SYMBOLIC POWERS AND MATROIDS

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Preliminaries and notation

Simplicial complexes

Throughout $n \in \mathbb{N}$ and $\mathcal{N} := \{1, \ldots, n\}$.

A simplicial complex on $\mathcal{N}$ is a collection $\Delta$ of subsets of $\mathcal{N}$ satisfying:

- $F \in \Delta$ and $G \subseteq F \Rightarrow G \in \Delta$;
- $\{i\} \in \Delta$ for all $i \in \mathcal{N}$.

The subsets of $\Delta$ are called faces and the faces maximal by inclusion are called facets. We will denote the set of facets of $\Delta$ by $F(\Delta)$.

The dimension of a face $F$ is $\dim F := |F| - 1$.

The dimension of $\Delta$ is $\dim \Delta := \max \{\dim F : F \in F(\Delta)\}$.

A simplicial complex $\Delta$ is called pure if $\dim F = \dim \Delta$ for all $F \in F(\Delta)$. 
Throughout $n \in \mathbb{N}$ and $[n] := \{1, \ldots, n\}$. 
Throughout $n \in \mathbb{N}$ and $[n] := \{1, \ldots, n\}$. A **simplicial complex** on $[n]$ is a collection $\Delta$ of subsets of $[n]$ satisfying:

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Preliminaries and notation

Stanley-Reisner ideals

We can associate a simplicial complex to any ideal $I \subseteq S := k[x_1, \ldots, x_n]$:

$$\Delta(I) := \{F \subseteq \{n\} : k[x_i : i \in F] \cap I = \{0\}\}.$$  

$\Delta(I)$ is called the independence complex of $I$.

In the other direction, we can associate an ideal to any simplicial complex on $\{n\}$:

$$I_{\Delta} := (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \in \Delta) \subseteq S.$$  

$I_{\Delta}$ is called the Stanley-Reisner ideal of $\Delta$.

Such a relationship leads to a one-to-one correspondence:

$$\{\text{Square-free monomial ideals of } S\} \leftrightarrow \{\text{Simplicial complexes on } \{n\}\}.$$  

To relate combinatorial properties of $\Delta$ with algebraic ones of $I_{\Delta}$ caught the attention of several mathematicians. For example, denoting by $\mathcal{P}(A) := (x_i : i \in A) \subseteq S$ (where $A \subseteq \{n\}$), it is easy to show that:

$$I_{\Delta} = \bigcap_{F \in \Delta} \mathcal{P}(\{n\} \setminus F).$$  

This fact implies that $\dim_k [\Delta] = \dim \Delta + 1$, where $k[\Delta] := S/I_{\Delta}$. 
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For example, denoting by $\wp_A := (x_i : i \in A) \subseteq S$ (where $A \subseteq \mathbb{k}[n]$), it is easy to show that:

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Matroids

A simplicial complex $\Delta$ is a matroid if:

$\forall \ F, G \in \mathcal{F}(\Delta), \forall i \in F, \exists j \in G: (F \setminus \{i\}) \cup \{j\} \in \mathcal{F}(\Delta)$

**Examples:**

- Let $V$ be a $k$-vector space and $v_1, \ldots, v_n$ some vectors of $V$. $\Delta := \{F \subseteq \{1, \ldots, n\} : |F| = \dim_k V < v_i : i \in F\}$ such a $\Delta$ is easily seen to be a matroid.

- Actually, the concept of matroid is an "abstraction of linear independence".

- Let $\Delta$ be the $i$-skeleton of the $(n-1)$-simplex: $\Delta := \{F \subseteq \{1, \ldots, n\} : |F| \leq i\}$. Such a $\Delta$ is obviously a matroid.

- If an ideal $I \subseteq S$ is prime, then $\Delta(I)$ is a matroid.
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Such a $\Delta$ is obviously a matroid.

- If an ideal $I \subseteq S$ is prime, then $\Delta(I)$ is a matroid.
Preliminaries and notation

Symbolic powers

The $k$th symbolic power of an ideal $I \subseteq S$ is the ideal of $S$: $I^{(k)} := (I^k \cdot W - 1) \cap S$, where $W$ is the multiplicative system $S \setminus \bigcup \mathcal{P} \in \text{Ass}(I) \mathcal{P}$.

$I^k \subseteq I^{(k)}$, and equality holds if $I^k$ has no embedded primes.

If $I = I_{\Delta}$ is a square-free monomial ideal, then it is easy to show: $I^{(k)}_{\Delta} = \bigcap F \in F(\Delta) \mathcal{P}^k[n] \setminus F$. 
Preliminaries and notation

**Symbolic powers**

The *kth symbolic power* of an ideal \( I \subseteq S \) is the ideal of \( S \):

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I^k := \left( I^k \cdot W^{-1} \right) \cap S,
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Preliminaries and notation

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$$I_\Delta^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \wp^k [n] \setminus F.$$
The problem

Cohen-Macaulay combinatorial counterpart

Reisner, in 1976, gave a characterization in terms of the topological realization of $\Delta$ of the Cohen-Macaulay property of $k[\Delta]$. However, a characterization in a combinatorial fashion still misses. A related question is when all the rings $S/I_\Delta$ are Cohen-Macaulay. An answer is provided by a general result of Cowsik and Nori (1977): $S/I_\Delta$ is Cohen-Macaulay $\forall k \in \mathbb{Z}^+$ $\iff$ $I_\Delta$ is a complete intersection. Notice that $S/I_\Delta$ Cohen-Macaulay $\implies$ $\text{Ass}(I_\Delta) = \text{Min}(I_\Delta) \implies I_\Delta = I(k\Delta)$. Therefore it is natural to ask: When is $S/I(k\Delta)$ Cohen-Macaulay for all $k \in \mathbb{Z}^+$?
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Notice that $S/I^k_\Delta$ Cohen-Macaulay $\Rightarrow$ $\text{Ass}(I^k_\Delta) = \text{Min}(I^k_\Delta) \Rightarrow I^k_\Delta = I^{(k)}_\Delta$. 

