \neq 0$, and $t_i^R(M) = 0$ otherwise. Here $\operatorname{Tor}_i^R(M,K)_j$ denotes the $j-$th graded piece of $\operatorname{Tor}_i^R(M,K)$. Then the Backelin rate of $R$ is defined as

$$\operatorname{rate}(R) = \sup_{i \geq 2}\{t_i^R(K) - 1/i - 1\}.$$

It turns out that the rate of any standard graded algebra is finite, see [A, ABH]. We have $\operatorname{rate}(R) \geq 1$ and the equality holds if and only if $R$ is Koszul, so that $\operatorname{rate}(R)$ can be taken as a measure of how much $R$ deviates from being Koszul, see [Ba].

Consider a minimal presentation of $R$ as a quotient of a polynomial ring, i.e.

$$R \simeq S/I$$

where $S = K[x_1, \ldots, x_n]$ is a polynomial ring and $I$ is an ideal generated by homogeneous elements of degree $> 1$. Let $m(I)$ be the maximum of the degree of a minimal homogeneous generator of $I$. It follows from (the graded version of) [BH, Thm. 2.3.2] that $t_2(K) = m(I)$, thus one has

$$\operatorname{rate}(S/I) \geq m(I) - 1.$$

Let $\tau$ be a monomial order on $S$. Denote by $\operatorname{in}(I)$ the initial ideal of $I$ with respect to $\tau$. In [BHV, Cor. 2.4, Thm.2.2] it is proved that:

$$\operatorname{rate}(S/I) \leq \operatorname{rate}(S/\operatorname{in}(I)).$$

Further for any monomial ideal $J$ it is known (see [Ba, ERT] and also Example 1.4) that $\operatorname{rate}(S/J) = m(J) - 1$. Summing up, one has

$$m(I) - 1 \leq \operatorname{rate}(S/I) \leq m(\operatorname{in}(I)) - 1.$$

In particular, it follows that if $I$ is generated by a Gröbner basis of forms of degree $\leq m(I)$ with respect to some coordinate system and some term order, then the rate is minimal, that is, $\operatorname{rate}(S/I) = m(I) - 1$.

An other interesting class of standard graded $K$-algebras with minimal rate is given by the generic toric rings, see [GPW].

The goal of this paper is to determine the rate of the coordinate ring of some finite sets of points in the projective space and of some algebras defined by a space of forms of small codimension.

Let $X$ be a set of points of the projective space $\mathbf{P}^n$ and let $R$ be its coordinate ring and $I$ be its defining ideal. In Section 3 we show that $\operatorname{rate}(R) = m(I) - 1$ under the assumption that the cardinality of $X$ is $\binom{n+t-1}{n} + c$ with $0 \leq c < n$ and that every subset of $X$ has the expected Hilbert function. Under these assumptions it follows also that $t = m(I)$.

Let $R$ be an algebra defined by a space $V$ of forms of degree $t$ whose codimension (in the space of forms of degree $t$) is $d$. In Section 4 we show that $\operatorname{rate}(R) = t - 1$ if either $d \leq 1$ or $d < \dim R_1$ and $V$ is generic.

Our approach is based on the notion of generalized Koszul filtration. It is an extension of the notion of Koszul filtration (see [CTV]) and it can be applied to the study of the rate of algebras whose defining equations have arbitrary degrees. We show that if an algebra $R$ has a generalized Koszul filtration whose ideals have generators in degree $< t$, then $\operatorname{rate}(R) \leq t - 1$.

Indeed we show that, under the above mentioned assumptions and notation, the coordinate ring $R$ of a set of points has a generalized Koszul filtration of ideals generated in degree less than $t$. This result can be seen as an extension of a theorem of Kempf [K] who proved that the coordinate ring of $s \leq 2n$ points in general linear position in $\mathbf{P}^n$ is Koszul. Under the assumption of Kempf's theorem it has been shown in [CRV] that $R$ has also a special kind of filtration, called a Gröbner flag, which implies that $R$ is defined by a Gröbner basis of quadrics.

We do not know whether there exists an analogue of the notion of Gröbner flag which works for algebras defined by polynomials of degree higher than 2. We also do not know whether the defining ideal of a set of general points in $\mathbf{P}^n$ of cardinality $\binom{n+t-1}{n} + c$ with $c < n$ has a Gröbner basis of forms of degree $t$.

