Regularity of line configurations

Bruno Benedetti [1]
Dept. of Mathematics
University of Miami
bruno@math.miami.edu

Michela Di Marca [2]
Dip. di Matematica
Università di Genova
dimarca@dima.unige.it

Matteo Varbaro [3]
Dip. di Matematica
Università di Genova
varbaro@dima.unige.it

August 6, 2016

Dedicated to the memory of Tony Geramita

Abstract

We show that in arithmetically-Gorenstein line arrangements with only planar singularities, each line intersects the same number of other lines. This number has an algebraic interpretation: it is the Castelnuovo–Mumford regularity of the coordinate ring of the arrangement.

We also prove that every $(d-1)$-dimensional simplicial complex whose 0-th and 1-st homologies are trivial is the nerve complex of a suitable d-dimensional standard graded algebra of depth ≥ 3. This provides the converse of a recent result by Katzman, Lyubeznik and Zhang.

Introduction

The study of lines on smooth surfaces of $\mathbb{P}^3$ has a fascinating history. Lines on smooth cubics were investigated in the Nineteenth century by Cayley [Ca1849], Salmon [Sa1849], and Clebsch [Cl1861], among others. Every smooth cubic contains exactly 27 lines, whose pattern of intersection is independent of the chosen cubic. In 1910 Schoute showed that the 27 lines can be put into a one-to-one correspondence with the vertices of a 6-dimensional polytope, so that all incidence relations between the lines are mirrored in the combinatorial structure of the polytope [Sch1910] [DuV1933].

The situation changes drastically for surfaces of larger degree. In fact, the generic surface of degree $d \geq 4$ contains no line at all. However, special surfaces do contain line arrangements. An example of a smooth quartic containing 64 lines was found in 1882 by Schur [Sc1882]:

$$x_0^4 + x_0x_1^3 = x_2^4 + x_2x_3^3.$$

[1]Supported by NSF Grant 1600741, 'Geometric Combinatorics and Discrete Morse Theory'
[2]Supported by PRIN 2010S47ARA 003, 'Geometria delle Varietà Algebriche.'
[3]Supported by PRIN 2010S47ARA 003, 'Geometria delle Varietà Algebriche.'

In 1943 B. Segre [Se1943] claimed that a smooth quartic may contain at most 64 lines, which is precisely the number achieved by Schur's quartic. Segre's proof, however, was based on the erroneous assumption that a line on a smooth quartic can meet at most 18 other lines on it. In 2015 the work of Rams and Schütt [RS2015] exhibited smooth quartics in which some line actually meets 20 other lines. Using a deeper argument, however, Rams and Schütt were able to salvage Segre's conclusion that indeed no smooth quartic contains more lines than Schur's.

We are naturally intrigued by the many symmetries that these line configurations on smooth surfaces seem to enjoy. For example:

(i) each line on a smooth cubic meets exactly 10 of the others;
(ii) in Schläfli's "double-six" configuration of 12 lines on a smooth cubic, each line meets exactly 5 of the others;
(iii) each line on the degree-d Fermat surface $x_0^d + x_1^d + x_2^d + x_3^d = 0$ (which contains $3d^2$ lines) meets exactly $4d - 2$ of the others.

Where does this regularity come from? Schoute's "polytopal bijection" offers an explanation only for cubic surfaces. The main result of the present paper provides a general, two-line answer to this question.

Theorem I (Theorem 1.3)**.** *In any arithmetically-Gorenstein line arrangement $X \subseteq \mathbb{P}^n$ where all singularities are planar, each line intersects exactly* $\operatorname{reg} X - 1$ *of the other lines.*

The planarity of all singularities is necessary: Without it, we can only claim that each line intersects *at least* $\operatorname{reg} X - 1$ of the other lines, cf. [BV2015, Theorem 3.8]. But curves on a smooth surface have by definition only planar singularities. In particular, if a line arrangement $X \subseteq \mathbb{P}^3$ is the complete intersection of a smooth surface of degree d and another surface (not necessarily smoooth) of degree e, Theorem I reveals that each line of the arrangement must intersect exactly $d + e - 2$ of the other lines. This explains cases (i), (ii) and (iii) above.

From Theorem I one can easily infer the following:

Corollary II (Corollary 1.4)**.** Let X be a smooth surface, H a very ample divisor, $X \subseteq \mathbb{P}^n$ the embedding given by the complete linear system $|H|$. Let $D_1, \ldots, D_s$ be lines on $X \subseteq \mathbb{P}^n$ such that $D_1 + \ldots + D_s \sim dH$ for some $d \in \mathbb{Z}_{>0}$.

(i) If $n = 3$, then $|\{j \neq i : D_j \cap D_i \neq \emptyset\}| = \deg H + d - 2$ for each $i = 1, \ldots, s$.
(ii) If X is a $K3$ surface, then $|\{j \neq i : D_j \cap D_i \neq \emptyset\}| = d + 2$ for each $i = 1, \ldots, s$.

