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PARTITIONS OF SINGLE EXTERIOR TYPE

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Abstract. We characterize the irreducible representations of the general linear group $\mathrm{GL}(V)$ that have multiplicity 1 in the direct sum of all Schur modules of a given exterior power of V. These have come up in connection with the relations of the lower order minors of a generic matrix. We show that the minimal relations conjectured by Bruns, Conca and Varbaro are exactly those coming from partitions of single exterior type.

1. Introduction

The main motivation for this note was the desire to provide further evidence for a conjecture of Conca and the authors [BCV, Conj. 2.12] on the polynomial relations between the t-minors of a generic matrix. With the notation in [BCV], let $X = (x_{ij})$ denote an $m \times n$ matrix of indeterminates over a field K of characteristic 0, $R = K[X]$ the polynomial ring over the variables x_{ij} and $A_t \subseteq R$ the K-subalgebra of R generated by the t-minors of X. With respect to a choice of bases in K-vector spaces V and W of dimension m and n, respectively, one has a natural action of the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ on R, induced by

$$(A, B) \cdot X = AXB^{-1} \qquad \forall\ A \in \mathrm{GL}(V), B \in \mathrm{GL}(W).$$

This action restricts to A_t, making A_t a G-algebra. Since the G-decomposition of A_t can be deduced from the work of De Concini, Eisenbud and Procesi [DEP], it is natural to exploit such an action. A presentation of A_t as a quotient of a polynomial ring is provided by the natural projection

$$\pi : S_t \to A_t,$$

where $S_t = \mathrm{Sym}(\bigwedge^t V \otimes \bigwedge^t W^*)$. Also S_t is a G-algebra, and the map π is G-equivariant. Therefore the ideal of relations $J_t = \mathrm{Ker}(\pi)$ is a G-module as well.

The conjecture [BCV, Conj. 2.12] predicts a minimal list of irreducible G-modules generating J_t, or, by Nakayama's lemma, the decomposition of

$$J_t \otimes_{S_t} K$$

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where we identify K and the residue class field of R with respect to the irrelevant maximal ideal generated by the indeterminates. In particular, the conjecture predicts that J_t is generated in degrees 2 and 3.

In the assignment of partitions to Young diagrams and to irreducible representations of $\mathrm{GL}(V)$ we follow Weyman [We]: a partition of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \geq \cdots \geq \lambda_k$, is pictorially represented by k rows of boxes of lengths $\lambda_1, \dots, \lambda_k$ with coordinates in the fourth quadrant, and a single row of length m represents $\bigwedge^m V$. The highest weight of the representation is then given by the transpose partition ${}^{\mathrm{t}}\lambda$ in which rows and columns are exchanged: $({}^{\mathrm{t}}\lambda)_i = |\{j : \lambda_j \geq i\}|$. With this convention, we denote the Schur module associated with the partition λ and the vector space V by $L_\lambda V$.

Because S_t is a quotient of

$$T_t = \bigoplus_{d\geq 0}\Big(\bigotimes^d\Big(\bigwedge^t V \otimes \bigwedge^t W^*\Big)\Big) = \bigoplus_{d\geq 0}\Big(\bigotimes^d \bigwedge^t V \otimes \bigotimes^d \bigwedge^t W^*\Big),$$

Pieri's rule implies that the irreducible summands of $J_t \otimes_{S_t} K$ must be of the form

$$L_\gamma V \otimes L_\lambda W^*$$

where γ and λ are partitions satisfying the following conditions:

(i) $\gamma, \lambda \vdash dt$,
(ii) both γ and λ have at most d rows.

We call such partitions (or *bipartitions* $(\gamma|\lambda)$) (t,d)*-admissible* (just t-admissible if we do not need to emphasize the degree). In [BCV] a set A of $(t,2)$-admissible bipartitions $(\gamma|\lambda)$ and a set B of $(t,3)$-admissible bipartitions $(\gamma|\lambda)$ were found such that

$$\bigoplus_{(\gamma|\lambda)\in A} L_\gamma V \otimes L_\lambda W^* \;\oplus\; \bigoplus_{(\gamma|\lambda)\in B} L_\gamma V \otimes L_\lambda W^* \subseteq J_t \otimes_{S_t} K. \tag{1.1}$$

Conjecture 2.12 in [BCV] states that the inclusion in Equation (1.1) is an equality. For the convenience of the reader and since it is crucial for the following we recall how A and B are defined.