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Notice that $S/I^k$ Cohen-Macaulay $\Rightarrow$ $\text{Ass}(I^k) = \text{Min}(I^k) \Rightarrow I^k = I^{(k)}$.

Therefore it is natural to ask:

When is $S/I^{(k)}$ Cohen-Macaulay for all $k \in \mathbb{Z}_+$???
The result

In this talk, we are going to answer the above question:

\[ S/I(k) \Delta \text{is Cohen-Macaulay} \forall k \in \mathbb{Z}^+ \iff \Delta \text{is a matroid} \]

It is fair to say that Minh and Trung proved at the same time the same result. However, the two proofs are completely different.
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In this talk, we are going to answer the above question:

\[ \frac{S}{I_{\Delta}^{(k)}} \text{ is Cohen-Macaulay} \quad \forall k \in \mathbb{Z}_+ \iff \Delta \text{ is a matroid} \]

It is fair to say that Minh and Trung proved at the same time the same result. However the two proofs are completely different.
THE PROOF
Stanley-Reisner ideals $\longrightarrow$ cover ideals

Properties of matroids

(i) If $\Delta$ is a matroid, then $\Delta$ is pure.

(ii) Exchange property.

If $\Delta$ is a matroid, then

$$\forall F, G \in F(\Delta), \forall i \in F(\Delta), \exists j \in G : (F \{i\} \cup \{j\}) \in F(\Delta)$$

and

$$\forall F, G \in F(\Delta), \forall i \in F(\Delta), \exists j \in G : (G \{j\} \cup \{i\}) \in F(\Delta)$$

(iii) Duality.

For any simplicial complex $\Delta$ on $[n]$, we have

$\Delta$ is a matroid $\iff$ $\Delta^c$ is a matroid.
Stanley-Reisner ideals $\rightarrow$ cover ideals

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Stanley-Reisner ideals → cover ideals

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The dual simplicial complex of $\Delta$ is the complex $\Delta^c$ on $[n]$ s. t.:
Stanley-Reisner ideals $\rightarrow$ cover ideals

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(i) If $\Delta$ is a matroid, then $\Delta$ is pure.

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The dual simplicial complex of $\Delta$ is the complex $\Delta^c$ on $[n]$ s. t.:

$$\mathcal{F}(\Delta^c) := \{[n] \setminus F : F \in \mathcal{F}(\Delta)\}.$$
Stanley-Reisner ideals → cover ideals

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(iii) Duality.
Stanley-Reisner ideals $\rightarrow$ cover ideals

**Properties of matroids**

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(ii) *Exchange property.* If $\Delta$ is a matroid, then $\forall F, G \in \mathcal{F}(\Delta), \forall i \in F,$
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Stanley-Reisner ideals \(\rightarrow\) cover ideals

**Properties of matroids**

(i) If \(\Delta\) is a matroid, then \(\Delta\) is pure.

(ii) *Exchange property.* If \(\Delta\) is a matroid, then \(\forall F, G \in \mathcal{F}(\Delta), \forall i \in F, \exists j \in G : (F \setminus \{i\}) \cup \{j\} \in \mathcal{F}(\Delta)\) and \((G \setminus \{j\}) \cup \{i\} \in \mathcal{F}(\Delta)!\)

The dual simplicial complex of \(\Delta\) is the complex \(\Delta^c\) on \([n]\) s. t.:

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\mathcal{F}(\Delta^c) := \{[n] \setminus F : F \in \mathcal{F}(\Delta)\}.
\]

(iii) *Duality.* For any simplicial complex \(\Delta\) on \([n]\), we have

\(\Delta\) is a matroid \(\Leftrightarrow\) \(\Delta^c\) is a matroid.
Stanley-Reisner ideals $\rightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I(\Delta^c)$, so:

$$J(\Delta) = \bigcap_{F \in F(\Delta)} \wp F,$$

$A \subseteq [n]$ is a vertex cover of $\Delta$ if $A \cap F \neq \emptyset \forall F \in F(\Delta)$.

One can easily check that:

$$J(\Delta) = (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is a vertex cover of } \Delta).$$
For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

\[
J(\Delta) = \bigcap_{F \in F(\Delta)} \mathcal{P}(F),
\]

where $A \subseteq [n]$ is a vertex cover of $\Delta$ if $A \cap F \neq \emptyset$ for all $F \in F(\Delta)$. One can easily check that $J(\Delta) = \langle x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is a vertex cover of } \Delta \rangle$. 

**Stanley-Reisner ideals $\rightarrow$ cover ideals**
For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

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where $A \subseteq [n]$ is a vertex cover of $\Delta$ if $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$. 
For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) \coloneqq I_{\Delta^c}$, so:

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For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

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**EXAMPLES:**

![Diagram of a hexagon with vertices labeled 1 to 6 and edges connecting them, illustrating a vertex cover example.](image-url)
Stanley-Reisner ideals $\mapsto$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := l_{\Delta^c}$, so:

$$J(\Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} \wp F,$$

$A \subseteq [n]$ is a vertex cover of $\Delta$ if $A \cap F \neq \emptyset \ \forall \ F \in \mathcal{F}(\Delta)$.

EXAMPLES: $\{1, 2, 3, 4, 5, 6\}$ is a vertex cover
For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

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EXAMPLES: \{1, 2, 4, 5\} is a vertex cover.
Stanley-Reisner ideals $\rightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

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EXAMPLES: $\{1, 2, 4\}$ is not a vertex cover
Stanley-Reisner ideals $\longrightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

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**EXAMPLES:**

$\{1, 3, 5\}$ is a vertex cover
Stanley-Reisner ideals $\longrightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) \coloneqq I_{\Delta^c}$, so:

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One can easily check that:

$$J(\Delta) = (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is a vertex cover of } \Delta).$$
Stanley-Reisner ideals $\rightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

$$J(\Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} \wp_F,$$

where $\wp_F$ is the ideal generated by the monomials of $F$.