The second section of the present paper is devoted to the related problem of understanding the geometry of a simplicial complex that is attached to any standard graded $\mathbb{C}$-algebra R, called the *Lyubeznik complex* or the *nerve complex* of R, and denoted by $\mathfrak{L}(R)$. A recent result by Katzman, Lyubeznik and Zhang [KLZ2015] states that when $\operatorname{depth} R \geq 3$ one has $\widetilde{H}_0(\mathfrak{L}(R); \mathbb{C}) = \widetilde{H}_1(\mathfrak{L}(R); \mathbb{C}) = 0$. We show the converse:

Theorem III (Theorem 2.4)**.** *For any $(d-1)$-dimensional simplicial complex Δ with $\widetilde{H}_0(\Delta; \mathbb{C}) = \widetilde{H}_1(\Delta; \mathbb{C}) = 0$, there exists a d-dimensional standard graded $\mathbb{C}$-algebra R such that* $\operatorname{depth} R \geq 3$ *and $\mathfrak{L}(R) = \Delta$.*

Our construction is compatible with the dual graph notion, in the sense that the dual graph of R is simply the 1-skeleton of Δ.

Notation

Let K be a field, and X a projective scheme over K. Fix a closed embedding $X \subseteq \mathbb{P}^n$, and let $I_X \subseteq K[x_0, \dots, x_n] =: S$ the corresponding saturated ideal. Let $R_X := S/I_X$ be the corresponding coordinate ring. We say that $X \subseteq \mathbb{P}^n$ is *arithmetically-Gorenstein* if R_X is Gorenstein. Furthermore, the *Castelnuovo-Mumford regularity* of $X \subseteq \mathbb{P}^n$ is

$$\operatorname{reg} X := \max\{j - i : \operatorname{Tor}_i^S(I_X, K)_j \neq 0\}.$$

We say that $X \subseteq \mathbb{P}^n$ is a *subspace arrangement* if $I_X = \bigcap_{i=1}^s \mathfrak{p}_i$ where the $\mathfrak{p}_i$'s are generated by linear forms. We say that $X \subseteq \mathbb{P}^n$ is a *line arrangement* if it is a subspace arrangement and $\dim S/\mathfrak{p}_i = 2$ for all $i = 1, \dots, s$.

By the *dual graph* of X, denoted by G_X, we mean the simple graph whose vertices are the s irreducible components $X_1, \dots, X_s$ and whose edges are

$$\{\{i, j\} : \dim X_i \cap X_j = \dim X - 1\}$$

The *valency* of a vertex is the number of vertices adjacent to it. (This is usually called "degree" in graph theory, but we refrain from this notation to avoid confusion.) We denote by $\delta(G_X)$ the minimum valency of a vertex. By $\Delta(G_X)$ we denote the maximum valency. If all vertices have the same valency (say, k), we say that the graph G_X *is regular* (or *has regularity* k). The graph G_X is *k-connected* if it has at least $k+1$ vertices and the deletion of less than k vertices from G_X, however chosen, yields a connected graph. It is easy to see that if G_X is k-connected, then $\delta(G_X) \geq k$. The *distance* of two vertices is the smallest number of edges of a path connecting them. The maximum distance in a graph is called *(graph) diameter*. It is well known that any k-connected graph with n vertices has diameter at most $\lfloor \frac{n+k-2}{k} \rfloor$.

1 Two notions of regularity

Let us start by showing that the dual graph of an arithmetically-Gorenstein line arrangement need not be regular. Recall that by [BV2015, Theorem 3.8] we have the inequality

$$\operatorname{reg} X - 1 \leq \delta(G_X) \leq \Delta(G_X).$$

Proposition 1.1. *There is a complete-intersection line arrangement X in $\mathbb{P}^4$ of Castelnuovo–Mumford regularity 4 whose dual graph has $\delta(G_X) = 3$ and $\Delta(G_X) = 4$.*

Proof. Let $I \subset \mathbb{Q}[x, y, z, t, w]$ be the following ideal:

$$I = (x^2 - y^2 + z^2 - t^2,\ xz - yt,\ xw).$$

With the help of the computer software Macaulay2, one can immediately verify that I is the intersection of the following 8 prime ideals:

$$\begin{array}{llll} \mathfrak{p}_1 = (t, y - z, x) & \mathfrak{p}_3 = (z - t, y, x) & \mathfrak{p}_5 = (w, z - t, x - y) & \mathfrak{p}_7 = (w, y + z, x + t) \\ \mathfrak{p}_2 = (t, y + z, x) & \mathfrak{p}_4 = (z + t, y, x) & \mathfrak{p}_6 = (w, z + t, x + y) & \mathfrak{p}_8 = (w, y - z, x - t) \end{array}$$

Hence, I defines an arrangement X of 8 lines. One can check that the dual graph is

$$G_X = 12, 13, 14, 18, 23, 24, 27, 34, 35, 46, 57, 58, 67, 68.$$

While the first four vertices have valency 4, the remaining four vertices have valency 3. Being a complete intersection of three quadrics, $\operatorname{reg} X = 4$. □

The graph of Proposition 1.1 is "very close" to being 3-regular. In fact, it has minimum valency equal to $\operatorname{reg} X - 1$ and maximum valency equal to $\operatorname{reg} X$. This triggers two questions:
(1) Is perhaps $\delta(G_X)$ always equal to $\operatorname{reg} X - 1$?
(2) Is there an upper bound on how large the gap $\Delta(G_X) - \operatorname{reg} X$ can be?
Our next step is to provide a negative answer to both questions. It turns out that even when the dual graph of X is regular, the graph-theoretic regularity of G_X may be *arbitrarily larger* then the Castelnuovo–Mumford regularity of X.

