AN ALTERNATIVE ALGORITHM FOR COMPUTING THE BETTI TABLE OF A MONOMIAL IDEAL

MARIA-LAURA TORRENTE AND MATTEO VARBARO

ABSTRACT. In this paper we describe an alternative algorithmic procedure to compute the Betti table of a monomial ideal. We present a prototype implementation of the algorithm and we perform numerical experiments suggesting a very promising efficiency. On the way of describing the method, we also prove new constraints on the shape of the possible Betti tables of a monomial ideal.

1. INTRODUCTION

Since many years syzygies, and more generally free resolutions, are central in purely theoretical aspects of algebraic geometry; more recently, after the connection between algebra and statistics have been initiated by Diaconis and Sturmfels in [DS98], free resolutions have also become an important tool in statistics (for instance, see [D11, SW09]). As a consequence, it is fundamental to have efficient algorithms to compute them. The usual approach uses Gröbner bases and exploits a result of Schreyer (for more details see [Sc80, Sc91] or [Ei95, Chapter 15, Section 5]). The packages for free resolutions of the most used computer algebra systems, like [Macaulay2, Singular, CoCoA], are based on these techniques. However, at least when dealing with monomial ideals in a polynomial ring, there exists an alternative way to compute the minimal free resolution. The aim of this paper is to describe this method, especially focusing on the theoretical aspects, and to present the resulting algorithm. A first implementation of the algorithm has already been tested and has led to promising results. It is our opinion that, by accurately refining the implementation, it is possible to achieve substantially better performance/timings than the already good ones shown in Table 1.

Monomial ideals are, essentially, combinatorial objects, however the combinatorics of their minimal free resolutions is still mysterious. In the literature, the minimal free resolution of several classes of monomial ideals have been studied and fully understood. Many of these resolutions are *cellular*, i.e. they come from a (labeled) regular cell complex built by the generators of the monomial ideal (see [BS98] or [MS04, Chapter 4]). Due to this reason, cellular resolutions have been intensively studied in the recent literature, giving rise to a new interesting area of research. However, not every monomial ideal admits a minimal cellular resolution, though all monomial ideals admit some (not necessarily minimal) cellular resolution. In this paper, we use algebraic discrete Morse theory to algorithmically reduce a cellular resolution of a given monomial ideal to its minimal free resolution. We explicitly describe the algorithm when the starting cellular resolution is the free resolution found by Lyubeznik in [Ly88], which is actually simplicial (i.e. the regular cell complex is simplicial). Our implementation of the algorithm is provided in this case.

On the way of writing the algorithm, we prove some rigidity properties, like the following vanishing for monomial ideals generated in degrees $\leq d$:

$$\beta_{i,k}(S/I) = 0 \ \ \forall \ k = j, \ldots, j + d - 1 \quad \Longrightarrow \quad \beta_{i+1,j+d}(S/I) = 0.$$

This strengthens the recent result proved by Herzog and Srinivasan in [HS14] saying that, if $\beta_{i,k}(S/I) = 0 \ \forall \ k \geq j$, then $\beta_{i+1,k+d}(S/I) = 0 \ \forall \ k \geq j$.

Finally, a similar idea could be used to compute the simplicial homology with coefficients in a field of a finite simplicial complex (see Remark 4.5 for more details). Already many algorithms are available for this task; nevertheless, we think it would be worth to inquire on the potentiality of this idea.

2. REDUCING RESOLUTIONS

Let R be a ring, and $\mathbf{F}_\bullet = (F_i, \partial_i)_{i \in \mathbb{N}}$ a complex of free R-modules, where $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$. For any $i \in \mathbb{N}$ let Δ_i be a set to index the basis of F_i, that is:

$$F_i = \bigoplus_{\sigma \in \Delta_i} R \cdot \sigma.$$

With respect to such a basis, we can write the differentials $\partial_j : F_j \to F_{j-1}$ as:

$$1_\sigma \mapsto \sum_{\tau \in \Delta_{j-1}} [\sigma : \tau] \cdot 1_\tau,$$

where $[\sigma : \tau]$ is an element of the ring R.

We are going to describe a special case of the so-called Algebraic Discrete Morse Theory, an useful tool to reduce the size of $\mathbf{F}_\bullet$ (when possible). Let $\alpha \in \Delta_j$ and $\beta \in \Delta_{j-1}$ such that $[\alpha : \beta]$ is an *invertible element in R*. Then construct the following complex $\tilde{\mathbf{F}}_\bullet = (\tilde{F}_i, \tilde{\partial}_i)_{i \in \mathbb{N}}$ of free R-modules:

$$\tilde{F}_i = \begin{cases} \bigoplus_{\sigma \in \Delta_j \setminus \{\alpha\}} R \cdot \sigma & \text{if } i = j, \\ \bigoplus_{\tau \in \Delta_{j-1} \setminus \{\beta\}} R \cdot \tau & \text{if } i = j - 1, \\ F_i & \text{otherwise} \end{cases}$$