## 1. Generalized Koszul filtration

The notion of Koszul filtration of a standard graded $K-$algebra $R$ has been introduced by Conca, Trung and Valla in [CTV]. It was inspired by the work of Herzog, Hibi and Restuccia [HHR] on strongly Koszul algebras. We extend this notion to be able to deal also with algebras whose defining equations have arbitrary degrees. Let $R$ be a standard graded $K$-algebra. Let $J$ be a homogeneous ideal of $R$. We will denote by $m(J)$ the highest degree of a minimal homogeneous generator of $J$.

**Definition 1.1.** Let $R$ be a standard graded $K$-algebra. A family $\mathbf{F}$ of ideals of $R$ is said to be a **generalized Koszul filtration** of $R$ if:

1) The ideal $(0)$ and the maximal homogeneous ideal $\mathcal{M}$ of $R$ belong to $\mathbf{F}$.

2) For every $J \in \mathbf{F}$ different from $(0)$ there exists $W \in \mathbf{F}$ such that $W \subset J$, $J/W$ is cyclic, $W : J \in \mathbf{F}$ and $m(W) \leq m(J)$.

One has

**Proposition 1.2.** *Let $\mathbf{F}$ be a generalized Koszul filtration of $R$ and let $d$ be an integer such that $m(J) \leq d$ for every ideal $J \in \mathbf{F}$. Then*

$$t_i^R(R/J) \leq m(J) + d(i-1)$$

*for all $i \geq 1$ and for all $J \in \mathbf{F}$. In particular, we have*

$$\operatorname{rate}(R) \leq d.$$

*Proof.* The second statement follows immediately from the first. In order to prove the first statement it is enough to show that for all $J \in \mathbf{F}$ and for all $j > m(J) + d(i-1)$ one has $\operatorname{Tor}_i^R(R/J, K)_j = 0$. Since every ideal in $\mathbf{F}$ is generated in degree $\leq d$, there are no infinite descending chains of ideals in $\mathbf{F}$. Therefore we may argue by induction on $i \geq 1$ and on $J$ (by inclusion). If $i = 1$, then $\operatorname{Tor}_1^R(R/J, K)_j = 0$ for all $j > m(J)$. If $J = (0)$, then $\operatorname{Tor}_i^R(R, K) = 0$ for all $i \geq 1$ and the assertion clearly holds.

So we may assume that $i > 1$ and $J \neq (0)$. Then there exists $W \in \mathbf{F}$ such that $W \subset J$, $J/W$ is cyclic, $W : J \in \mathbf{F}$ and $m(W) \leq m(J)$. Set $H = W : J$. Let $x \in J$ such that $J = W + (x)$, and set $c = \deg x$. Note that by the Nakayama Lemma one has $x \in J - J\mathcal{M}$. Hence

$c \leq m(J) \leq d$. By construction, $J/W \simeq R/H(-c)$. The short exact sequence

$$0 \to R/H(-c) \simeq J/W \to R/W \to R/J \to 0$$

yields the exact sequence

$$\mathrm{Tor}_i^R(R/W, K)_j \to \mathrm{Tor}_i^R(R/J, K)_j \to \mathrm{Tor}_{i-1}^R(R/H, K)_{j-c}.$$

Now $W \subset J$, so by the inductive assumption, $\mathrm{Tor}_i^R(R/W, K)_j = 0$ for every $j > m(W) + d(i-1)$ and, in particular, for $j > m(J) + d(i-1)$. Moreover $\mathrm{Tor}_{i-1}^R(R/H, K)_{j-c}$ vanishes for $j - c > m(H) + d(i-2)$. In particular, it vanishes for $j > m(J) + d(i-1)$ since $c \leq m(J)$ and $m(H) \leq d$. Now the first and the third terms of the sequence vanish for all $j > m(J) + d(i-1)$, so the middle term vanishes in the same degrees. □

**Remark 1.3.** Let $R = K[x_1, \ldots, x_n]/I$. We have already said that $\mathrm{rate}(R) \geq m(I) - 1$. We remark that if $R$ has a generalized Koszul filtration consisting of ideals $J$ generated by elements of maximal degree $d$, then by the above proposition we get $d \geq m(I) - 1$.