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By the duality for matroids, because $I_\Delta = J(\Delta^c)$, we can pass
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$$S/I_{\Delta}^{(k)} \text{ is CM for any } k \in \mathbb{Z}_+ \iff \Delta \text{ is a matroid}$$
Stanley-Reisner ideals $\rightarrow$ cover ideals

For a simplicial complex $\Delta$, its cover ideal is $J(\Delta) := I_{\Delta^c}$, so:

$$J(\Delta) = \bigcap_{F \in F(\Delta)} \wp_{\varnothing},$$

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By the duality for matroids, because $I_\Delta = J(\Delta^c)$, we can pass to

$$S/J(\Delta)^{(k)} \text{ is CM for any } k \in \mathbb{Z}_+ \iff \Delta \text{ is a matroid}$$
Symbolic powers and $k$-covers

We have $\mathcal{J}(\Delta)(k) = \bigcap_{F \in \mathcal{F}(\Delta)} \wp_k F \forall k \in \mathbb{N}$.

We want to describe which monomials belong to $\mathcal{J}(\Delta)(k)$.

For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \rightarrow \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if:

$$\sum_{i \in F} \alpha(i) \geq k \forall F \in \mathcal{F}(\Delta).$$

A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$. 
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in F(\Delta)} \mathfrak{s}_F^k \quad \forall \ k \in \mathbb{N}$. 
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{O}_F^k \quad \forall \ k \in \mathbb{N}.$

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We have \( J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{O}_F^k \) \( \forall k \in \mathbb{N} \).

We want to describe which monomials belong to \( J(\Delta)^{(k)} \). For each \( k \in \mathbb{N} \), a nonzero function \( \alpha : [n] \to \mathbb{N} \) is called a \textit{k-cover} of a simplicial complex \( \Delta \) on \( [n] \) if:

\[ \sum_{i \in F} \alpha(i) \geq k \forall F \in \mathcal{F}(\Delta). \]
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \varnothing_F^k \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \to \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \quad \forall \ F \in \mathcal{F}(\Delta)$. 

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Symbolic powers and $k$-covers

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We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \to \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: \[
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\]

A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

**EXAMPLES:**

\[
\begin{array}{c}
\text{EXAMPLES:} \\
\begin{array}{c}
\text{Diagram of a simplicial complex}\end{array}
\end{array}
\]
Symbolic powers and \( k \)-covers

We have \( J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{P}_F^k \quad \forall \ k \in \mathbb{N}. \)

We want to describe which monomials belong to \( J(\Delta)^{(k)} \). For each \( k \in \mathbb{N} \), a nonzero function \( \alpha : [n] \to \mathbb{N} \) is called a \( k \)-cover of a simplicial complex \( \Delta \) on \([n]\) if: \( \sum_{i \in F} \alpha(i) \geq k \quad \forall \ F \in \mathcal{F}(\Delta). \)

A \( k \)-cover \( \alpha \) is basic if there is not a \( k \)-cover \( \beta \) with \( \beta < \alpha \).

**EXAMPLES:**

```
  vertex cover
```
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \phi^k_F \quad \forall k \in \mathbb{N}$.

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A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

**EXAMPLES:**

```
1 0
```

1-cover
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in F(\Delta)} \wp^k_F \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \to \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \ \forall \ F \in F(\Delta)$.

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**EXAMPLES:**

```
1 0 1
```

1-cover
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \varnothing_F^k \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \to \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \quad \forall \ F \in \mathcal{F}(\Delta)$.

A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

**EXAMPLES:**

```
1  2
\hline
\hline
2  3
```

3-cover
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \phi_F^k \quad \forall \ k \in \mathbb{N}$.

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A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

EXAMPLES:

```
2  1  2
2 3  1
3-cover
```
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \phi^k_F \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \rightarrow \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \quad \forall \ F \in \mathcal{F}(\Delta)$.

A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

**EXAMPLES:**

```
1 2 2
2 1
```

basic 3-cover
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \wp^k_F \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \rightarrow \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \ \forall \ F \in \mathcal{F}(\Delta)$.

A $k$-cover $\alpha$ is basic if there is not a $k$-cover $\beta$ with $\beta < \alpha$.

It is not difficult to show:

$$J(\Delta)^{(k)} = (x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} : \alpha \text{ is a } k\text{-cover}).$$
Symbolic powers and $k$-covers

We have $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{F}^{k}_F \quad \forall \ k \in \mathbb{N}$.

We want to describe which monomials belong to $J(\Delta)^{(k)}$. For each $k \in \mathbb{N}$, a nonzero function $\alpha : [n] \to \mathbb{N}$ is called a $k$-cover of a simplicial complex $\Delta$ on $[n]$ if: $\sum_{i \in F} \alpha(i) \geq k \quad \forall \ F \in \mathcal{F}(\Delta)$.

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It is not difficult to show:

$J(\Delta)^{(k)} = (x_1^{\alpha(1)} \cdots x_n^{\alpha(n)} : \alpha \text{ is a basic } k\text{-cover})$. 
The algebra of basic covers

Definition

The symbolic Rees algebra of $J(\Delta)$, i.e. $A(\Delta) := \bigoplus_{k \in \mathbb{N}} J(\Delta)(k)$, has an obvious interpretation in terms of $k$-covers. It has been introduced by Herzog, Hibi and Trung, and it is called the vertex cover algebra of $\Delta$. We need to deal with the special fiber of $A(\Delta)$: $\overline{A}(\Delta) := A(\Delta) / m_A(\Delta) = \bigoplus_{k \in \mathbb{N}} J(\Delta)(k) / m_J(\Delta)(k)$, where $m := (x_1, \ldots, x_n) \subseteq S$. For all $k \in \mathbb{Z}^+$, we have: $\overline{A}(\Delta)_k = \langle x^\alpha : \alpha \text{ is a basic } k\text{-cover} \rangle$. For this reason, $\overline{A}(\Delta)$ is called the algebra of basic covers of $\Delta$. 
The algebra of basic covers

Definition

The symbolic Rees algebra of $J(\Delta)$,
The algebra of basic covers

Definition

The symbolic Rees algebra of $J(\Delta)$, i.e. $A(\Delta) := \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)}$,
The algebra of basic covers

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The symbolic Rees algebra of $J(\Delta)$, i.e. $A(\Delta) := \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)}$, has an obvious interpretation in terms of $k$-covers.
The algebra of basic covers