The differentials $\tilde{\partial}_i : \tilde{F}_i \to \tilde{F}_{i-1}$ are defined as:

$$\tilde{\partial}_i(1_\sigma) = \begin{cases} \partial_i(1_\sigma) & \text{if } i \notin \{j+1, j\}, \\ \sum_{\tau \in \Delta_j \setminus \{\alpha\}} [\sigma : \tau] \cdot 1_\tau & \text{if } i = j + 1, \\ \sum_{\tau \in \Delta_{j-1} \setminus \{\beta\}} \left([\sigma : \tau] - \frac{[\sigma : \beta][\alpha : \tau]}{[\alpha : \beta]} \right) \cdot 1_\tau & \text{if } i = j \end{cases}$$

The following lemma is a particular case of [JW09, Theorem 2.2].

Lemma 2.1. $\tilde{\mathbf{F}}_\bullet = (\tilde{F}_i, \tilde{\partial}_i)_{i \in \mathbb{N}}$ *is a complex of free R-modules homotopically equivalent to* $\mathbf{F}_\bullet = (F_i, \partial_i)_{i \in \mathbb{N}}$. *In particular,*

$$H_i(\mathbf{F}_\bullet) \cong H_i(\tilde{\mathbf{F}}_\bullet) \quad \forall \ i \in \mathbb{N}.$$

Let us remind that $\mathbf{F}_\bullet$ is a free resolution of an R-module M if $H_i(\mathbf{F}_\bullet) = 0$ for any $i > 0$ and $H_0(\mathbf{F}_\bullet) \cong M$. Therefore the above lemma implies that, if $\mathbf{F}_\bullet$ is a free resolution of M, then $\tilde{\mathbf{F}}_\bullet$ is a free resolution of M as well.

We are interested in the case that R is a polynomial ring over a field $\Bbbk$ and $\mathbf{F}_\bullet$ is a graded free resolution of a finitely generated graded R-module M. In this case, $\mathbf{F}_\bullet$ is not minimal if and only if there exist $\alpha \in \Delta_j$ and $\beta \in \Delta_{j-1}$ such that $[\alpha : \beta] \in \Bbbk^\times$. Therefore Lemma 2.1 provides an iterative procedure to get the minimal graded free resolution of M if some graded free resolution of M is known a priori.

3. Simplicial resolutions

Let $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\Bbbk$, and $I \subseteq S$ be a monomial ideal generated by monomials $\underline{u} = u_1, \ldots, u_r$. Let Δ be a (labeled) simplicial complex on the set $\{1, \ldots, r\} =: [r]$, where the label of a face $\sigma \in \Delta$ is the monomial $m_\sigma \in S$, where:

$$(1) \qquad m_\sigma := \operatorname{lcm}(u_i | i \in \sigma) \in S.$$

Let us remind that the reduced simplicial homology of Δ (with coefficients in $\Bbbk$) is the homology of the complex of $\Bbbk$-vector spaces $\mathbf{C}_\bullet(\Delta; \Bbbk) = (C_i, \partial_i)_{i=-1,\ldots,d-1}$, where $d-1$ is the dimension of Δ,

$$C_i = \bigoplus_\sigma \Bbbk \cdot \sigma$$

where the sum runs over the i-dimensional faces of Δ, and the differentials are defined as:

$$1_\sigma \mapsto \sum_{v \in \sigma} \operatorname{sign}(v, \sigma) \cdot 1_{\sigma \setminus \{v\}},$$

where

$$(2) \qquad \operatorname{sign}(v, \sigma) := (-1)^{q-1} \in \Bbbk$$

if v is the qth element of σ. By imitating this construction, let us consider the graded complex of free S-modules $\mathbf{F}_\bullet(\Delta; \underline{u}) = (F_i, \partial_i)_{i=0,\ldots,d}$, where

$$F_i = \bigoplus_\sigma S(-\deg(m_\sigma))$$

where the sum runs over the $(i-1)$-dimensional faces of Δ, and the differentials are defined as:

$$1_\sigma \mapsto \sum_{v \in \sigma} \operatorname{sign}(v, \sigma) \frac{m_\sigma}{m_{\sigma \setminus \{v\}}} \cdot 1_{\sigma \setminus \{v\}}.$$

Since any $v \in [r]$ is a vertex of Δ, one has that $H_0(\mathbf{F}_\bullet(\Delta; \underline{u})) = S/I$. Furthermore, because the differential are graded of degree 0, one can check that $\mathbf{F}_\bullet(\Delta; \underline{u})$ is a graded free resolution of S/I if and only if $\Delta_{\leq u}$ is an acyclic simplicial complex for any monomial $u \in S$, where $\Delta_{\leq u}$ consists in the faces $\sigma \in \Delta$ such that $m_\sigma | u$ (cf. [BPS98, Lemma 2.2]). We will call this kind of resolutions, introduced for the first time by Bayer, Peeva and Sturmfels in [BPS98], *simplicial*. In particular, if $\Delta = \mathcal{K}$ is the $(r-1)$-simplex, then $\mathbf{F}_\bullet(\mathcal{K}; \underline{u})$ is a graded free resolution of S/I: indeed in this case $\Delta_{\leq u}$ is the simplex on the vertices i such that $u_i | u$,

which is contractible. That $\mathbf{F}_\bullet(\mathcal{K}; \underline{u})$ was a resolution was proved, for the first time, by Taylor in [Ta60].