Proposition 1.2. *For any integers $n, d \geq 2$, there is a complete intersection $X \subseteq \mathbb{P}^{n+1}$ of Castelnuovo–Mumford regularity $nd - n + 1$ whose dual graph is $(d^n - 1)$-regular and $(d^n - 1)$-connected.*

Proof. Let $f_1, \ldots, f_n$ be generic homogeneous polynomials of degree d in $K[x_0, \ldots, x_n]$. Let $Y \subseteq \mathbb{P}^n$ be the projective scheme defined by the ideal $J = (f_1, \ldots, f_n)$. Since the f_i's are generic, the ideal J is a radical complete intersection, so $Y \subseteq \mathbb{P}^n$ consists of d^n distinct points. Consider the cone $X \subseteq \mathbb{P}^{n+1}$ of Y. This is an arrangement of d^n lines in $\mathbb{P}^{n+1}$. Being a complete intersection, X is arithmetically-Gorenstein and $\operatorname{reg} X = nd - n + 1$. Since all lines in X pass through $[0, \ldots, 0, 1] \in \mathbb{P}^{n+1}$, the dual graph of X is the complete graph on d^n vertices. □

We are now ready for our main result. A curve $(X, \mathcal{O}_X)$ over K has *only planar singularities* if

$$\widehat{\mathcal{O}_{X,P}} \cong K[[x, y]]/I_P \quad \text{for all } P \in X \quad \text{and for some } I_P \subseteq K[[x, y]].$$

For line arrangements in $\mathbb{P}^n$, this boils down to the following condition: whenever three or more lines come together at a single point, all those lines must belong to the same plane. Of course, every line arrangement where *no* three lines meet at a common point, yields an example of a curve with only planar singularities. By definition, any curve lying on a smooth surface has only planar singularities.

Theorem 1.3. *Let $X \subseteq \mathbb{P}^n$ be an arithmetically-Gorenstein line arrangement with only planar singularities. The dual graph G_X is $(\operatorname{reg} X - 1)$-regular and $(\operatorname{reg} X - 1)$-connected.*

Proof. Choose an arbitrary ordering $L_1, \ldots, L_s$ of the lines of X. Let d be the number of lines intersecting the last line L_s. We are going to show that $d = \operatorname{reg} X - 1$, whence the conclusion follows by reshuffling the order.

Let $I = \bigcap_{i=1}^{s} \mathfrak{p}_i \subseteq S = K[x_0, \ldots, x_n]$ be the defining ideal of $X \subseteq \mathbb{P}^n$. Set $J = \bigcap_{i=1}^{s-1} \mathfrak{p}_i$. The ideal $J + \mathfrak{p}_s$ defines the scheme $L_s \cap (L_1 \cup L_2 \cup \ldots \cup L_{s-1})$, which is possibly not reduced, and whose underlying topological space consists of $k \leq d$ points of L_s. Hence we can find a polynomial $g \in S$ of degree h, where $k \leq h \leq d$, such that

$$J + \mathfrak{p}_s = \mathfrak{p}_s + (g).$$

Note that $h = d$: in fact, $g = \prod_{i=1}^{k} q_i^{a_i}$, where q_i is the equation of P_i in L_s; since P_i is a planar singularity of X, the number a_i counts the lines of X that are different from L_s and pass through P_i. In particular, $\operatorname{Tor}_n^S(S/(J + \mathfrak{p}_s), K)_{n-1+d} \neq 0$.

But $\mathrm{Proj}(S/J)$ is geometrically linked to L_s by X; it follows that S/J is a 2-dimensional Cohen–Macaulay ring, and in particular

$$\mathrm{Tor}_n^S(S/J, K) = \mathrm{Tor}_n^S(S/\mathfrak{p}_s, K) = 0. \tag{1}$$

Now, consider the short exact sequence

$$0 \to S/I \to S/J \oplus S/\mathfrak{p}_s \to S/(J + \mathfrak{p}_s) \to 0.$$

Applying Tor and using equation (1), we see that there is an injection

$$\mathrm{Tor}_n^S(S/(J + \mathfrak{p}_s), K)_{n-1+d} \hookrightarrow \mathrm{Tor}_{n-1}^S(S/I, K)_{n-1+d}.$$

So also $\mathrm{Tor}_{n-1}^S(S/I, K)_{n-1+d} \neq 0$. By definition of Castelnuovo–Mumford regularity, we obtain

$$\mathrm{reg}(S/I) \geq d.$$

On the other hand, the graph $G(X)$ is $\mathrm{reg}(S/I)$-connected by [BV2015, Theorem 3.8]. In any k-connected graph, no vertex can have less than k neighbors. This means that $\mathrm{reg}(S/I) \leq d$. Hence $d = \mathrm{reg}(S/I) = \mathrm{reg}\, X - 1$. □

Corollary 1.4. *Let X be a smooth surface, H a very ample divisor, $X \subseteq \mathbb{P}^n$ the embedding given by the complete linear system $|H|$, and $D_1, \ldots, D_s$ lines on $X \subseteq \mathbb{P}^n$ such that $D_1 + \ldots + D_s \sim dH$ for some $d \in \mathbb{Z}_{>0}$:*