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For all $k \in \mathbb{Z}^+$, we have: $\bar{A}(\Delta)^k = \langle x^{\alpha} : \alpha \text{ is a basic } k\text{-cover} \rangle$. For this reason, $\bar{A}(\Delta)$ is called the algebra of basic covers of $\Delta$. 
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The algebra of basic covers

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$$\bar{A}(\Delta) := A(\Delta)/mA(\Delta) = \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)}/mJ(\Delta)^{(k)}.$$
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The algebra of basic covers
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For all $k \in \mathbb{Z}^+$, we have:

$$\bar{A}(\Delta)_k = \langle x^\alpha : \alpha \text{ is a basic } k\text{-cover} \rangle.$$
The algebra of basic covers

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The symbolic Rees algebra of $J(\Delta)$, i.e. $A(\Delta) := \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)}$, has an obvious interpretation in terms of $k$-covers. It has been introduced by Herzog, Hibi and Trung, and it is called the vertex cover algebra of $\Delta$. We need to deal with the special fiber of $A(\Delta)$:

$$\tilde{A}(\Delta) := A(\Delta)/mA(\Delta) = \bigoplus_{k \in \mathbb{N}} J(\Delta)^{(k)}/mJ(\Delta)^{(k)},$$

where $m := (x_1, \ldots, x_n) \subseteq S$.

For all $k \in \mathbb{Z}_+$, we have:

$$\tilde{A}(\Delta)_k = \langle x^\alpha : \alpha \text{ is a basic } k\text{-cover} \rangle$$

For this reason, $\tilde{A}(\Delta)$ is called the algebra of basic covers of $\Delta$. 
The algebra of basic covers

How \( \bar{A}(\Delta) \) comes into play
The algebra of basic covers

(HHT). $A(\Delta)$ is a Cohen-Macaulay, finitely generated $S$-algebra.
The algebra of basic covers

How $\tilde{\mathcal{A}}(\Delta)$ comes into play

$(HHT)$. $\mathcal{A}(\Delta)$ is a Cohen-Macaulay, finitely generated $S$-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

$$\dim \mathcal{A}(\Delta) = \dim \mathcal{A}(\Delta) + 1$$

Whenever $\Delta$ is a matroid.
The algebra of basic covers

\( \tilde{A}(\Delta) \) comes into play

\( (HHT) \). \( A(\Delta) \) is a Cohen-Macaulay, finitely generated \( S \)-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

\[
\dim \tilde{A}(\Delta) = n - \min_{k \in \mathbb{N}_{>0}} \{ \text{depth}(S/J(\Delta)^{(k)}) \}
\]
The algebra of basic covers

How $\bar{A}(\Delta)$ comes into play

$(HHT)$. $A(\Delta)$ is a Cohen-Macaulay, finitely generated $S$-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

$$\dim \Delta + 1 = \text{ht}(J(\Delta)) \leq \dim \bar{A}(\Delta) = n - \min_{k \in \mathbb{N}_0} \{\text{depth}(S/J(\Delta)^{(k)})\}.$$
The algebra of basic covers

The algebra of basic covers

How $\tilde{A}(\Delta)$ comes into play

(HHT). $A(\Delta)$ is a Cohen-Macaulay, finitely generated $S$-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

$$\dim \Delta + 1 = \text{ht}(J(\Delta)) \leq \dim \tilde{A}(\Delta) = n - \min_{k \in \mathbb{N}_{>0}} \{ \text{depth}(S/J(\Delta)^{(k)}) \}.$$  

Therefore, since $\dim S/J(\Delta) = n - \dim \Delta - 1$, we get
The algebra of basic covers

**How \( \bar{A}(\Delta) \) comes into play**

\((HHT)\). \( A(\Delta) \) is a Cohen-Macaulay, finitely generated \( S \)-algebra.

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\[
\dim \Delta + 1 = \text{ht}(J(\Delta)) \leq \dim \bar{A}(\Delta) = n - \min_{k \in \mathbb{N}_0} \{ \text{depth}(S/J(\Delta)^{(k)}) \}.
\]

Therefore, since \( \dim S/J(\Delta) = n - \dim \Delta - 1 \), we get

\[
S/J(\Delta)^{(k)} \text{ is CM for any } k \in \mathbb{Z}_+ \iff \dim \bar{A}(\Delta) = \dim \Delta + 1.
\]
The algebra of basic covers

How \( A(\Delta) \) comes into play

\((HHT)\). \( A(\Delta) \) is a Cohen-Macaulay, finitely generated \( S \)-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

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\]

Therefore, since \( \dim S/J(\Delta) = n - \dim \Delta - 1 \), we get

\[
S/J(\Delta)^{(k)} \text{ is CM for any } k \in \mathbb{Z}_+ \iff \dim \tilde{A}(\Delta) = \dim \Delta + 1.
\]

In the next slides we are going to show that:
(HHT). $\mathcal{A}(\Delta)$ is a Cohen-Macaulay, finitely generated $S$-algebra.

Using a theorem of Eisenbud and Huneke, the above result yields

$$\dim \Delta + 1 = \text{ht}(J(\Delta)) \leq \dim \mathcal{A}(\Delta) = n - \min_{k \in \mathbb{N}_0} \{\text{depth}(S/J(\Delta)^{(k)})\}.$$ 

Therefore, since $\dim S/J(\Delta) = n - \dim \Delta - 1$, we get

$S/J(\Delta)^{(k)}$ is CM for any $k \in \mathbb{Z}_+$ if and only if $\dim \mathcal{A}(\Delta) = \dim \Delta + 1$.