Definition 3.1. With the above notation, $\mathbf{F}_\bullet(\mathcal{K}; \underline{u})$ is called the *Taylor resolution.*

The Taylor resolution is far to be minimal in general: in fact, it tends to be huge. A smaller graded free resolution, still simplicial, was found by Lyubeznik in [Ly88]. For each $k \in [r]$ and $\sigma \subseteq [r]$, denote by

$$(3) \qquad\qquad \sigma_{>k} := \{i \in \sigma : i > k\}.$$

Let $\mathcal{L}$ be the simplicial complex on $[r]$ consisting of those faces $\sigma \subseteq [r]$ such that u_k does not divide $m_{\sigma_{>k}}$ for any $k \in [r]$. Then Lyubeznik proved in [Ly88] that $\mathbf{F}_\bullet(\mathcal{L}; \underline{u})$ is a graded free resolution of S/I.

Definition 3.2. With the above notation, $\mathbf{F}_\bullet(\mathcal{L}; \underline{u})$ is called the *Lyubeznik resolution.*

Notice that the Lyubeznik resolution depends on the way in which we order the monomials $u_1, \ldots, u_r$. Even if the Lyubeznik resolution tends to be much smaller than the Taylor resolution, it may happen (indeed it often happens) that it is not minimal for any order of the generators of I.

4. THE ALGORITHM

Let $I \subseteq S = \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal generated by monomials of positive degree $u_1, \ldots, u_r$ ($r > 0$). If $\sigma = \{i_1, \ldots, i_s\}$ and $\tau = \{j_1, \ldots, j_s\}$ are subsets of $[r]$ of the same cardinality, by $\sigma < \tau$ we mean that σ is less than τ lexicographically, i.e. that there is k such that $i_k < j_k$ and $i_p = j_p$ for all $p < k$. For any $i \in [r]$, define the following set:

$$(4) \qquad\qquad \mathrm{Lyu}_i := \{\sigma \subseteq [r] : |\sigma| = i \text{ and } \exists\, k \in [r] : u_k | m_{\sigma_{>k}}\},$$

where $\sigma_{>k}$ was defined in (3). Construct the graph $G(I)$ as follows:

(i) The set of vertices of $G(I)$ is $\Delta_0 \cup \Delta_1$ where

$$\Delta_i := \{\sigma \subseteq [r] : |\sigma| = i,\ \sigma \notin \mathrm{Lyu}_i\} \quad \forall\, i = 0, 1.$$

(ii) There are no edges.

The purpose of the following algorithm is to iteratively transform the weighted graph $G(I)$ in a way that will provide the Betti table of S/I (see sections 2 and 3). During the algorithm, the vertices of $G(I)$ will be always partitioned in 2 sets, namely Δ_i and Δ_{i+1}. The main loop of the algorithm works as follows. If either Δ_i or Δ_{i+1} is empty, the algorithm stops. Otherwise, the set of vertices Δ_i is stored and deleted, and a new set of vertices Δ_{i+2} is constructed together with a set of weighted edges Δ_{i+1}^{i+2}, going from Δ_{i+2} to Δ_{i+1}. Both sets Δ_{i+2} and Δ_{i+1}^{i+2} continually change, and eventually the edges of Δ_{i+1}^{i+2} will have weight 0. At the end of each loop, the counter i is increased by one. In the worst case, when $i = \min\{n, r\}$, the set Δ_{i+1} is empty, so the algorithm stops.

Remark 4.1. With the above notation, the following inequalities follow by the Taylor's resolution:

$$\text{projdim}(S/I) \leq \min\{r, n\} \quad \text{and} \quad \text{reg}(S/I) \leq \text{projdim}(S/I) \cdot \max_i \{\deg(u_i) - 1\}$$

In particular, the Betti table of S/I has size at most $a(I) \times b(I)$, where $a(I) := \min\{r, n\} + 1$ and $b(I) := \min\{r, n\} \cdot \max_i \{\deg(u_i) - 1\} + 1$.

Algorithm 4.2. Let $\Delta_0, \Delta_1, a(I), b(I)$ be defined as above; set $i := 0$, $V_1 := \Delta_1$, and let $\mathbb{B}$ be the $(a(I) \times b(I))$-matrix with all zero entries.