(i) *If $n = 3$, then $|\{j \neq i : D_j \cap D_i \neq \emptyset\}| = \deg H + d - 2$ for each $i = 1, \ldots, s$.*

(ii) *If X is a K_3 surface, then $|\{j \neq i : D_j \cap D_i \neq \emptyset\}| = d + 2$ for each $i = 1, \ldots, s$.*

Proof. Consider the line arrangement $Y := \bigcup_{i=1}^s D_i \subseteq \mathbb{P}^n$. Being contained in a smooth surface, it has only planar singularities. If $n = 3$, then $Y \subseteq \mathbb{P}^3$ is the scheme-theoretic intersection of X and some surface of degree d. In particular, it is arithmetically-Gorenstein of Castelnuovo–Mumford regularity $\deg H + d - 2$. This settles part (i).

For (ii): By definition the embedding $X \subseteq \mathbb{P}^n$ is linearly normal. So it is projectively normal by results of Saint-Donat [Sa1974]. We claim that $H^1(X, \mathcal{O}_X(k)) = 0$ for all $k \in \mathbb{Z}$. Since $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$, in characteristic 0 this claim would immediately follow by the Kodaira vanishing and Serre's duality. However one can easily argue in any characteristic, by considering the following short exact sequence:

$$0 \to \mathcal{O}_X(k-1) \to \mathcal{O}_X(k) \to \mathcal{O}_H(k) \to 0.$$

Because $H^0(X, \mathcal{O}_X(k)) \cong H^0(H, \mathcal{O}_H(k))$ for all $k \leq 0$ and $H^0(X, \mathcal{O}_X(k)) = 0$ for all $k < 0$, the map

$$H^1(X, \mathcal{O}_X(k-1)) \to H^1(X, \mathcal{O}_X(k))$$

is injective for all $k \leq 0$. Since $H^1(X, \mathcal{O}_X) = 0$ we therefore infer by induction that $H^1(X, \mathcal{O}_X(k)) = 0$ for all $k \leq 0$. Since $K_X \sim 0$, by Serre's duality we obtain that $H^1(X, \mathcal{O}_X(k)) = 0$ for all $k \in \mathbb{Z}$, and the claim is proven. Hence, the embedding $X \subseteq \mathbb{P}^n$ is arithmetically Cohen-Macaulay. Yet $K_X \sim 0$, so the embedding is also arithmetically-Gorenstein of (Castelnuovo–Mumford) regularity 4. Since $Y \subseteq \mathbb{P}^n$ is scheme-theoretically the intersection of X with a hypersurface of degree d in $\mathbb{P}^n$, it is arithmetically-Gorenstein of regularity $d + 3$. □

Applications to line arrangements in $\mathbb{P}^3$

Having codimension 2, for curves in $\mathbb{P}^3$ the properties of being a complete intersection and that of being arithmetically-Gorenstein are the same. Before digging into examples, we would like to clarify in what sense it is restrictive for a line arrangement to live in a three-dimensional ambient space.

Lemma 1.5. *Let $X \subseteq \mathbb{P}^3$ be a line arrangement. If X is the complete intersection of two surfaces of degree d and e respectively, then the diameter of G_X is at most* $\min\{d, e\}$.

Proof. By [BV2015, Theorem 3.8], G_X is a $(d + e - 2)$-connected graph on de vertices. Hence the diameter is at most $\lfloor \frac{de+(d+e-2)-2}{d+e-2} \rfloor$, a quantity never larger than $\min\{d, e\}$. □

Proposition 1.6. *A graph G is the dual graph of a line arrangement in $\mathbb{P}^n$, for some $n \geq 3$, if and only if it is the dual graph of some line arrangement in $\mathbb{P}^3$. However, some graphs are dual to complete intersection line arrangements $Y \subseteq \mathbb{P}^n$ for some $n \geq 4$, but cannot be realized as dual graph of any complete intersection line arrangement $X \subseteq \mathbb{P}^3$.*

Proof. Let $Y \subseteq \mathbb{P}^n$ be a line arrangement such that $G = G_Y$. The secant variety Z of $Y \subseteq \mathbb{P}^n$ has dimension at most 3, so if $n \geq 4$ there is a point $P \in \mathbb{P}^n \setminus Z$. Denoting by π the projection from P to some $\mathbb{P}^3 \subsetneq \mathbb{P}^n$ not containing P, then $X := \pi(Y) \subseteq \mathbb{P}^3$ is a line arrangement isomorphic to Y. This settles the first part of the claim. However, the complete intersection property is not preserved under projections. In fact, consider

$$I = (x_1y_1,\ x_2y_2,\ x_3y_3) = \bigcap_{\sigma \subseteq \{1,2,3\}} (x_i,\ y_j : i \in \sigma, j \notin \sigma) \subseteq K[x_i, y_i : i = 1, 2, 3].$$

Clearly, I defines an arrangement of 8 lines in $\mathbb{P}^5$ which is a complete intersection. Note that its dual graph has diameter 3. Now, suppose by contradiction that some complete intersection line arrangement $X \subseteq \mathbb{P}^3$ had the same dual graph. Then X would consist of 8 lines. Let f, g be the two homogeneous polynomials such that $(f, g) = I_X$. From the fact that $\deg f \cdot \deg g = 8$ we infer that $\min\{\deg(f), \deg(g)\}$ is either 1 or 2. Yet the diameter of G_X is 3, which contradicts Lemma 1.5. □

We are now ready to discuss a few famous examples of line arrangements in $\mathbb{P}^3$. For some of them, the property of being a complete intersection is not obvious. For this reason and for didactical purposes, we examine each example in detail. The expert reader may skip directly to the next section.

