SYMBOLIC POWERS AND MATROIDS

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Preliminaries and notation

Simplicial complexes

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This fact implies that $\dim \mathbb{k}[\Delta] = \dim \Delta + 1$, where $\mathbb{k}[\Delta] := S/I_\Delta$.

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When is $S/I_\Delta^{(k)}$ Cohen-Macaulay for all $k \in \mathbb{Z}_+$???

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It is fair to say that *Minh* and *Trung* proved at the same time the same result. However the two proofs are completely different.

THE PROOF

Stanley-Reisner ideals $\longrightarrow$ cover ideals

Properties of matroids

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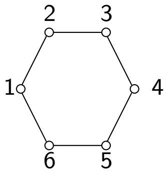

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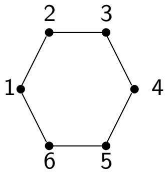

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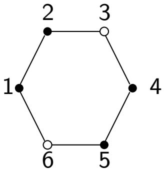

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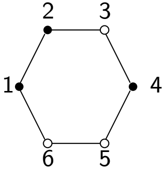

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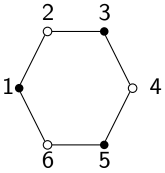

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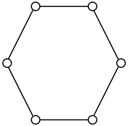

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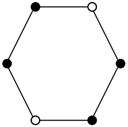

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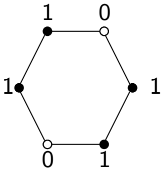

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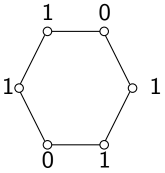

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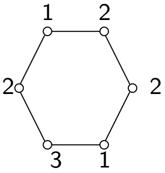

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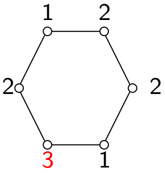

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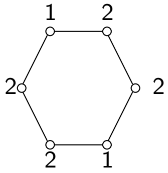

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In the next slides we are going to show that:

$$\dim \bar{A}(\Delta) = \dim \Delta + 1 \text{ whenever } \Delta \text{ is a matroid.}$$

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Claim: $|\{\text{basic } k\text{-covers of } \Delta\}| = O(k^{d-1})$.

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IF $S/J(\Delta)^{(k)}$ IS COHEN-MACAULAY FOR ALL $k \in \mathbb{Z}_+$...

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The polarization of a monomial ideal

Given a monomial $u \in S$, say $u := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, its *polarization* is:

$$\widetilde{u} := \prod_{j=1}^{\alpha_1} x_{1,j} \cdot \prod_{j=1}^{\alpha_2} x_{2,j} \cdots \prod_{j=1}^{\alpha_n} x_{n,j} \in \Bbbk[x_{i,j} : i \in [n],\ j \in [\deg u]].$$

If $I \subseteq S$ is a monomial ideal with minimal monomial generators $u_1, \ldots, u_m$, its polarization is the square-free monomial ideal:

$$\widetilde{I} := (\widetilde{u_1}, \ldots, \widetilde{u_m}) \subseteq \widetilde{S} := \Bbbk[x_{i,j} : i \in [n],\ j \in [\max_i\{\deg u_i\}]].$$

$\widetilde{I}$ has the same height and graded Betti numbers of I. In particular:

$$S/I \text{ is Cohen-Maculay} \Leftrightarrow \widetilde{S}/\widetilde{I} \text{ is Cohen-Macaulay}.$$

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Already in this case things are complicated!

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