S1: Let B_i be the $(a(I) \times b(I))$-matrix whose entries are all zero except for the $(i+1)$-th column: the $(j, i+1)$-th entry will be

$$|\{\sigma \in \Delta_i : \deg(m_\sigma) = i + j - 1\}|.$$

Analogously, let B_{i+1} be the $(a(I) \times b(I))$-matrix whose entries are all zero except for the $(i+2)$-th column: the $(j, i+2)$-th entry will be

$$|\{\sigma \in \Delta_{i+1} : \deg(m_\sigma) = i + j\}|.$$

S2: If $i = r - 1$ then set $\mathbb{B} := \mathbb{B} + B_i + B_{i+1}$ and stop;
else if $i = n$ then set $\mathbb{B} := \mathbb{B} + B_i$ and stop;
else if $\Delta_i = \emptyset$ then stop;
else if $\Delta_{i+1} = \emptyset$ then set $\mathbb{B} := \mathbb{B} + B_i$ and stop;
else set $\mathbb{B} := \mathbb{B} + B_i$ and remove the set of vertices Δ_i from $G(I)$. Change the weighted graph $G(I)$ as described in the procedure **The deformation of $G(I)$** below, set $i := i + 1$ and go to step **S1**.

The deformation of $G(I)$

Set $V_{i+2} := \{\tau \cup \{v\} : \tau \in V_{i+1}, v \in [r] \setminus \tau\}$, $\Delta_{i+2} := \emptyset$ and $\Delta_{i+1}^{i+2} := \emptyset$. Iteratively transform Δ_{i+2} and Δ_{i+1}^{i+2} by lexicographically running on the elements $\sigma \in V_{i+2}$ in increasing order as follows:

- If $\sigma \in \text{Lyu}_{i+2}$, then $V_{i+2} := V_{i+2} \setminus \{\sigma\}$ and go to the next σ.
- Otherwise, construct the following sets:

$$(\Delta_{i+1}^\sigma)_1 := \{\sigma \setminus \{v\} \in \Delta_{i+1} : v \in \sigma, m_{\sigma \setminus \{v\}} = m_\sigma \text{ and } (\sigma, \sigma \setminus \{v\}) \notin \Delta_{i+1}^{i+2}\}$$
$$(\Delta_{i+1}^\sigma)_2 := \{\beta \in \Delta_{i+1} : (\sigma, \beta) \in \Delta_{i+1}^{i+2}\}$$
$$(\Delta_{i+1}^\sigma)_3 := \{\beta \in (\Delta_{i+1}^\sigma)_2 : [\sigma : \beta] \neq 0\}$$

- If $\Delta_{i+1}^\sigma := (\Delta_{i+1}^\sigma)_1 \cup (\Delta_{i+1}^\sigma)_3 = \emptyset$, then set $\Delta_{i+2} := \Delta_{i+2} \cup \{\sigma\}$ and go to the next σ.
- Otherwise, for every $\beta \in (\Delta_{i+1}^\sigma)_1$, set $[\sigma : \beta] := \text{sign}(\sigma \setminus \beta, \beta)$ (see (2)).
Let τ be the smallest element of Δ_{i+1}^σ. Reset $\Delta_{i+1}^\sigma := \Delta_{i+1}^\sigma \setminus \{\tau\}$.
Set $\Delta_\tau^{i+2} := (\Delta_\tau^{i+2})_1 \cup (\Delta_\tau^{i+2})_3$, where:

$$(\Delta_\tau^{i+2})_1 := \{\alpha = \tau \cup \{v\} : v \in [r] \setminus \tau, \alpha > \sigma, u_v | m_\sigma, \alpha \notin \text{Lyu}_{i+2}\}$$
$$(\Delta_\tau^{i+2})_2 := \{\alpha \in \Delta_{i+2} : \alpha > \sigma, (\alpha, \tau) \in \Delta_{i+1}^{i+2}\}$$
$$(\Delta_\tau^{i+2})_3 := \{\alpha \in (\Delta_\tau^{i+2})_2 : [\alpha, \tau] \neq 0\}$$

If $\alpha \in (\Delta_\tau^{i+2})_1 \setminus (\Delta_\tau^{i+2})_2$, define $[\alpha : \tau] := \text{sign}(\alpha \setminus \tau, \alpha)$.

Set $\Delta_{i+2} := \Delta_{i+2} \cup (\Delta_\tau^{i+2})_1$ and, for all $(\alpha, \beta) \in \Delta_\tau^{i+2} \times \Delta_{i+1}^\sigma$, procede as follows:

    – If $(\alpha, \beta) \in \Delta_{i+1}^{i+2}$, then $[\alpha, \beta] := [\alpha : \beta] - \frac{[\alpha:\tau]\cdot[\sigma:\beta]}{[\sigma:\tau]}$.

