Square-free Gröbner degenerations

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joint work with Aldo Conca

New Trends in Syzygies, Banff, 28/6/2018

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If $I \subseteq S$, we denote the initial ideal of I w.r.t. $\prec$ with $\text{in}(I)$. If I is homogeneous, denoting by

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the following is well-known:

Theorem

$$h^{ij}(S/I) \leq h^{ij}(S/\,\mathrm{in}(I)).$$

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It is easy to produce examples in which the inequalities above are strict, but equalities hold in a special and important case...

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Theorem (Bayer-Stillman, 1987)

If $\prec$ is a degree reverse lexicographic monomial order and the coordinates are generic (with respect to I), then

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On a different perspective, **Algebras with Straightening Laws (ASL)** were introduced in the eighties by De Concini, Eisenbud and Procesi. This notion arose as an axiomatization of the underlying combinatorial structure observed by many authors in classical algebras like coordinate rings of flag varieties, their Schubert subvarieties and various kinds of rings defined by determinantal equations.

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Conjecture (Herzog)

Let $I \subseteq S$ be a homogeneous ideal such that $\operatorname{in}(I)$ is a square-free monomial ideal. Then

$$\operatorname{depth} S/I = \operatorname{depth} S/\operatorname{in}(I), \qquad \operatorname{reg} S/I = \operatorname{reg} S/\operatorname{in}(I).$$

The main result

Theorem (Conca-_ , 2018)

Let $I \subseteq S$ be a homogeneous ideal such that $\text{in}(I)$ is a square-free monomial ideal. Then

$$h^{ij}(S/I) = h^{ij}(S/\text{in}(I)) \quad \forall\ i, j \in \mathbb{Z}.$$

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As a consequence, we get Herzog's conjecture and, in particular, the following:

Corollary

For any ASL A, we have $\operatorname{depth} A = \operatorname{depth} A_D$. In particular, A is Cohen-Macaulay if and only if A_D is Cohen-Macaulay.

The main result

Further consequences are:

Corollary

Let $I \subseteq S$ be a homogeneous ideal such that $\text{in}(I)$ is a square-free monomial ideal. Then

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1. S/I is generalized Cohen-Macaulay if and only if $S/\text{in}(I)$ is generalized Cohen-Macaulay.
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The analogs for the generic initial ideal of the two statements above are false.

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that is $\text{depth}\, S/I \geq \text{depth}\, S/J$.

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By Grothendieck duality, then $H_{\mathfrak{m}}^i(S/J) \to H_{\mathfrak{m}}^i(S/I)$ is surjective for all $i \in \mathbb{N}$.

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By Grothendieck duality, then $H_{\mathfrak{m}}^i(S/J) \to H_{\mathfrak{m}}^i(S/I)$ is surjective for all $i \in \mathbb{N}$. This fact yields that S/I is cohomologically full, in the sense of Dao, De Stefani and Ma (2018).

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As it turns out, the differences $h^{ij}(S/I) - h^{ij}(S/\text{in}(I))$ are measured by the t-torsion of $\text{Ext}_P^{n-i}(A, P)$. Indeed, one has:

$$h^{ij}(S/I) = h^{ij}(S/\text{in}(I)) \ \forall\, j \iff \text{Ext}_P^{n-i}(A, P) \text{ has no } t\text{-torsion.}$$

Sketch of the proof

For all $m \in \mathbb{N}$, set $A_m = \dfrac{P}{\hom_w(I) + (t^{m+1})}$, $R_m = K[t]/(t^{m+1})$ and $P_m = P/(t^{m+1})$. Then $A_0 \cong S/\operatorname{in}(I)$.

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We show that point 3 holds true by induction on m provided that $\operatorname{in}(I)$ is a square-free monomial ideal ($m = 0$ is obvious because R_0 is a field):

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We show that point 3 holds true by induction on m provided that $\operatorname{in}(I)$ is a square-free monomial ideal ($m = 0$ is obvious because R_0 is a field): the idea to prove it comes from a recent work of Kollár and Kovács on deformations of Du Bois singularities (2018)...

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So, since $tA_m \cong A_{m-1}$, for all m, h, the following is exact:

$$0 \to H^h_{(x_1,\ldots,x_n,t)}(A_{m-1}) \to H^h_{(x_1,\ldots,x_n,t)}(A_m) \to H^h_{(x_1,\ldots,x_n,t)}(A_0) \to 0.$$

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By Grothendieck duality, so, for all m, k the following is exact:

$$0 \to \text{Ext}^k_{P_0}(A_0, P_0) \to \text{Ext}^k_{P_m}(A_m, P_m) \to \text{Ext}^k_{P_{m-1}}(A_{m-1}, P_{m-1}) \to 0.$$

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Note that A_0 can be embedded in A_m by multiplication by t^m, so

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Together with the previous short exact sequence, this allows to show that $\mathrm{Ext}^k_{P_{m-1}}(A_{m-1}, P_{m-1}) \cong \dfrac{\mathrm{Ext}^k_{P_m}(A_m, P_m)}{t^m\,\mathrm{Ext}^k_{P_m}(A_m, P_m)}$

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Questions and answers

When we uploaded the paper on the arXiv, we concluded it proposing five questions. We received several comments concerning these questions and it turns out that only one is still open!

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Question 2

Let $\mathfrak{p} \subseteq S$ be a Knutson prime ideal. Is $S/\mathfrak{p}$ Cohen-Macaulay?

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Question 3

Let $I \subseteq S$ be an ideal such that $\mathrm{in}(I)$ is a square-free monomial ideal for degrevlex. If $\mathrm{char}(K) > 0$, is it true that S/I is F-pure?

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Question 4

Let $I \subseteq S$ be a prime ideal such that $\mathrm{in}(I)$ is a square-free monomial ideal for degrevlex. Is it true that S/I is normal?

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The only question remained, so far, unsolved, is the following:

Question 5

Let $\mathfrak{p} \subseteq S$ be a homogeneous prime ideal with a square-free initial ideal such that $\text{Proj}\, S/\mathfrak{p}$ is nonsingular. Is $S/\mathfrak{p}$ Cohen-Macaulay and with negative a-invariant?

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