In the next slides we are going to show that:

$$\dim \mathcal{A}(\Delta) = \dim \Delta + 1$$

whenever $\Delta$ is a matroid.
The algebra of basic covers

A combinatorial description of $\dim \tilde{A}(\Delta)$
The algebra of basic covers

A combinatorial description of $\dim \tilde{A}(\Delta)$

Since $A(\Delta)$ is Noetherian, $\tilde{A}(\Delta)$ is a finitely generated $k$-algebra.
The algebra of basic covers

A combinatorial description of \( \dim \tilde{A}(\Delta) \)

Since \( A(\Delta) \) is Noetherian, \( \tilde{A}(\Delta) \) is a finitely generated \( k \)-algebra. This implies that there exists a positive integer \( \delta \) such that:
The algebra of basic covers

A combinatorial description of $\dim \tilde{A}(\Delta)$

Since $A(\Delta)$ is Noetherian, $\tilde{A}(\Delta)$ is a finitely generated $k$-algebra. This implies that there exists a positive integer $\delta$ such that:

$$\tilde{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \tilde{A}(\Delta)_{\delta m}$$

is a standard graded $k$-algebra.
The algebra of basic covers

A combinatorial description of \( \dim \bar{A}(\Delta) \)

Since \( A(\Delta) \) is Noetherian, \( \bar{A}(\Delta) \) is a finitely generated \( \mathbb{k} \)-algebra. This implies that there exists a positive integer \( \delta \) such that:

\[
\bar{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \bar{A}(\Delta)_{m} \text{ is a standard graded } \mathbb{k}-\text{algebra.}
\]

\( \bar{A}(\Delta) \) fin. gen. \( \bar{A}(\Delta)^{(\delta)} \)-module \( \Rightarrow \) \( \dim \bar{A}(\Delta) = \dim \bar{A}(\Delta)^{(\delta)} \).
The algebra of basic covers

Since $A(\Delta)$ is Noetherian, $\bar{A}(\Delta)$ is a finitely generated $k$-algebra. This implies that there exists a positive integer $\delta$ such that:

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Because $\bar{A}(\Delta)^{(\delta)}$ is standard graded, it has a Hilbert polynomial.
The algebra of basic covers

A combinatorial description of \( \dim \tilde{A}(\Delta) \)

Since \( A(\Delta) \) is Noetherian, \( \tilde{A}(\Delta) \) is a finitely generated \( \mathbb{k} \)-algebra. This implies that there exists a positive integer \( \delta \) such that:

\[
\tilde{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \tilde{A}(\Delta)_{\delta m}
\]

is a standard graded \( \mathbb{k} \)-algebra.

\( \tilde{A}(\Delta) \) fin. gen. \( \tilde{A}(\Delta)^{(\delta)} \)-module \( \Rightarrow \) \( \dim \tilde{A}(\Delta) = \dim \tilde{A}(\Delta)^{(\delta)} \).

Because \( \tilde{A}(\Delta)^{(\delta)} \) is standard graded, it has a Hilbert polynomial.

I.e. a polynomial \( P \in \mathbb{Q}[T] \), of degree \( \dim \tilde{A}(\Delta)^{(\delta)} - 1 \), such that:
The algebra of basic covers

A combinatorial description of \( \dim \tilde{A}(\Delta) \)

Since \( A(\Delta) \) is Noetherian, \( \tilde{A}(\Delta) \) is a finitely generated \( \mathbb{k} \)-algebra. This implies that there exists a positive integer \( \delta \) such that:

\[
\tilde{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \tilde{A}(\Delta)_{\delta m} \text{ is a standard graded } \mathbb{k}\text{-algebra.}
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\( \tilde{A}(\Delta) \) fin. gen. \( \tilde{A}(\Delta)^{(\delta)} \)-module \( \Rightarrow \) \( \dim \tilde{A}(\Delta) = \dim \tilde{A}(\Delta)^{(\delta)} \).

Because \( \tilde{A}(\Delta)^{(\delta)} \) is standard graded, it has a Hilbert polynomial. I.e. a polynomial \( P \in \mathbb{Q}[T] \), of degree \( \dim \tilde{A}(\Delta)^{(\delta)} - 1 \), such that:

\[
P(m) = \dim_{\mathbb{k}}(\tilde{A}(\Delta)_{\delta m}) \quad \forall \ m \gg 0.
\]
Since $A(\Delta)$ is Noetherian, $\bar{A}(\Delta)$ is a finitely generated $k$-algebra. This implies that there exists a positive integer $\delta$ such that:

\[ \bar{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \bar{A}(\Delta)_{\delta m} \text{ is a standard graded } k\text{-algebra.} \]

Because $\bar{A}(\Delta)^{(\delta)}$ is standard graded, it has a Hilbert polynomial. I.e. a polynomial $P \in \mathbb{Q}[T]$, of degree $\dim \bar{A}(\Delta)^{(\delta)} - 1$, such that:

\[ P(m) = \dim_k(\bar{A}(\Delta)_{\delta m}) \quad \forall \ m \gg 0. \]

Therefore, since $\dim_k(\bar{A}(\Delta)_{\delta m}) = |\{\text{basic } \delta m\text{-cover of } \Delta\}|$, 

The algebra of basic covers

\[ A \text{ combinatorial description of } \dim \bar{A}(\Delta) \]
The algebra of basic covers

A combinatorial description of \( \dim \tilde{A}(\Delta) \)

Since \( A(\Delta) \) is Noetherian, \( \tilde{A}(\Delta) \) is a finitely generated \( k \)-algebra. This implies that there exists a positive integer \( \delta \) such that:

\[
\tilde{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \tilde{A}(\Delta)_{\delta m} \quad \text{is a standard graded } k \text{-algebra.}
\]

Because \( \tilde{A}(\Delta)^{(\delta)} \) is standard graded, it has a Hilbert polynomial. I.e. a polynomial \( P \in \mathbb{Q}[T] \), of degree \( \dim \tilde{A}(\Delta)^{(\delta)} - 1 \), such that:

\[
P(m) = \dim_k(\tilde{A}(\Delta)_{\delta m}) \quad \forall \ m \gg 0.
\]

Therefore, since \( \dim_k(\tilde{A}(\Delta)_{\delta m}) = |\{ \text{basic } \delta m \text{-cover of } \Delta \}| \), if \( |\{ \text{basic } k \text{-covers} \}| = O(k^{s-1}) \),
The algebra of basic covers

A combinatorial description of $\dim \mathcal{A}(\Delta)$

Since $\mathcal{A}(\Delta)$ is Noetherian, $\mathcal{A}(\Delta)$ is a finitely generated $\mathbb{k}$-algebra. This implies that there exists a positive integer $\delta$ such that:

$$\mathcal{A}(\Delta)^{(\delta)} := \bigoplus_{m \in \mathbb{N}} \mathcal{A}(\Delta)_{\delta m}$$

is a standard graded $\mathbb{k}$-algebra.

