ON THE DUAL GRAPH OF COHEN-MACAULAY ALGEBRAS

Joint with Bruno Benedetti

Matteo Varbaro

Università degli Studi di Genova

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Throughout, we assume that X is equidimensional, i.e. $\dim(X_i) = \dim(X)$ for all $i = 1, \ldots, s$.

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- What is the length of such a path?
- How many paths are there between two vertices?

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(Adiprasito and Benedetti, 2013): True if I is a monomial ideal.

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Hirsch conjecture, 1957: If Δ is the boundary of a d-polytope on n vertices, then

$$\operatorname{diam}(G(\Delta)) \leq n - d.$$

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In view of this history, we say that $X \subseteq \mathbb{P}^n$ is Hirsch if

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Theorem: If X is a union of lines, no three of which meet at the same point, and the embedding $X \subseteq \mathbb{P}^n$ is provided by the canonical series of X, then X is Hirsch.

Subspace arrangements

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In a sense, this justifies to take care, for our aims, especially of subspace arrangements.

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Corollary (Balinski): If Δ is the triangulation of a d-sphere, then $G(\Delta)$ is $(d+1)$-connected.

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- A theorem due to Hartshorne-Schenzel in Liaison Theory.
- The following result of Derksen and Sidman: If

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s$$

where the $\mathfrak{p}_i$'s are generated by linear forms, then

$$\operatorname{reg}(S/I) \leq s.$$