**Example 1.4.** An important class of rings with a generalized Koszul filtration are rings defined by monomials. Let $R = K[x_1, \ldots, x_n]/I$ where $I$ is generated by monomials. Let $a$ be an integer with $a \geq m(I)$. Consider the following family of ideals of $R$:

$$\mathbf{F} = \{ \text{ ideals generated by classes of monomials of degree } \leq a-1\}.$$

We claim that $\mathbf{F}$ is a generalized Koszul filtration of $R$. To prove the claim, let $I = (n_1, ..., n_p)$ and let $J$ be an ideal in $\mathbf{F}$ generated by the classes of the monomials, say, $m_1, ..., m_k$. Take $W \subset J$ generated by the classes of the monomials $m_1, ..., m_{k-1}$. Since $W : J$ is equal to

$$(n_i/\gcd(n_i, m_k) : i = 1, ..., p) + (m_j/\gcd(m_j, m_k) : j = 1, ..., k-1)/I,$$

it follows easily that $W : J$ is in $\mathbf{F}$.

By 1.2 one has that, for every ideal $J \in \mathbf{F}$, we have

$$t_i^R(R/J) \leq m(J) + (a-1)(i-1)$$

for all $i \geq 1$. In particular, by taking $a = m(I)$, one has

$$\mathrm{rate}(R) = m(I) - 1.$$

The results of Example 1.4 have been observed also in [ERT, Sect. 4].

## 2. The rate of some sets of points

Kempf proved in [K] that the coordinate ring $R$ of a set $X$ of $s$ points of $\mathbf{P}^n$ in linear general position is Koszul provided $s \leq 2n$. Later Conca, Trung and Valla [CTV] showed that, under the same assumption, the algebra $R$ has a Koszul filtration (i.e. a generalized Koszul filtration whose ideals are generated in degree 1).

The goal of this section is to extend the above results to a larger number of points. We will show that, under certain assumptions, the rate of a set of general points $X$ is exactly one less than the degree of the generators of the defining ideal of $X$.

We recall some notation. Let $S = K[x_0, \dots, x_n]$ be the coordinate ring of $\mathbf{P}^n$. For every set of points $X$ in $\mathbf{P}^n$ denote by $I_X$ the defining ideal of $X$, by $H_X(i)$ the Hilbert function of $S/I_X$ and by $P_X(z)$ the Hilbert series of $S/I_X$. We have:

**Theorem 2.1.** *Let $X$ be a set of points in $\mathbf{P}^n$ of cardinality $|X| = \binom{n+t-1}{n} + c$ where $t$ is an integer and $0 \leq c < n$. Assume that the Hilbert function of $S/I_X$ is maximal and that the points of $X$ are in uniform position, that is,*

$$H_Y(i) = \min\left\{|Y|, \binom{n+i}{n}\right\}$$

*for every $Y \subseteq X$ and for every $i$. Let $R$ be the coordinate ring of $X$. Then*

$$\operatorname{rate}(R) = m(I_X) - 1 = t - 1.$$

To prove the theorem we will show that $R$ has a generalized Koszul filtration of ideals generated in degree $< t$. To this end we need some notation and some preliminary results. Set $s = |X|$ and denote by $P_1, \dots, P_s \in \mathbf{P}^n$ the points in $X$. Let $\wp_1, \dots, \wp_s$ be the corresponding prime ideals of $S$. The defining ideal of $X$ is $I_X = \cap_{i=1}^s \wp_i$. Let $Z$ be the set of points $P_1, \dots, P_{c+1}$ and $Y = X \setminus Z$. Hence $X = Z \cup Y$ and

$$|Y| = \binom{n+t-1}{n} - 1, \qquad |Z| = c + 1 \leq n.$$

Under the assumption of 2.1 there exists an hyperplane $L = 0$ passing through the points of $Z$ and avoiding the points of $Y$. Similarly there exists an hypersurface $F = 0$ of degree $t-1$ passing through the points of $Y$ and avoiding the points of $Z$.