Example A. (Lines From Two Rulings) The complete bipartite graph $K_{m,n}$ is the dual graph of a line arrangement. This can be realized in any smooth quadric in $\mathbb{P}^3$, by picking m lines from one of the two rulings of the quadric, and then by picking n further lines from the other ruling. The resulting arrangement $A \subseteq \mathbb{P}^3$ was studied by Geramita–Weibel [GW1985] and later by Teitler–Torrance [TT2015], who computed the Castelnuovo-Mumford regularity and classified the arithmetical Cohen–Macaulayness of A in terms of m and n. Assuming $m, n \geq 3$, A is arithmetically Cohen–Macaulay if and only if the integers m and n differ by 0 or 1 [TT2015, Theorem 1.2].

We claim that A is a complete intersection if and only if $m = n$. The "only if" part follows from Theorem 1.3 (or from a direct argument). For the "if" part: when $m = n$,

the arrangement A is the complete intersection of the quadric with a union of n planes. Since any edge in the dual graph corresponds to a pair of coplanar lines, the choice of the n planes correspond to the choice of a complete matching of the graph.

Example B (Twenty-Seven Lines)**.** Any smooth cubic surface Y in $\mathbb{P}^3$ contains exactly 27 lines. Any such cubic Y is isomorphic to the blow-up of the projective plane $\mathbb{P}^2$ along 6 points $P_1, \ldots, P_6$. The 27 lines can be described as follows:

(a) the exceptional divisor E_i corresponding to P_i, with $i \in \{1, \ldots, 6\}$ (for a total of 6 lines of this type);

(b) the strict transform L_{ij} of the line in $\mathbb{P}^2$ that passes through P_i and P_j, with $i, j \in \{1, \ldots, 6\}$ and $i < j$ (which yields a total of 15 further lines);

(c) the strict transform C_i of the unique conic in $\mathbb{P}^2$ that passes through all points except P_i, with $i \in \{1, \ldots, 6\}$ (6 further lines).

Let B be the line arrangement given by the $6 + 15 + 6 = 27$ lines and let G_B be the dual graph of B. Consistently with the above notation, we denote by E_i, L_{ij} and C_i the vertices of G_B. Then, by construction, G_B consists of the following edges:

- $\{E_i, L_{hk}\}$ if and only if $i = h$ or $i = k$;
- $\{L_{ij}, C_k\}$ if and only if $i = k$ or $j = k$;
- $\{E_i, C_j\}$ if and only if $i \neq j$.
- $\{L_{ij}, L_{hk}\}$ if and only if $\{i, j\} \cap \{h, k\} = \emptyset$.

It is straightforward to check that G_B is a 10-connected 10-regular graph. Theorem 1.3 correctly predicts this fact. Indeed, B has only planar singularities and is the complete intersection of the cubic with the union of 9 planes. (Each triangle in G_B corresponds to a plane, the one containing the three lines; G_B can be thus partitioned into 9 triangles.)

Example C (Steiner set)**.** A Steiner set is a line arrangement given by 3 sets of 3 lines each, where each line is incident with 2 from each of the other two sets. In the notation of Example B, let G_B be the dual graph of an arrangement B of 27 lines on some smooth cubic Y. Let G_C be the subgraph of G_B induced on the following vertices:

$$\{E_i \,|\, i = 1, \ldots, 3\} \;\cup\; \{L_{ij} \,|\, 1 \leq i < j \leq 3\} \;\cup\; \{C_j \,|\, j = 1, \ldots, 3\}.$$

This G_C is 4-regular, it has 9 vertices and 18 edges. The subarrangement C of B dual to G_C is a Steiner set. We can partition G_C in 3 triangles given by the triples of lines:

$$\{E_1, L_{12}, C_2\},\ \{E_2, L_{23}, C_3\},\ \{E_3, L_{13}, C_1\}.$$

Each triple of lines lies on a plane, so the line arrangement C lies on the union Z of the three planes. So we have $C \subseteq Y \cap Z$, and for degree reasons the inclusion must be an equality. Thus $C \subseteq \mathbb{P}^3$ is a complete intersection of the smooth cubic Y and a union Z of three planes. Theorem 1.3 correctly claims that G_C is $(3 + 3 - 2)$-regular.

Example D (Schläfli's double-six)**.** Let G_D be the bipartite graph on $\{a_1, \ldots, a_6\} \cup \{b_1, \ldots, b_6\}$ such that $\{a_i, b_j\}$ is an edge if and only if $i \neq j$. Clearly G_D is a 5-regular graph. Schläfli's double-six is a line arrangement $D \subseteq \mathbb{P}^3$ having G_D as dual graph; it consists of 12 of the 27 lines on a smooth cubic Y, and precisely $E_1, \ldots, E_6, C_1, \ldots, C_6$, with the notation of Example B.