    – If $(\alpha, \beta) \notin \Delta_{i+1}^{i+2}$, then $\Delta_{i+1}^{i+2} := \Delta_{i+1}^{i+2} \cup \{(\alpha, \beta)\}$ and:

        * If $\alpha \supseteq \beta$, then $[\alpha, \beta] := \text{sign}(\alpha \setminus \beta, \alpha) - \frac{[\alpha:\tau]\cdot[\sigma:\beta]}{[\sigma:\tau]}$.

        * Otherwise, $[\alpha, \beta] := - \frac{[\alpha:\tau]\cdot[\sigma:\beta]}{[\sigma:\tau]}$.

Set $\Delta_{i+2} := \Delta_{i+2} \setminus (\{\sigma\} \cap \Delta_{i+2})$ and $\Delta_{i+1} := \Delta_{i+1} \setminus \{\tau\}$. Put

$$\Delta_{i+1}^{i+2} := \Delta_{i+1}^{i+2} \setminus (\{(\alpha, \tau) : \alpha \in (\Delta_\tau^{i+2})_2\} \cup \{(\sigma, \beta) : \beta \in (\Delta_{i+1}^\sigma)_2\})$$

and go to the next σ.

Remark 4.3. Note that the above algorithm can be modified to provide the multigraded Betti numbers of S/I: namely, given a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$,

$$\beta_{i,\boldsymbol{a}}(S/I) = |\{\sigma \in \Delta_i : m_\sigma = x_1^{a_1} \cdots x_n^{a_n}\}| \quad \forall \, i \in \mathbb{N},$$

where Δ_i is taken at the step i.

From the algorithm described above one can immediately infer some interesting upper/lower bounds for the depth/Castelnuovo-Mumford regularity of S/I. With the notation above, let $I = (u_1, \ldots, u_r) \subseteq S = \mathbb{k}[x_1, \ldots, x_n]$. We say that a subset $\sigma \subseteq [r]$ is *critical (for I)* if:

$$m_\sigma \neq m_{\sigma \setminus \{v\}} \ \forall \, v \in \sigma \quad \text{and} \quad m_\sigma \neq m_{\sigma \cup \{v\}} \ \forall \, v \in [r] \setminus \sigma.$$

We introduce the following two invariants of a monomial ideal I:

(5) $$p(S/I) := \max\{|\sigma| : \sigma \text{ is critical}\}$$

(6) $$r(S/I) := \max\{\deg(m_\sigma) - |\sigma| : \sigma \text{ is critical}\}$$

Proposition 4.4. *If σ is a critical set for I, then $\beta_{|\sigma|,\deg(m_\sigma)}(S/I) \neq 0$. In particular:*

$$\text{depth}(S/I) \leq n - p(S/I) \quad \text{and} \quad \text{reg}(S/I) \geq r(S/I).$$

As a special case, we recover a result of Katzman [Ka06]: let G be a simple graph of n vertices and $I = (x_i x_j : \{i, j\} \text{ is an edge of } G)$. He noticed that, if G has an induced subgraph H consisting of t disjoint edges, then $\text{reg}(S/I) \geq t$. Clearly, in such a situation the set σ corresponding to the edges in H is critical for I, so that $r(S/I) \geq \deg(m_\sigma) - |\sigma| = t$.

Remark 4.5. To compute the multigraded Betti numbers of a monomial ideal, we can always reduce ourselves to square-free monomial ideals by polarization. A square-free monomial ideal $I = (u_1, \ldots, u_r) \subseteq S$ is associated to a simplicial complex Δ on n vertices via the Stanley-Reisner correspondence. Thanks to the Hochster's formula (cf. [MS04, Corollary 5.12]), to know the multigraded Betti numbers of S/I is equivalent to know the simplicial homology with coefficients in $\mathbb{k}$ of all the 2^n induced subcomplexes of Δ.

If one is only interested in the simplicial homology of Δ, it is enough to know the multigraded Betti numbers $\beta_{i,(1,\ldots,1)}(S/I)$, more precisely:

$$\dim_{\mathbb{k}} \widetilde{H}_i(\Delta; \mathbb{k}) = \beta_{n-i-1,(1,\ldots,1)}(S/I)$$

Being our algorithm "local", to know such multigraded Betti numbers is not necessary to compute the minimal free resolution of S/I completely. In fact, it is possible to implement a faster version of the algorithm aimed at the computation of the simplicial homology with coefficients in a field. Already several algorithms are available for this task, with applications, for example, in network theory. Nevertheless, we wish to study the potentiality of such an algorithm in a forthcoming paper.