**Lemma 2.2.** *With the above notation, for every $j \geq t$ we have*

$$[I_X + (L)]_j = [I_Z]_j \quad \textit{and} \quad [I_X + (F)]_j = [I_Y]_j.$$

*Proof.* Since the inclusions $I_X + (F) \subseteq I_Y$ and $I_X + (L) \subseteq I_Z$ are obvious, it is enough to prove that $H_{S/I_X+(L)}(j) = H_{S/I_Z}(j)$ and that $H_{S/I_X+(F)}(j) = H_{S/I_Y}(j)$ for every $j \geq t$. One has the short exact sequences

$$0 \to (S/I_X : L)[-1] \to S/I_X \to S/I_X + (L) \to 0,$$

$$0 \to (S/I_X : F)[-t+1] \to S/I_X \to S/I_X + (F) \to 0.$$

Note that

$$I_X : L = (\cap_{i=1}^s \wp_i) : L = \cap_{i=1}^s (\wp_i : L) = \cap_{i=c+2}^s \wp_i = I_Y.$$

In the same way

$$I_X : F = (\cap_{i=1}^s \wp_i) : F = \cap_{i=1}^s (\wp_i : F) = \cap_{i=1}^{c+1} \wp_i = I_Z.$$

Hence:

$$H_{S/I_X+(L)}(j) = H_{S/I_X}(j) - H_{S/I_Y}(j-1) \text{ and}$$

$$H_{S/I_X+(F)}(j) = H_{S/I_X}(j) - H_{S/I_Z}(j-t+1).$$

The conclusion follows since by assumption $H_X(j) = |X|$ for $j \geq t$, $H_Y(j) = |Y|$ for $j \geq t-1$ and $H_Z(j) = |Z|$ for $j \geq 1$. □

In the following we let $T \in S_1$ be a linear form which is a non-zerodivisor on $R = S/I_X$. We denote by $x$, $y$ and $f$ the residue classes of $T$, $L$ and $F$ respectively in $R$. Since $LF \in I_X$, we have $yf = 0$ in $R$.

**Lemma 2.3.** *With the above notation we have*

$$H_{R/(x,y)}(j) = H_{R/(x,f)}(j) = 0 \quad \text{for every} \quad j \geq t.$$

*Proof.* By Lemma 2.2, we have to prove that

$$H_{S/I_Z+(T)}(j) = H_{S/I_Y+(T)}(j) = 0$$

for every $j \geq t$. By the uniform position property, we know that

$$P_Y(z) = \frac{\sum_{i=0}^{t-2} \binom{n+i-1}{i} z^i + [\binom{n+t-2}{t-1} - 1] z^{t-1}}{1-z} \quad \text{and} \quad P_Z(z) = \frac{1+cz}{1-z}.$$

Now the result follows since by assumption $T \notin \cup_{i=1}^{s} \wp_i$, and hence $T$ is also a non-zerodivisor modulo $I_Z$ and modulo $I_Y$. □

As a corollary we have:

**Corollary 2.4.** *Let $J$ be a homogeneous ideal of $R$. If $J$ contains either $x$ and $y$, or $x$ and $f$, then it is generated by forms of degree at most $t-1$.*

*Proof.* Suppose that $J$ contains $x$ and $y$ (resp. $x$ and $f$). By virtue of Lemma 2.3 the ideal $(x, y)$ (resp. $(x, f)$) contains $R_t$, hence the minimal generators of $J$ have degree $\leq t-1$. □

Now we are in the position to present the generalized Koszul filtration for $R$.

**Proposition 2.5.** *Consider the following families of homogeneous ideals of $R$:*

$\mathbf{F}_1 = \{0, (x)\}$,

$\mathbf{F}_2 = \{J$ *such that* $J$ *contains* $(x, y)\}$,

$\mathbf{F}_3 = \{J$ *such that* $J$ *is generated by* $x, f$ *and forms of degree* $t-1\}$.

*Set* $\mathbf{F} = \mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3$. *Then* $\mathbf{F}$ *is a generalized Koszul filtration of* $R$ *and every* $J$ *in* $\mathbf{F}$ *is generated in degree* $\leq t-1$.

*Proof.* The second statement follows directly from 2.4. Condition 1) of Definition 1.1 is clearly satisfied and also condition 2) for the ideal $(x)$ is trivially satisfied. So we have just to check condition 2) for the ideals in $\mathbf{F}_2$ and $\mathbf{F}_3$.