Since G_D is 5-regular and triangle-free, by picking the intersection points of D we get a set S of 30 distinct points such that any line of D passes through exactly 5 points of S. Next, choose 4 non-coplanar points x, y, v, w outside the cubic, and consider the set $S' = S \cup \{x, y, v, w\}$. Since the linear system of quartics of $\mathbb{P}^3$ has dimension 34, we can find a quartic Z passing through all points of S'. Since Z contains 5 points for each line of D, then it must contain D. In addition, Z cannot contain the cubic Y (otherwise Z would be Y union a plane, against the choice of x, y, v, w). Then $Y \cap Z$ is a complete intersection of degree 12 containing D; and the equality $D = Y \cap Z$ follows because $\deg D = 12$. This is in accordance with Theorem 1.3, which correctly predicts that G_D is $(4+3-2)$-regular.

Contrarily to the previous examples, the graph G_D has diameter $3 > 2$. This is somehow unexpected to us, so we would like to pose the following:

Question 1.7. *Given a positive integer d, is there always some complete intersection line arrangement $X \subseteq \mathbb{P}^3$ such that G_X has diameter d?*

Notice that a line arrangement as above should be the complete intersection of two surfaces of degree $\geq d$, by Lemma 1.5.

Higher-degree surfaces. As mentioned in the introduction, the generic smooth surface of degree $d \geq 4$ contains no line. Furthermore, by a result of B. Segre [Se1943], any smooth surface of degree $d \geq 4$ cannot contain more than $(d-2)(11d-6)$ lines. Examples of smooth surfaces in $\mathbb{P}^3$ with many lines are those of equation

$$F(x_0, x_1, x_2, x_3) = \phi(x_0, x_1) - \psi(x_2, x_3),$$

where ϕ and ψ are two arbitrary homogenous polynomials of degree d. As shown in [BS2007], in this case the maximal number of lines N_d is given, for $d \geq 3$, by the formula

$$N_d = \begin{cases} 64 & \text{if } d = 4, \\ 180 & \text{if } d = 6, \\ 256 & \text{if } d = 8, \\ 864 & \text{if } d = 12, \\ 1600 & \text{if } d = 20, \\ 3d^2 & \text{otherwise.} \end{cases}$$

The maximum is almost always achieved by the degree-d Fermat surfaces, which contains exactly $3d^2$ lines, as we review below.

Example E (Lines on Fermat surfaces)**.** Let $\mathfrak{F}_d \subseteq \mathbb{P}^3$ be the Fermat surface of degree d given by the equation $x_0^d + x_1^d + x_2^d + x_3^d = 0$. Let us denote by $\zeta_1, \zeta_2, \ldots, \zeta_d$ the d-th roots of unity. Let us also fix a $2d$-th root of unity ω. Adapting the notations in [SSV2010] to our convenience, we list the lines of $\mathfrak{F}_d$ as follows (both i and j range from 1 to d):

$$\begin{aligned} l_1(i,j) &= \{[\lambda, \omega\zeta_i\lambda, \mu, \omega\zeta_j\mu]\}, \\ l_2(i,j) &= \{[\lambda, \mu, \omega\zeta_i\lambda, \omega\zeta_j\mu]\}, \\ l_3(i,j) &= \{[\lambda, \mu, \omega\zeta_j\mu, \omega\zeta_i\lambda]\}. \end{aligned}$$

This yields an arrangement E of $3 \cdot d \cdot d = 3d^2$ lines. The dual graph G_E consists of the following edges:

1. $\{l_a(i,j), l_a(h,k)\}$ iff $i = h$ or $j = k$, $a = 1, 2, 3$;
2. $\{l_1(i,j), l_2(h,k)\}$ iff $i - j = h - k$ (mod d);
3. $\{l_1(i,j), l_3(h,k)\}$ iff $i + j = h - k$ (mod d);
4. $\{l_2(i,j), l_3(h,k)\}$ iff $i + j = h + k$ (mod d).

We can partition the vertex set of G_E into $3d$ copies of the complete graph K_d. In fact, if we fix $(a,i) \in \{1,2,3\} \times \{1,\ldots,d\}$, the induced subgraph corresponding to the set of lines $E_{(a,i)} := \{l_a(i,j), j = 1,\ldots,d\}$ is complete. So each set of lines $E_{(a,i)}$ spans a plane $\pi_{a,i}$ in $\mathbb{P}^3$. Hence, E can be seen as the complete intersection of the surface $\mathfrak{F}_d$ with the union of the $3d$ planes $\pi_{a,i}$ just described. In particular, one has

$$\operatorname{reg} X = d + 3d - 1 = 4d - 1.$$

One can see from the description above that G_E is a $(4d-2)$-regular graph. This is consistent with Theorem 1.3: being $\mathfrak{F}_d$ smooth, the arrangement E has only planar singularities. Note also that the Fermat surface of degree $d = 3$ is a smooth cubic. Since the dual graph does not depend on the chosen cubic, the dual graph of "27 lines on a smooth cubic" is thus a particular case of the dual graph of "$3d^2$ lines on the Fermat surface".