To give a preview of how such an algorithm would work, let $\{G_1, \ldots, G_r\}$ be the set of minimal nonfaces of Δ. Essentially, one should perform the "deformation of $G(I)$" only on the sets of vertices

$$V_i = \{\sigma \subseteq [r] \setminus \mathrm{Lyu}_i : |\sigma| = i \text{ and } \bigcup_{j \in \sigma} G_j = [n]\} \quad i = n - \dim \Delta, \ldots, n.$$

Then $\dim_{\Bbbk} \widetilde{H}_i(\Delta; \Bbbk)$ will be the cardinality of the set Δ_{n-i-1} (with the same notation of the algorithm) at the end of the deformation. A remarkable feature is that to compute the simplicial homology in this way, one needs as input the minimal nonfaces of the simplicial complex, rather than its facets. This fact is convenient in certain situations. For example, in many cases is of interest to compute the homology of the clique complex of a given graph: while the facets of such a simplicial complex are the maximal cliques of the graph, its minimal nonfaces are simply the pair of nonadjacent vertices, which are easier to store if only the graph is given (as it typically happens in network theory).

5. Experiments

In this section we present some examples to show the effectiveness of our algorithm. We implemented it using the C++ language and some routines available in CoCoALib, a GPL C++ library which is the mathematical kernel for the computer algebra system CoCoA-5 [CoCoA]. All computations are performed on an Intel Core 2 i5 processor (at 1.4 GHz) using both our new algorithm and the function *BettiDiagram* included in CoCoA-4. We are not comparing our timings with CoCoA-5 itself because the implementation for minimal free resolutions has not yet been optimized in the current version CoCoA-5.1.1. We perform three experimental tests: in the first case we consider monomial ideals of degree $d = 2$ (results summarized in Table 1.(a)); in the second case we consider monomial ideals of degree $d = 4$ (results summarized in Table 1.(b)); in the third case we consider monomial ideals of degree d, with $5 \leq d \leq 8$ (results summarized in Table 1.(c)).

Tables 1.(a), 1.(b) and 1.(c) consist of four columns: the first column, labeled with n, contains the number of variables of the polynomial ring; the second column, labeled with r, contains the number of minimal generators of the monomial ideal; the last two columns, labeled with "New" and "CoCoA", contain the average time to compute the Betti tables of 10 monomial ideals using our new algorithm and the software CoCoA-4 respectively. In all tested numerical experiments our algorithm reveals to be more efficient than CoCoA-4, since it returns the Betti tables in a shorter computational time. We also performed some comparisons of our results with computations done using the software Macaulay-2 (see [Macaulay2]),

---

[1]this average is made on 7 tests out of 10, since in 3 experiments the computation ran out of memory

(a) Monomial ideals of degree $d = 2$.

		Timings	
n	r	New	CoCoA
10	8	0.001	0.02
10	10	0.005	0.04
10	12	0.014	0.11
15	12	0.023	0.19
15	15	0.177	1.32
15	18	0.512	2.33
18	18	2.111	13.14
18	20	9.608	32.87

(b) Monomial ideals of degree $d = 4$.

		Timings	
n	r	New	CoCoA
10	8	0.002	0.02
10	10	0.004	0.05
10	12	0.012	0.11
15	12	0.029	0.43
15	15	0.208	2.91
15	18	0.786	9.37
18	18	3.282	78.88
18	20	24.194	158.27

(c) Monomial ideals of degree d, with $5 \leq d \leq 8$.

		Timings	
n	r	New	CoCoA
10	8	0.002	0.04
10	10	0.007	0.09
10	12	0.014	0.23
15	12	0.035	0.81
15	15	0.261	7.51
15	18	1.247	58.84
18	18	4.064	271.5
18	20	63.00	482.73[1]

TABLE 1. Tests on monomial ideals of degree $d = 2$, $d = 4$ and $5 \leq d \leq 8$

in which case the computational timings are comparable. In our opinion, though, by accurately refining the implementation, the efficiency of our algorithm has substantial room for improvement.

6. A VANISHING INSPIRED BY THE ALGORITHM

In the last section of the paper we prove a vanishing result arisen while developing the algorithm. Before stating it, for the convenience of the reader we recall how to get the *Mayer-Victories spectral sequence* for simplicial homology:

Remark 6.1. Let Δ be a simplicial complex on $[n]$, and $\Delta_0, \ldots, \Delta_s$ be subcomplexes such that

$$\Delta = \bigcup_{i=0}^{s} \Delta_i.$$

For any sequence of integers $0 \leq a_0 < \ldots < a_p \leq s$, let us denote by

$$\Delta_{a_0,\ldots,a_p} = \bigcap_{k=0}^{p} \Delta_{a_k}.$$

Fixed a ring R, for any simplicial complex Γ we denote by $\mathbf{C}_\bullet(\Gamma; R)$ the complex of R-modules defined as in Section 3 in the particular case that R was the field $\Bbbk$, and by $\mathbf{C}^\bullet(\Gamma; R) :=$ $\mathrm{Hom}_R(\mathbf{C}_\bullet(\Gamma; R))$. One can check that the following is an exact sequence of complexes of R-modules:

$$0 \to \mathbf{C}^\bullet(\Delta; R) \xrightarrow{d^0} \mathcal{C}^{\bullet,0}(R) \xrightarrow{d^1} \mathcal{C}^{\bullet,1}(R) \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} \mathcal{C}^{\bullet,s-1}(R) \xrightarrow{d^s} \mathcal{C}^{\bullet,s}(R) \to 0,$$

where $\mathcal{C}^{\bullet,p}(R) = \bigoplus_{a_0 < \ldots < a_p} \mathbf{C}^\bullet(\Delta_{a_0,\ldots,a_p}; R)$ and, for a given j, if $\alpha = (\alpha_{a_0,\ldots,a_p})_{a_0 < \ldots < a_p}$ is an element of $\mathcal{C}^{j,p}(R)$, the $(b_0, \ldots, b_{p+1})$th component of $d^{p+1}(\alpha) \in \mathcal{C}^{j,p+1}(R)$ is:

$$\sum_{k=0}^{p+1} (-1)^k \left(\alpha_{b_0,\ldots,\widehat{b_k},\ldots,b_{p+1}} \right) \Big|_{C^j(\Delta_{b_0,\ldots,b_{p+1}})}$$

(considering Δ as $\Delta_\emptyset$). Let us consider the double complex of R-modules $\mathcal{C}^{\bullet,\bullet}(R)$ where the differentials

$$\partial^p : \mathcal{C}^{p-1,q}(R) \to \mathcal{C}^{p,q}(R),$$

for a given natural number q, are the ones used to define the simplicial cohomology of $\Delta_{a_0,\ldots,a_q}$ with coefficients in R. By the general theory of spectral sequences arising from double complexes (c.f. [GM03, Chapter III.7, Proposition 10]), we get the following spectral sequence:

$$\tag{7} \bigoplus_{0 \leq a_0 < \ldots < a_p \leq s} H^q\left(\Delta_{a_0\ldots,a_p}; R\right) \implies H^{p+q}(\Delta; R).$$

Given two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of $\mathbb{Z}^n$, for $\boldsymbol{a} \leq \boldsymbol{b}$ we mean $a_i \leq b_i$ for all $i \in [n]$. Furthermore, for $\boldsymbol{a} < \boldsymbol{b}$ we mean $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a} \neq \boldsymbol{b}$.

Theorem 6.2. *Let $I = (u_1, \ldots, u_r)$ be a monomial ideal of $S = \Bbbk[x_1, \ldots, x_n]$, and let $\boldsymbol{a_\ell} = \deg_{\mathbb{Z}^n}(u_\ell)$ for all $\ell \in [r]$. For any $\ell \in [r]$, $i \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{Z}^n$:*

$$\beta_{i,\boldsymbol{b}}(S/I) = 0 \ \ \forall \ \boldsymbol{a} \leq \boldsymbol{b} < \boldsymbol{a} + \boldsymbol{a_\ell} \quad \implies \quad \beta_{i+1,\boldsymbol{a}+\boldsymbol{a_\ell}}(S/I) = 0.$$

In particular, if $d = \max_\ell\{\deg_{\mathbb{Z}}(u_\ell)\}$, for any $j \in \mathbb{Z}$:

$$\beta_{i,k}(S/I) = 0 \ \ \forall \ k = j, \ldots, j+d-1 \quad \implies \quad \beta_{i+1,j+d}(S/I) = 0.$$

Proof. It is not difficult to see that, by polarizing, one can reduce himself to the square-free case (though one has to be a bit careful because by polarizing the multi-degree changes). In such a situation, for any $s \in \mathbb{N}$, $\beta_{s,\boldsymbol{c}}(S/I) \neq 0$ only if $\boldsymbol{c} = (c_1, \ldots, c_n)$ is a square-free vector, that is $c_j \in \{0,1\}$ for any $j \in [n]$. It is convenient to think at such square-free vectors as subsets of $[n]$, namely

$$\boldsymbol{c} \longleftrightarrow \sigma(\boldsymbol{c}) = \{j \in [n] : c_j = 1\}.$$

Notice that $\boldsymbol{c_1} \leq \boldsymbol{c_2}$ if and only if $\sigma(\boldsymbol{c_1}) \subseteq \sigma(\boldsymbol{c_2})$. Let α, α_ℓ and $\gamma = \alpha \cup \alpha_\ell$ be the subsets of $[n]$ corresponding to, respectively, $\boldsymbol{a}$, $\boldsymbol{a_\ell}$ and $(\boldsymbol{a} + \boldsymbol{a_\ell})(\mathrm{mod}2)$. Furthermore, let Δ be the simplicial complex such that $I = I_\Delta$ and set $\Gamma = \Delta_\gamma$. Notice that our vanishing hypothesis can be written as:

$$\beta_{i,\beta}(S/I) = 0 \ \ \forall \ \alpha \subseteq \beta \subsetneq \gamma.$$