Let us do it first for an ideal $J$ in $\mathbf{F}_2$. First assume that $J = (x, y)$, then we take $W = (x) \in \mathbf{F}_1$ and we have to check that $x : y \in \mathbf{F}$. Since $yf = 0$ we have $(x, f) \subset x : y$. Thus, by virtue of Corollary 2.4, the ideal $x : y$ is generated by forms of degree at most $t-1$. Since $x$ is not a zero-divisor of degree 1 on $R$ and since $R_j = S_j$ for

$j \leq t-1$, we may conclude that

$$(x):(y) = (x, f, g_1, \ldots, g_h)$$

where $g_1, \ldots, g_h$ are elements of degree $t-1$. Hence $(x):(y)$ is in $\mathbf{F}_3$. If instead $J$ contains properly $(x, y)$, say $J = (x, y, a_1, \ldots, a_k)$, then just take $W = (x, y, a_1, \ldots, a_{k-1})$. One has that $W$ and $W : J$ are both in $\mathbf{F}_2$ since they contain $(x, y)$.

It remains to check condition 2) for an ideal $J$ in $\mathbf{F}_3$. If $J = (x, f)$, then we take $W$ to be $(x)$. Then by construction we have that $W : J = (x) : (f)$ contains $(x, y)$ and hence $W : J \in \mathbf{F}_2$. If instead $J$ contains properly $(x, f)$, say $J = (x, f, g_1, \ldots, g_k)$ with $\deg g_i = t-1$, then take $W = (x, f, g_1, \ldots, g_{k-1})$. One has that $W \in \mathbf{F}_3$ and $W : J = W : g_k = \mathcal{M}$ since $R_t \subset (x, f)$. $\square$

*Proof.* [of Theorem 2.1] As a consequence of Proposition 2.5 and Proposition 1.2, we get $\text{rate}(R) \leq t-1$. Since $m(I_X) \geq t$, we have $\text{rate}(R) = t-1 = m(I_X) - 1$. $\square$

**Remark 2.6.** The proof of the theorem holds under weaker assumptions. It is enough to assume that $H_Y(i) = \min\left\{|Y|, \binom{n+i}{n}\right\}$ for $Y = X$ and for every $Y \subset X$ with $|Y| \leq \binom{n+t-1}{n}$.

As a consequence of Theorem 2.1 one has that the ideal $I_X$ is generated in degree $t$. This has been shown already in [CaRV, Thm.2.5] and it gives a positive answer to the "Ideal generation conjecture" (see [L]) for a set $X$ of points in generic position with $|X| = \binom{n+t-1}{n} + c$ with $0 \leq c < n$.

In [CRV] it has been shown that the coordinate ring $R$ of a set of $s \leq 2n$ points in $\mathbf{P}^n$ in linear general position has a special kind of filtration, called a Gröbner flag. This implies that $R$ is defined by a Gröbner basis of quadrics. We do not know whether these results can be extended to a set of points defined by equations of higher degree. More precisely, we do not know whether there exists an analogue of the notion of Gröbner flag for algebras defined by polynomials of degree higher than 2 and we do not know whether the defining ideal of a set of general points of $\mathbf{P}^n$ of cardinality $\binom{n+t-1}{n} + c$ with $c < n$ has a Gröbner basis of forms of degree $t$.

## 3. Rate of algebras defined by spaces of forms of small codimension

Let $S = K[x_1, \ldots, x_n]$ and let $R = S/I$ be a standard graded $k$-algebra. We say that $R$ has a **$G_t$-basis** if its defining ideal has a Gröbner basis of forms of degree less than or equal to $t$ with respect to some term order and some system of coordinates. As we have already noticed in the introduction, if $R$ has a $G_t$-basis, then $\operatorname{rate}(R) \leq t-1$, but the converse does not hold (see the examples in [ERT, Sect.6]).

If $R$ has a $G_2$-basis, then $R$ is said to be G-quadratic. In [C] the first author studies the problem of whether an ideal $I$ in $S$ generated by a space of quadrics of low codimension has a Gröbner basis of quadrics. We study the corresponding problem for an ideal $I$ generated by a space of forms of degree not necessarily two. Following the approach of [C], we will first of all establish a sufficient criterion for an algebra $R$ to have a $G_t$-basis.