Because $\mathcal{A}(\Delta)^{(\delta)}$ is standard graded, it has a Hilbert polynomial. I.e. a polynomial $P \in \mathbb{Q}[T]$, of degree $\dim \mathcal{A}(\Delta)^{(\delta)} - 1$, such that:

$$P(m) = \dim_{\mathbb{k}}(\mathcal{A}(\Delta)_{\delta m}) \quad \forall \ m \gg 0.$$ 

Therefore, since $\dim_{\mathbb{k}}(\mathcal{A}(\Delta)_{\delta m}) = |\{\text{basic } \delta m\text{-cover of } \Delta\}|$, if $|\{\text{basic } k\text{-covers}\}| = O(k^{s-1})$, then $\dim(\mathcal{A}(\Delta)) \leq s$. 

Since $\mathcal{A}(\Delta)$ is finitely generated, $\mathcal{A}(\Delta)$ fin. gen. $\mathcal{A}(\Delta)^{(\delta)}$-module $\Rightarrow \dim \mathcal{A}(\Delta) = \dim \mathcal{A}(\Delta)^{(\delta)}$. 

Therefore, since $\dim_{\mathbb{k}}(\mathcal{A}(\Delta)_{\delta m}) = |\{\text{basic } \delta m\text{-cover of } \Delta\}|$, if $|\{\text{basic } k\text{-covers}\}| = O(k^{s-1})$, then $\dim(\mathcal{A}(\Delta)) \leq s$. 

If $\Delta$ is a matroid ...

The claim

$\text{Set}_d := \dim(\Delta) + 1$.

Being a matroid, $\Delta$ is pure, so:

$d$ is the cardinality of each facet of $\Delta$.

In order to show that $S/J(\Delta)$ is Cohen-Macaulay $\forall k \in \mathbb{Z}^+$, we have to prove that $\dim \bar{A}(\Delta) = d$.

So:

Claim: $|\{\text{basic } k\text{-covers of } \Delta\}| = O(k^d - 1)$. 
If $\Delta$ is a matroid ...

The claim

Set $d := \dim(\Delta) + 1$. 

$\forall k \in \mathbb{Z}^+$, we have to prove that $\dim \bar{A}(\Delta) = d$. 

Claim: $\left| \{ \text{basic } k\text{-covers of } \Delta \} \right| = O(k^{d-1})$. 

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In order to show that $S/J(\Delta)^k$ is Cohen-Macaulay $\forall \ k \in \mathbb{Z}_+$,
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In order to show that \( S/J(\Delta)^k \) is Cohen-Macaulay \( \forall \ k \in \mathbb{Z}_+ \), we have to prove that \( \dim \overline{A}(\Delta) = d \).
If $\Delta$ is a matroid ...

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Claim: $|\{\text{basic } k\text{-covers of } \Delta\}| = O(k^{d-1})$. 
If $\Delta$ is a $(d - 1)$-dimensional matroid ...
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If $\Delta$ is a $(d - 1)$-dimensional matroid ... 

The rigidity of the basic covers of a matroid

Let $\alpha$ be a basic $k$-cover of $\Delta$. Since $\alpha$ is basic, $\exists F \in \mathcal{F}(\Delta)$:
If $\Delta$ is a $(d - 1)$-dimensional matroid ...

Let $\alpha$ be a basic $k$-cover of $\Delta$. Since $\alpha$ is basic, $\exists \ F \in \mathcal{F}(\Delta)$:

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Sub-claim: $F$ fixes $\alpha$. I.e., all the values of $\alpha$ are determined by those on $F$. 

If $\Delta$ is a $(d - 1)$-dimensional matroid ...
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In fact, let $j_0$ be in $[n] \setminus F$. 

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In fact, let $j_0$ be in $[n] \setminus F$. Again, since $\alpha$ is basic, $\exists G \in \mathcal{F}(\Delta)$:
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Exchange property for matroids $\Rightarrow$ there exists $i_0 \in F$ such that
If $\Delta$ is a $(d-1)$-dimensional matroid ...

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$$j_0 \in G \quad \text{and} \quad \sum_{j \in G} \alpha(j) = k.$$

Exchange property for matroids $\Rightarrow$ there exists $i_0 \in F$ such that

(I) $F' := (F \setminus \{i_0\}) \cup \{j_0\} \in \mathcal{F}(\Delta)$ and (II) $G' := (G \setminus \{j_0\}) \cup \{i_0\} \in \mathcal{F}(\Delta)$. 
If $\Delta$ is a $(d - 1)$-dimensional matroid ...

The rigidity of the basic covers of a matroid

Let $\alpha$ be a basic $k$-cover of $\Delta$. Since $\alpha$ is basic, $\exists F \in \mathcal{F}(\Delta)$:

$$\sum_{i \in F} \alpha(i) = k.$$ 

Sub-claim: $F$ fixes $\alpha$. I.e., all the values of $\alpha$ are determined by those on $F$.

In fact, let $j_0$ be in $[n] \setminus F$. Again, since $\alpha$ is basic, $\exists G \in \mathcal{F}(\Delta)$:

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Exchange property for matroids $\Rightarrow$ there exists $i_0 \in F$ such that

(I) $F' := (F \setminus \{i_0\}) \cup \{j_0\} \in \mathcal{F}(\Delta)$ and (II) $G' := (G \setminus \{j_0\}) \cup \{i_0\} \in \mathcal{F}(\Delta)$.

(I) $\Rightarrow$ $\sum_{i \in F'} \alpha(i) \geq k$
If $\Delta$ is a $(d - 1)$-dimensional matroid ...

The rigidity of the basic covers of a matroid

Let $\alpha$ be a basic $k$-cover of $\Delta$. Since $\alpha$ is basic, $\exists F \in \mathcal{F}(\Delta)$:

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Therefore (I) and (II) together yield $\alpha(j_0) = \alpha(i_0)$. 

