SYMBOLIC POWERS AND MATROIDS

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Preliminaries and notation

Stanley-Reisner ideals

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$$I^{(k)} = \bigcap_{A \in \mathcal{F}} \wp_A^k.$$

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Therefore it is natural to ask:

When is $S/I_\Delta^{(k)}$ Cohen-Macaulay for all positive integers k???

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SKETCH OF THE PROOF

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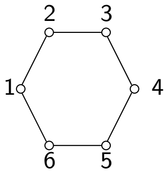

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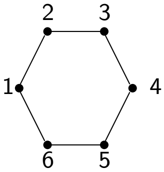

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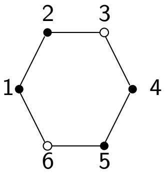

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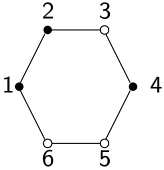
EXAMPLES: 

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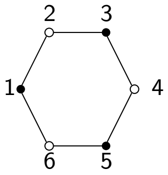

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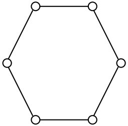

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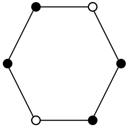

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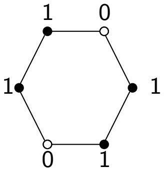

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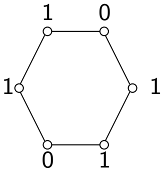

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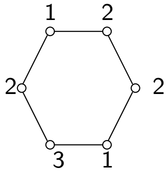

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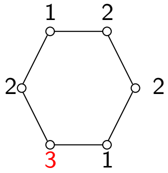

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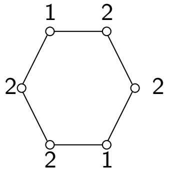

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D e f i n i t i o n

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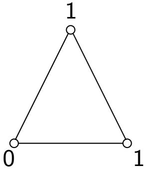

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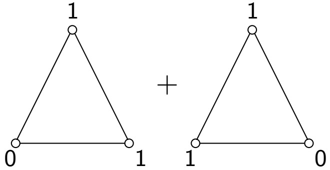

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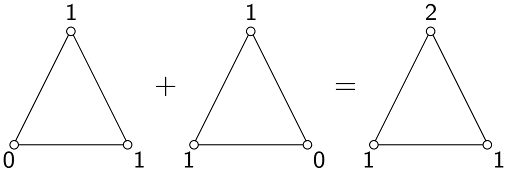

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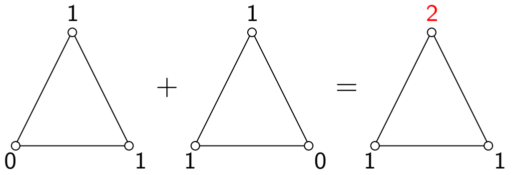

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Using tools and result from blow-up algebras, we have:

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Notice that $\dim S/J(\Delta)^{(k)} = \dim S/J(\Delta) = n - \dim \Delta - 1$. So to show that all the rings $S/J(\Delta)^{(k)}$ are Cohen-Macaulay, that is equivalent to show that $\min_{k \in \mathbb{N}_{>0}} \{\mathrm{depth}(S/J(\Delta)^{(k)})\} = n - \dim \Delta - 1$, we have to show that $\dim \bar{A}(\Delta) \leq \dim \Delta + 1$.

In the next slides we will show that:

$\dim \bar{A}(\Delta) \leq \dim \Delta + 1$ whenever Δ is a matroid.

The algebra of basic covers

A combinatorial description of $\dim \bar{A}(\Delta)$

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we have to say that the basic k-covers of matroids are "few":

More precisely, our purpose is to prove that if Δ is a matroid of dimension $d-1$, then $|\{\text{basic } k\text{-covers of } \Delta\}| = O(k^d)$.

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Therefore (I) and (II) together yield $\alpha(j_0) = \alpha(i_0)$.

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Hence $S/J(\Delta)^{(k)}$ is Cohen-Macaulay for any k !!!