**Lemma 3.1.** *Let $R$ be a standard $K$-algebra and let $x$ be a non-zero linear form in $R$. Suppose that there exist two integers $t, s$, with $1 \leq s < t$, such that $x^{s+1}R_{t-s-1} = 0$ and $x^s R_{t-s} = R_t$. Then $R$ has a $G_t$-basis, and $R_i = 0$ for $i > t$.*

*Proof.* We complete $x$ to a basis of $R_1$ with $x_1, ..., x_{n-1}$, and set $x_n = x$. Let $S = K[y_1, ..., y_n]$. We consider the presentation of $R$ obtained by sending $y_i$ to $x_i$ for every $i$. Let $I$ be the presentation ideal. By assumption $I$ contains $y_n^{s+1}S_{t-s-1}$. Moreover for every $b < s$, and for every monomial $q$ of degree $t-b$ in $K[y_1, ..., y_{n-1}]$, the ideal $I$ contains a polynomial of the form $y_n^b q - y_n^s p$ where $p$ is a polynomial of degree $t-s$ in $S$.

Fix on $S$ the degree lexicographical order with $y_1 > y_2 > ... > y_n$. Then $\operatorname{in}(y_n^b q - y_n^s p) = y_n^b q$. Summing up, $\operatorname{in}(I)$ contains all the monomials of the form $y_n^b q$ with $b \neq s$ and where $q$ is a monomial of degree $t-b$ in $y_1, \ldots, y_{n-1}$.

It follows that $\operatorname{in}(I)_t S_1 = S_{t+1}$, hence $\operatorname{in}(I)$ is generated in degree less than or equal to $t$ and $R_{t+1} = 0$. □

To apply the criterion of Lemma 3.1 one has to find a non-zero linear form $x$ in $R$, with $x^t = 0$.

**Lemma 3.2.** *Let $R$ be a standard $K$-algebra over an algebraically closed field $K$, and let $t$ be a positive integer. If $V$ is a subspace* *of $R_1$ such that $\dim V > \dim R_t$, then there exists a non-zero linear form $x \in V$ such that $x^t = 0$.*

*Proof.* Let $n = \dim V$, $m = \dim R_t$, and fix bases $x_1, ..., x_n$ and $y_1, ..., y_m$ respectively of $V$ and $R_t$. Let $x$ be a non-zero element in $V$. Then $x = \sum_{i=1}^{n} a_i x_i$, with $a_i \in K$ for every $i$. It follows that we may write $x^t = \sum_{k=1}^{m} F_k(a_1, ..., a_n) y_k$, where $F_k(a_1, ..., a_n)$ is an hypersurface of degree $t$ in the $a_i$'s, for every $k$. Thus $\{x \in V, x \neq 0 \colon x^t = 0\}$ is the zero-locus of the hypersurfaces $F_1(a_1, ..., a_n), ..., F_m(a_1, ..., a_n)$, and hence it is a projective variety whose dimension is bigger than or equal to $(n-1) - m \geq 0$. □

**Proposition 3.3.** *Let $R = S/I$ be a standard $K$-algebra. Assume that $I$ is generated in degree $\leq t$, $K$ algebraically closed and $\dim R_t \leq 1$. Then $R$ has a $G_t$-basis.*

*Proof.* Set $n = \dim R_1$. We may assume $n > 1$. If $\dim R_t = 0$, then the conclusion is trivially true.

Let $\dim R_t = 1$. By Lemma 3.2, every 2-dimensional subspace of $R_1$ contains a non-zero element $x$ with $x^t = 0$. Therefore there exist $n-1$ linearly independent elements, say $x_1, ..., x_{n-1}$, in $R_1$ such that $x_i^t = 0$ for $i = 1, ..., n-1$. If for some $i$ we have $x_i R_{t-1} \neq 0$, then take $s$ the largest integer such that $x_i^s R_{t-1} \neq 0$, and since $\dim R_t = 1$, the assertion follows by Lemma 3.1. Otherwise $x_i R_{t-1} = 0$ for $i = 1, .., n-1$. If one presents $R$ as $K[y_1, ..., y_n]/I$ by sending $y_i$ to $x_i$ for every $i$, then one has that $I_t$ is generated by the component of degree $t$ of the ideal $(y_1, \ldots, y_{n-1})$. Since $I$ is generated in degree $\leq t$, it follows that $I \subseteq (y_1, ..., y_{n-1})$, and hence $R$ is not Artinian. Then $\dim R_i = 1$ for $i \geq t$. Since the component of degree $t$ of the ideal $(y_1, \ldots, y_{n-1})$ is a monomial space whose Hilbert function coincides with that of $R$ from degree $t$ on, we may conclude that $R$ has a $\mathrm{G}_t$-basis. □