If $\Delta$ is a $(d - 1)$-dimensional matroid ...
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The conclusion

Let us recall that, since $F \in \mathcal{F}(\Delta)$, $|F| = d$. 
If \( \Delta \) is a \((d-1)\)-dimensional matroid ...

**The conclusion**

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\[
|\{(a_1, \ldots, a_d) \in \mathbb{N}^d : a_1 + \ldots + a_d = k\}| = \binom{k + d - 1}{k}.
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Therefore $\dim \tilde{A}(\Delta) = d = \dim \Delta + 1$. 
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So

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Therefore $\dim \bar{A}(\Delta) = d = \dim \Delta + 1$.

Hence $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for any $k \in \mathbb{Z}_+$.!
IF $S/J(\Delta)^{(k)}$ IS COHEN-MACAULAY FOR ALL $k \in \mathbb{Z}_+$...
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The polarization of a monomial ideal
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The polarization of a monomial ideal

Given a monomial $u \in S$, say $u := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, its polarization is:
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If $I \subseteq S$ is a monomial ideal with minimal monomial generators $u_1, \ldots, u_m$, ...
If $S/J(\Delta)(k)$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

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If $I \subseteq S$ is a monomial ideal with minimal monomial generators $u_1, \ldots, u_m$, its polarization is the square-free monomial ideal:
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If $I \subseteq S$ is a monomial ideal with minimal monomial generators $u_1, \ldots, u_m$, its polarization is the square-free monomial ideal:

$$\tilde{I} := (\tilde{u}_1, \ldots, \tilde{u}_m) \subseteq \tilde{S} := \mathbb{k}[x_{i,j} : i \in [n], j \in [\max_i\{\deg u_i\}]].$$
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Given a monomial $u \in S$, say $u := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, its polarization is:

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$\widetilde{I}$ has the same height and graded Betti numbers of $I$. 
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The polarization of a monomial ideal

Given a monomial $u \in S$, say $u := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, its polarization is:

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If $I \subseteq S$ is a monomial ideal with minimal monomial generators $u_1, \ldots, u_m$, its polarization is the square-free monomial ideal:

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$\tilde{I}$ has the same height and graded Betti numbers of $I$. In particular:

$S/I$ is Cohen-Macaulay $\iff \tilde{S}/\tilde{I}$ is Cohen-Macaulay.
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The associated primes of $\widehat{J(\Delta)}^{(k)}$
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The associated primes of $\overline{J(\Delta)^{(k)}}$

The trick is in understanding the polarization of $J(\Delta)^{(k)}$;
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ... 

The associated primes of $\widetilde{J(\Delta)}^{(k)}$

The trick is in understanding the polarization of $J(\Delta)^{(k)}$;

Since $\widetilde{J(\Delta)}^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \tilde{\mathfrak{p}}_F^k$, 

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The trick is in understanding the polarization of $J(\Delta)^{(k)}$;

Since $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \overline{\varnothing}_F^k$, we can focus in understanding:

$$\overline{\varnothing}_F^k = (\prod_{j=1}^k x_{i_1,j}, \prod_{j=1}^{k-1} x_{i_1,j} \cdot x_{i_2,1}, \ldots, \prod_{j=1}^k x_{i_d,j}),$$
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The trick is in understanding the polarization of $J(\Delta)^{(k)}$;

Since $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \widetilde{\wp}_F^k$, we can focus in understanding:

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\]

\( F := \{i_1, \ldots, i_d\} \ (d = \dim \Delta + 1) \).
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$F := \{i_1, \ldots, i_d\}$ ($d = \dim \Delta + 1$). We need to describe $\text{Ass}(\widetilde{\wp}_F^k)$:
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The associated primes of $\widetilde{J(\Delta)}^{(k)}$

The trick is in understanding the polarization of $J(\Delta)^{(k)}$;

Since $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \tilde{\wp}_F^k$, we can focus in understanding:

\[ \tilde{\wp}_F^k = \left( \prod_{j=1}^{k} x_{i_1,j}, \prod_{j=1}^{k-1} x_{i_1,j} \cdot x_{i_2,1}, \ldots, \prod_{j=1}^{k} x_{i_d,j} \right), \]

$F := \{i_1, \ldots, i_d\}$ ($d = \dim \Delta + 1$). We need to describe $\text{Ass}(\tilde{\wp}_F^k)$:

For each vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ with $1 \leq a_i \leq k$, set

\[ \wp_{F,a} := (x_{i_1,a_1}, x_{i_2,a_2}, \ldots, x_{i_d,a_d}) \subseteq \tilde{S}. \]
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ... 

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One can prove that for any prime ideal $\wp \subseteq \widetilde{S}$,
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The associated primes of $\mathcal{J}(\Delta)^{(k)}$

The trick is in understanding the polarization of $J(\Delta)^{(k)}$;

Since $J(\Delta)^{(k)} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathcal{S}^k_F$, we can focus in understanding:

$$\mathcal{S}^k_F = (\prod_{j=1}^k x_{i_1,j}, \prod_{j=1}^{k-1} x_{i_1,j} \cdot x_{i_2,1}, \ldots, \prod_{j=1}^k x_{i_d,j}),$$

$F := \{i_1, \ldots, i_d\}$ ($d = \text{dim} \Delta + 1$). We need to describe $\text{Ass}(\mathcal{S}^k_F)$:

For each vector $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$ with $1 \leq a_i \leq k$, set

$$\mathcal{S}_{F,a} := (x_{i_1,a_1}, x_{i_2,a_2}, \ldots, x_{i_d,a_d}) \subseteq \mathcal{S}.$$

One can prove that for any prime ideal $\mathfrak{p} \subseteq \mathcal{S}$,

$$\mathfrak{p} \in \text{Ass}(\mathcal{S}^k_F) \iff \mathfrak{p} = \mathcal{S}_{F,a} \text{ with } |a| = a_1 + \ldots + a_d \leq k + d - 1.$$
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The lack of connectedness
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The lack of connectedness

Assume by contradiction that $\Delta$ is not a matroid.
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Then there exist $F, G \in \mathcal{F}(\Delta), i \in F$, such that:
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$$(F \setminus \{i\}) \cup \{j\}$$

is not a facet of $\Delta$ for every $j \in G$. 