Example F (A 12-line arrangement different than Schläfli's)**.** For $d \geq 3$, let $\mathfrak{F}_d$ be the degree-d Fermat surface and let $\pi_{a,i}$ be the $3d$ planes described in Example E. For some integer $h \in \{1,\ldots,3d\}$, choose h of these $3d$ planes and let Π_h be their union. By the construction explained in Example E, the intersection $F_h = \mathfrak{F}_d \cap \Pi_h$ is a subarrangement of E, consisting of exactly hd lines. Clearly F_h is a complete intersection and the dual graph G_{F_h} has regularity $h + d - 2$. If we choose the degree-3 Fermat surface as smooth cubic, the "Steiner set" can be viewed as the case $h = 3$, $d = 3$ of this construction. Let us focus instead on the case $h = 4$, $d = 3$. This yields a 12-line arrangements with 30 intersection points that is different than Schläfli's double six. In fact, the graph G_D of Example D has diameter 3 and it is bipartite, whereas G_{F_4} has diameter 2 and contains triangles. However, because of Theorem 1.3, both graphs G_D and G_{F_4} are 5-regular.

2 The nerve complex

Let R be a standard graded ring with $R_0 = \mathbb{C}$. Let $\mathfrak{m}$ be its homogenous maximal ideal. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ be the minimal primes of R. The *nerve complex* or *Lyubeznik complex* $\mathfrak{L}(R)$ is the simplicial complex on the vertices $1,\ldots s$ described by

$$\{i_1,\ldots,i_k\} \text{ is a face } \iff \sqrt{\mathfrak{p}_{i_1} + \ldots + \mathfrak{p}_{i_k}} \neq \mathfrak{m}.$$

Equivalently, if $X = \operatorname{Proj} R$ and $X_1,\ldots,X_s$ are its irreducible components, one could describe the nerve complex as follows:

$$\{i_1,\ldots,i_k\} \text{ is a face } \iff X_{i_1} \cap \ldots \cap X_{i_k} \text{ is nonempty.}$$

It is a straightforward consequence of Borsuk's Nerve Lemma that any simplicial complex Δ is homotopy equivalent to $\mathfrak{L}(\mathbb{C}[\Delta])$, where $\mathbb{C}[\Delta]$ is the usual notation for the Stanley–Reisner ring of Δ. However, Δ and $\mathfrak{L}(\mathbb{C}[\Delta])$ are not homeomorphic in general: for example, when Δ is a simplex, $\mathfrak{L}(\mathbb{C}[\Delta])$ consists of a single point.

In 1962, Hartshorne [Har1962, Prop. 2.1] showed that if R is a standard graded ring and $\mathrm{depth} R \geq 2$, then $\operatorname{Proj} R$ is connected; or equivalently,

$$\widetilde{H}_0(\mathfrak{L}(R); \mathbb{C}) = 0.$$

Recently, Katzman, Lyubeznik and Zhang proved the following beautiful extension:

Theorem 2.1 ([KLZ2015])**.** *If R is a standard graded ring and* $\mathrm{depth} R \geq 3$*, then*

$$\widetilde{H}_0(\mathfrak{L}(R); \mathbb{C}) = \widetilde{H}_1(\mathfrak{L}(R); \mathbb{C}) = 0.$$

It is not clear whether the obvious generalization to higher depth is true: the proof given in [KLZ2015] relies on the fact that, because $\mathrm{depth} R \geq 3$, the cohomological dimension of the complement of $\operatorname{Proj} R$ in $\mathbb{P}^n$ (embedded by the linear system R_1) is no more than $n-2$, as proven by the third author [Va2013]. However, the latter fact is false for higher depths.

The theorem of [KLZ2015] naturally suggests the question: what about the converse?

Question 2.2. *Given a simplicial complex Δ with vanishing 0-th and 1-st homology, is it the Lyubeznik complex of a ring of depth ≥ 3?*

Without any "depth request", the answer is known. Every simplicial complex Δ is the nerve of some ring: in fact, even of some Stanley–Reisner ring. Here is a simple construction illustrating this fact that was suggested to us by Alessio D'Alì.

Lemma 2.3. *Let Δ be a simplicial complex on n vertices and dimension $d-1$. Let N be the number of facets of Δ. Let M be the maximum, taken over all vertices v of Δ, of the number of facets containing v. There is a simplicial complex Γ on s vertices, with $N \leq s \leq n+N$, and of dimension either $M-1$ or M, such that*

$$\mathfrak{L}(\mathbb{C}[\Gamma]) = \Delta.$$

(Or in the words of combinatorialists: The nerve of Γ coincides with Δ.)

Proof. Let $1, \ldots, n$ be the vertices of Δ, and let $F_1, \ldots, F_N$ be its facets. Let $A_1, \ldots, A_n$ be subsets of $[N]$ defined as follows:

$$A_i = \{j \in [N] \text{ s.t. } i \in F_j\}.$$

These A_i's might not be facets of some simplicial complex, because a priori it can happen that some A_i is contained in some other $A_{i'}$. This can be fixed as follows: if A_i is contained in some $A_{i'}$, with $i \neq i'$, we update the set A_i by adding to it the integer $N+i$. Let now Γ be the simplicial complex generated by the $A_1, \ldots, A_n$. By construction, the intersection of $A_{i_1}, \ldots, A_{i_k}$ is non-empty if and only if $\{i_1, \ldots, i_k\}$ is contained in a facet of Δ. □

If we try to use the construction above to gather information on the depth of Γ, however, we are doomed. In fact, from Γ one cannot even recover the dual graph of Δ. Moreover, the dimension of Γ can be arbitrarily larger (or also smaller) than $\dim \Delta$.