In general, if we consider an algebra such that $\dim R_t < \dim R_1$, then by Lemma 3.2, there exists an element $x \in R_1 \setminus \{0\}$ such that $x^t = 0$. Moreover, by Lemma 3.1, $R$ has a $\mathrm{G}_t$-basis provided $x^{t-1} R_1 = R_t$. Since $\dim R_t < \dim R_1$ we may expect that the above condition holds for an algebra which is general enough. This is what happens in the case $t = 2$ (see [C, Sect.4]). The next result shows that the same holds for any $t$. The proof is an extension of that of [C, Sect.4], hence we give only a sketch and refer the reader to [C] for more details.

Let $m, t$ and $n$ be integers and let $S = K[x_1, \dots, x_n]$. We denote by $\mathrm{Grass}(m, S_t)$ the Grassmannian of the spaces of forms of degree $t$ and codimension $m$ in $S_t$. The set $\mathrm{Grass}(m, S_t)$ is indeed a projective variety embedded via the Plücker map in $\mathbf{P}^N$, where $N = \binom{\dim S_t}{m} - 1$. We identify $\mathrm{Grass}(m, S_t)$ with the family of the algebras of the form $S/I$ where $I$ is generated by a space of forms of degree $t$ and codimension $m$. We say that a property $\mathcal{P}$ holds for a generic algebra in $\mathrm{Grass}(m, S_t)$ if there exists a non-empty Zariski open subset $U$ of $\mathrm{Grass}(m, S_t)$ such that the algebra $R = S/(V)$ has property $\mathcal{P}$ for every $V \in U$.

**Theorem 3.4.** *Let $K$ be an algebraically closed field. If $R$ is a generic algebra in* $\mathrm{Grass}(m, S_t)$ *with $m < n = \dim R_1$, then $R$ has a $G_t$-basis.*

*Proof.* The conclusion follows if we prove that there exists a non-empty Zariski open subset of $\mathrm{Grass}(m, S_t)$ where the property "there exists an element $x \in R_1 \setminus \{0\}$ such that $x^t = 0$, and $x^{t-1}R_1 = R_t$" holds. Let $V \in \mathrm{Grass}(m, S_t)$ and define

$$X_V = \{x \in S_1 \setminus \{0\} \colon x^t \in V\}, \quad Y_V = \{x \in S_1 \setminus \{0\} \colon x^{t-1}S_1 + V \neq S_t\}.$$

We show that there exists a non-empty open subset $U$ of $\mathrm{Grass}(m, S_t)$ such that $X_V \not\subseteq Y_V$ for every $V \in U$. If $S = K[x_1, ..., x_n]$, we may write $x \in S_1$ as $\sum_{i=1}^n \alpha_i x_i$. It follows easily that $X_V$ is the zero-set of $m$ polynomials $f_1, ..., f_m$ of degree $t$ in the $\alpha_i$'s. If we give an example of $V$ such that $X_V \not\subseteq Y_V$ and $f_1, ..., f_m$ are a regular sequence in $K[\alpha_1, ..., \alpha_n]$, then we may conclude arguing as in the proof of [C, Theorem 10].

We consider the subspace $V$ of $S_t$ generated by :

$$
\begin{aligned}
&x_j x_n^{t-1} && \text{with } m < j \leq n, \\
&x_1^{i_1} \cdots x_n^{i_n} && \text{with } \textstyle\sum_{j=1}^n i_j = t, \text{ and } i_n < t-1, \\
&x_j^t - x_j x_n^{t-1} && \text{for every } 1 \leq j < n.
\end{aligned}
$$

It is easy to see that $x_n \in X_V \setminus Y_V$. In this case $f_1 = \alpha_1^t + t\alpha_1\alpha_n^{t-1}$, $\dots$ ,$f_m = \alpha_m^t + t\alpha_m\alpha_n^{t-1}$ and clearly $f_1, \dots, f_m$ are a regular sequence in $K[\alpha_1, \dots, \alpha_n]$. □

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