By Hochster's formula, this is equivalent to say:

$$\tag{8} H^{|\beta|-i-1}(\Gamma_\beta; \Bbbk) = 0 \ \ \forall \ \alpha \subseteq \beta \subsetneq \gamma.$$

Since α_ℓ is not a face of Δ,

$$\bigcup_{v \in \alpha_\ell} \Gamma_{\gamma \setminus \{v\}} = \Gamma.$$

Obviously, for all subsets δ of α_ℓ:

$$\bigcap_{v \in \delta} \Gamma_{\gamma \setminus \{v\}} = \Gamma_{\gamma \setminus \delta}.$$

Therefore the Mayer-Vietoris spectral sequence (7) with respect to the covering $(\Gamma_{\gamma \setminus \{v\}})_{v \in \alpha_\ell}$ of Γ is:

$$\bigoplus_{\substack{\delta \subseteq \alpha_\ell \\ |\delta| = p+1}} H^q(\Gamma_{\gamma \setminus \delta}; \Bbbk) \implies H^{p+q}(\Gamma; \Bbbk).$$

So, by (8) we infer that $H^{|\gamma|-i-2}(\Gamma; \Bbbk) = 0$. We conclude by using Hochster's formula that

$$\beta_{i+1,\gamma}(S/I) = 0.$$

$\square$

Remark 6.3. As a particular case of the above theorem, we get:

$$\beta_{i,k}(S/I) = 0 \;\; \forall \; k \geq j \quad \implies \quad \beta_{i+1,k+d}(S/I) = 0 \;\; \forall \; k \geq j.$$

This fact has recently been proved by Herzog and Srinivasan in [HS14, Corollary 4]. A further consequence is the following result of Fernández-Ramos and Gimenez [FG14, Theorem 2.1]: if I is the edge ideal of a graph, then $\beta_{i,j}(S/I) = \beta_{i,j+1}(S/I) = 0 \implies \beta_{i+1,j+2}(S/I) = 0$.

References

[BPS98]   D. Bayer, I. Peeva, B. Sturmfels. *Monomial resolutions.* Mathematical Research Letters, Vol. 5 (1998), pp. 31-46.

[BS98]   D. Bayer, B. Sturmfels, *Cellular resolutions of monomial modules*, J. Reine Angew. Math. 502 (1998), 123-140.

[CoCoA]   CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*, available at `http://cocoa.dima.unige.it`.

[Singular] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular: A computer algebra system for polynomials computations*, available at `http://www.singular.uni-kl.de`.

[DS98]   P. Diaconis, B. Sturmfels, *Algebraic algorithms for sampling from conditional distributions*, Ann. Statist. 26 (1998), no. 1, 363-397.

[D11]   I. H. Dinwoodie, *Syzygies for Metropolis base chains*, Linear Algebra and its Applications, 434 (2011) 2176-2186.

[Ei95]   D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150, Springer-Verlag, New York (1995).

[FG14]   O. Fernández-Ramos, P. Gimenez, *Regularity 3 in edge ideals associated to bipartite graphs*, J. Algebraic Combin. 39, no. 4, 919-937 (2014).

[GM03]   S. Gelfand, YU. Manin, *Methods of Homological Algebra*, Springer Monographs in Mathematics (second edition), Springer-Verlag Berlin Heidelberg (2003).

[Macaulay2] D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`.

[JW09]   M. Jöllenbeck, V. Welker, *Minimal Resolutions Via Algebraic Discrete Morse Theory*, Memoirs of the American Mathematical Society, Vol. 197, Nr. 923 (2009).

[HS14]   J. Herzog, H. Srinivasan, *A note on the subadditivity problem for maximal shifts in free resolutions*, to appear in MSRI Proceedings.

[Ka06]   M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A 113, n. 3, pp. 435-454 (2006).

[Ly88]   G. Lyubeznik, *A new explicit free resolution of ideals generated by monomials in an R-sequence*, Journal of Pure and Applied Algebra, Vol. 51 (1988), 193-195.

[MS04]   E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, Springer, Vol. 227 (2004).

[SW09]   E. Sáenz-de-Cabezón, H. Wynn, *Betti numbers and minimal free resolutions for multi-state system reliability bounds*, Journal of Symbolic Computation, Vol. 44, no. 9 (2009), 1311-1325.

[Sc80]   F.-O. Schreyer, *Die Berechnung von Syzygien mit dem verallgemeinerten Weierstrassschen Divisionssatz.* Diplomarbeit, Hamburg (1980).

[Sc91]   F.-O. Schreyer, *A standard basis approach to syzygies of canonical curves*, J. reine angew. Math. 421, 83-123 (1991).

[Ta60]   D. Taylor, *Ideals generated by monomials in an R-sequence*, PhD thesis, University of Chicago (1960).

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, Genova 16146, Italy
E-mail address: `torrente@dima.unige.it`

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, Genova 16146, Italy
E-mail address: `varbaro@dima.unige.it`