To bypass these difficulties and tackle Question 2.2, we need to introduce a more geometric construction.

Theorem 2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex $d \geq 2$. There exists a d-dimensional standard graded $\mathbb{C}$-algebra $R = R(\Delta)$ such that:*

(i) $\mathfrak{L}(R) = \Delta$;

(ii) *the dual graph of R is the 1-skeleton of Δ.*

Furthermore, if $\widetilde{H}_0(\Delta;\mathbb{C}) = 0$, then one can choose R such that $\operatorname{depth}(R) \geq 2$*; and if $\widetilde{H}_0(\Delta;\mathbb{C}) = \widetilde{H}_1(\Delta;\mathbb{C}) = 0$, then one can choose R so that* $\operatorname{depth}(R) \geq 3$.

Proof. Let us fix a total order $\sigma_1, \ldots, \sigma_r$ of the minimal *non*-faces of Δ, so that

$$\dim \sigma_i < \dim \sigma_j \Rightarrow i > j,$$

that is, higher-dimensional non-faces are listed first.

Let us choose an arrangement of n hyperplanes in $\mathbb{P}^d$ (where n is the number of vertices of Δ) such that for each subset $A \subseteq [n] = \{1, \ldots, n\}$

$$\dim \bigcap_{i \in A} H_i = \max\{-1, d - |A|\}.$$

This condition is obviously met if the hyperplanes are generic. Let now $X(\sigma_1)$ be the blow-up of $\mathbb{P}^d$ along $\bigcap_{j\in\sigma_1} H_j$. For each $j \in [n]$, let $H_j(\sigma_1)$ be the strict transform of H_j in $X(\sigma_1)$. Recursively, for each $i \in \{2, \ldots, r\}$ we denote

- by $X(\sigma_i)$ the blow-up of $X(\sigma_{i-1})$ along $\bigcap_{j\in\sigma_i} H_j(\sigma_{i-1})$, and
- by $H_j(\sigma_i)$ the strict transform of $H_j(\sigma_{i-1})$ in $X(\sigma_i)$, for each $j \in [n]$.

Consider $Y := \bigcup_{j=1}^n H_j(\sigma_r)$. Being a blow-up of a projective scheme, by the blow-up lemma there exists a d-dimensional standard graded $\mathbb{C}$-algebra R such that $\operatorname{Proj} R = Y$. By construction, $\mathfrak{L}(R) = \Delta$. Furthermore, the dual graph of such an R is the 1-skeleton of Δ. This shows items (i) and (ii). Note also that the irreducible components of Y and their nonempty intersections are smooth rational varieties.

We are now left with the last two claims. First of all we observe that $\mathfrak{L}(R) = \Delta$ is the nerve complex of Y associated to the covering $\{H_j(\sigma_r)\}_{j=1,\ldots,n}$. A smooth rational variety over $\mathbb{C}$ is simply connected (see [De2001, Corollary 4.18]), then by the extended version of the nerve lemma [Bj2003, Theorem 6], we have

$$H^i(Y;\mathbb{C}) \cong H^i(\Delta;\mathbb{C}) \quad \text{for } i = 0, 1. \tag{2}$$

By the genericity of the hyperplanes $H_j \subseteq \mathbb{P}^d$, Y has only simple normal crossing singularities. So the natural maps

$$H^i(Y;\mathbb{C}) \longrightarrow H^i(Y;\mathcal{O}_Y)$$

are surjective for all i. Furthermore we have the Kodaira vanishing (e.g. [Ko1995, 9.12]):

$$H^i(Y;\mathcal{O}_Y(k)) = 0 \ \ \forall\ k < 0,\ i < d-1.$$

Denoting with $\mathfrak{m}$ the unique homogeneous maximal ideal of R, this translates into

$$H^1_{\mathfrak{m}}(R)_k = H^2_{\mathfrak{m}}(R)_k = 0 \ \ \forall\ k < 0.$$

Obviously we can choose R such that $H^0_{\mathfrak{m}}(R) = 0$. Finally, by replacing R with a high enough Veronese, we will also have that $H^1_{\mathfrak{m}}(R)_k = H^2_{\mathfrak{m}}(R)_k = 0$ for all $k > 0$. Since $\dim_{\mathbb{C}} H^1_{\mathfrak{m}}(R)_0 = \dim_{\mathbb{C}} H^0(Y;\mathbb{C}) - 1$ and $\dim_{\mathbb{C}} H^2_{\mathfrak{m}}(R)_0 = \dim_{\mathbb{C}} H^1(Y;\mathbb{C})$, the conclusion follows by (2). □

Acknowledgments. We are indebted to Winfried Bruns, for a question to the third author that eventually inspired Theorem 1.3; to Bernd Sturmfels, for drawing our attention to the 27 lines; to Daniele Faenzi, for suggesting applications to $K3$ surfaces; to Alessio D'Alí, for Lemma 2.3. The second and third author wish also to thank the math department of the University of Miami for the warm hospitality in February 2016.

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