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Let us assume that \( F = \{i_1, \ldots, i_d\}, G = \{j_1, \ldots, j_d\} \) and \( i = i_1 \).
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Let us assume that $F = \{i_1, \ldots, i_d\}, G = \{j_1, \ldots, j_d\}$ and $i = i_1$.

Eventually, consider $\mathcal{H} := J(\Delta)^{(d+1)}$. $(d + 1 + d - 1 = 2d)$. 
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

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Eventually, consider $\mathcal{H} := J(\Delta)^{(d+1)}$. ($d + 1 + d - 1 = 2d$).

By the previous slide, $\varnothing_{F,a}$ and $\varnothing_{G,b}$ belong to $\text{Ass}(\mathcal{H})$, where

$a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

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Let us assume that $F = \{i_1, \ldots, i_d\}$, $G = \{j_1, \ldots, j_d\}$ and $i = i_1$.

Eventually, consider $H := J(\Delta)^{(d+1)}$. ($d + 1 + d - 1 = 2d$).

By the previous slide, $\wp_{F,a}$ and $\wp_{G,b}$ belong to $\text{Ass}(H)$, where $a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).
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Let us assume that $F = \{i_1, \ldots, i_d\}$, $G = \{j_1, \ldots, j_d\}$ and $i = i_1$.

Eventually, consider $\mathcal{H} := J(\Delta)^{(d+1)}$. $(d + 1 + d - 1 = 2d)$.

By the previous slide, $\varphi_{F,a}$ and $\varphi_{G,b}$ belong to $\text{Ass}(\mathcal{H})$, where $a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).

We will show that $R := \tilde{S}/\mathcal{H}$ is not Cohen-Macaulay,
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$, then the lack of connectedness.

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Let us assume that $F = \{i_1, \ldots, i_d\}$, $G = \{j_1, \ldots, j_d\}$ and $i = i_1$.

Eventually, consider $\mathcal{H} := J(\Delta)^{(d+1)}$. ($d + 1 + d - 1 = 2d$).

By the previous slide, $\mathcal{H}$ belongs to $\text{Ass}(\mathcal{H})$, where

$a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).

We will show that $R := \tilde{S}/\mathcal{H}$ is not Cohen-Macaulay, contradicting the hypothesis that $S/J(\Delta)^{(d+1)}$ is Cohen-Macaulay.
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The lack of connectedness

Assume by contradiction that $\Delta$ is not a matroid.

Then there exist $F, G \in \mathcal{F}(\Delta)$, $i \in F$, such that:

$$(F \setminus \{i\}) \cup \{j\} \text{ is not a facet of } \Delta \text{ for every } j \in G.$$ Let us assume that $F = \{i_1, \ldots, i_d\}$, $G = \{j_1, \ldots, j_d\}$ and $i = i_1$.

Eventually, consider $\mathcal{H} := J(\Delta)^{(d+1)}$. ($d + 1 + d - 1 = 2d$).

By the previous slide, $\wp_{F,a}$ and $\wp_{G,b}$ belong to $\text{Ass}(\mathcal{H})$, where $a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).

We will show that $R := \tilde{S}/\mathcal{H}$ is not Cohen-Macaulay, contradicting the hypothesis that $S/J(\Delta)^{(d+1)}$ is Cohen-Macaulay. Were it, $R_{\wp_{F,a} + \wp_{G,b}}$ would be Cohen-Macaulay too.
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The lack of connectedness

Assume by contradiction that $\Delta$ is not a matroid.

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By the previous slide, $\mathcal{O}_{F,a}$ and $\mathcal{O}_{G,b}$ belong to $\text{Ass}(\mathcal{H})$, where $a := (d + 1, 1, \ldots, 1) \in \mathbb{N}^d$ and $b := (2, 2, \ldots, 2) \in \mathbb{N}^d$ ($|a| = |b| = 2d$).

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Were it, $R_{\mathcal{O}_{F,a} + \mathcal{O}_{G,b}}$ would be Cohen-Macaulay too.

Particularly, $R_{\mathcal{O}_{F,a} + \mathcal{O}_{G,b}}$ would be connected in codimension 1.
If $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for all $k \in \mathbb{Z}_+$ ...

The conclusion

So, there should be a prime $\wp \in \text{Ass}(\mathcal{H})$ such that:
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(i) $\wp \subseteq \wp F,a + \wp G,b$;
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So, there should be a prime $\mathfrak{p} \in \text{Ass}(\mathcal{H})$ such that:

(i) $\mathfrak{p} \subseteq \mathfrak{p}_{F,a} + \mathfrak{p}_{G,b}$;

(ii) $\text{ht}(\mathfrak{p} + \mathfrak{p}_{F,a}) = d + 1$.
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In other words, there should be $p, q \in [d]$ such that:

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\wp = (x_{is,a_s}, x_{jq,b_q} : s \in [d] \setminus \{p\}).
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If $p \neq 1$, then $(d + 1) + 1 + \ldots + 1 + 2 > 2d$, a contradiction.
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If $p \neq 1$, then $(d + 1) + \underbrace{1 + \ldots + 1}_{d-2} + 2 > 2d$, a contradiction.

So, $\Delta$ has to be a matroid!
A related problem

As we noticed, for any $\Delta$, we have $\dim \overline{A}(\Delta) \geq \dim \Delta + 1$. We showed that equality holds true exactly when $\Delta$ is a matroid. So, the following is a natural problem: Looking for a combinatorial characterization of $\dim \overline{A}(\Delta)$. Together with Constantinescu, we solved this problem when $\dim \Delta = 1$, that is when $\Delta = G$ is a graph. A little more precisely, we defined an invariant of $G$, called ordered matching number and denoted by $\nu^\circ(G)$, and we showed that $\dim \overline{A}(G) = \nu^\circ(G) + 1$. Already in this case things are complicated!
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References