(i) For $u \in \{0, \dots, t\}$ let:

$$\tau_u = (t+u, t-u).$$

(ii) For $u \in \{1, \dots, \lfloor t/2 \rfloor\}$ let

$$\gamma_u = (t+u,\ t+u,\ t-2u) \qquad \text{and} \qquad \lambda_u = (t+2u,\ t-u,\ t-u).$$

(iii) For each $u \in \{2, \dots, \lceil t/2 \rceil\}$ let

$$\rho_u = (t+u,\ t+u-1,\ t-2u+1) \qquad \text{and} \qquad \sigma_u = (t+2u-1,\ t-u+1,\ t-u).$$

With this notation,

$$\begin{aligned}A &= \big\{(\tau_u|\tau_v) : 0 \le u, v \le t,\ u+v \text{ even}, u \ne v\big\},\\ B &= \big\{(\gamma_u|\lambda_u), (\lambda_u|\gamma_u) : 1 \le u \le \lfloor t/2 \rfloor\big\}\\ &\qquad\qquad \cup \big\{((\rho_v|\sigma_v), (\sigma_v|\rho_v) : 2 \le v \le \lceil t/2 \rceil\big\}.\end{aligned}$$

Note that not all the partitions above are supported by the underlying vector spaces if their dimensions are too small: a partition λ can only appear in a representation of $\mathrm{GL}(V)$ if $\lambda_1 \le \dim V$. For simplicity we have passed this point over since it is essentially irrelevant. The reader is advised to remove all partitions from the statements that are too large for the vector spaces under consideration.

The decomposition of S_t as a module over the "big" group

$$H = \mathrm{GL}(E) \times \mathrm{GL}(F), \quad E = \bigwedge^t V,\ \ F = \bigwedge^t W,$$

is well known by Cauchy's rule:

$$S_t = \bigoplus_\mu L_\mu E \otimes L_\mu F^* \tag{1.2}$$

where μ is extended over all partitions. The $\mathrm{GL}(V)$-decomposition of $L_\mu E$ is an essentially unsolved plethysm. However, the partitions in the definition of A and B play a very special role in it, as was already observed in [BCV]:

Definition 1.1. Let $\lambda \vdash dt$ be t-admissible. Then λ is said to be of *single* $\bigwedge^t$*-type* μ if $\mu \vdash d$ is the only partition such that $L_\lambda V$ is a direct summand of $L_\mu(\bigwedge^t V)$ and, moreover, has multiplicity 1 in it. Without specifying μ, notice that λ is of single $\bigwedge^t$-type if and only if λ has multiplicity 1 in $\bigoplus_{\alpha\vdash d} L_\alpha(\bigwedge^t V)$.

In this note we will classify all partitions of single $\bigwedge^t$-type (or simply *single exterior type*) and show that the bi-partitions in the sets A and B are exactly those of single $\bigwedge^t$-type that occur in a minimal generating set of J_t. While this observation does certainly not prove the conjecture in [BCV], it provides further evidence for it.

2. Auxiliary results on partitions

In this section we discuss two transformations of partitions that preserve single exterior type. It was already observed in [BCV] that trivial extensions in the following sense are irrelevant: if a partition $\tilde\lambda$ arises from a t-admissible partition $\lambda \vdash dt$ by prefixing λ with columns of length d, then $\tilde\lambda$ is called a *trivial extension* of λ. We quote [BCV, 1.16] (e_λ denotes the multiplicity of λ):

Proposition 2.1. *Let μ be a partition of d and consider partitions $\lambda=(\lambda_1,\dots,\lambda_k)$ $\vdash td$ with $k \le d$ and $\tilde\lambda = (\lambda_1+1,\dots,\lambda_k+1,1,\dots,1) \vdash dt+d$. If $\dim_K V \ge \lambda_1+1$, then*

$$\mathrm{e}_\lambda\Big(L_\mu\Big(\bigwedge^t V\Big)\Big) = \mathrm{e}_{\tilde\lambda}\Big(L_\mu\Big(\bigwedge^{t+1} V\Big)\Big).$$

In particular, λ is of single $\bigwedge^t$-type μ if and only if $\tilde\lambda$ is of single $\bigwedge^{t+1}$-type μ.

Next we want to show that a similar result holds for dualization, in the sense that $\bigwedge^{n-t} V$, $n = \dim V$, is dual to $\bigwedge^t V$ (up to tensoring with the determinant). Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash td$ be t-admissible; then we set

$$\lambda^{*,n} = (n - \lambda_d, \dots, n - \lambda_1) \vdash (n-t)d.$$

Evidently $\lambda^{*,n}$ is $(n-t)$-admissible. Note that λ and $\lambda^{*,n}$ rotated by 180° degrees complement each other to a $d \times n$ rectangle (representing the d-th tensor power of the determinant $\det V = \bigwedge^n V$ when $n = \dim V$).

Notice that $\lambda^{*,n}$ is a trivial extension of λ^{*,λ_1}. In view of this we will denote λ^{*,λ_1} just with λ^*, calling it simply the *dual* of λ. Also, note that if $k = d$, so that λ is a trivial extension of some γ, then $\lambda^{*,n} = \gamma^{*,n}$. Therefore, when speaking of dual partitions, we will usually assume that $n = \lambda_1$ and $k < d$.

Proposition 2.2. *Let μ be a partition of d and consider a t-admissible partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash td$. Suppose $\dim V = \lambda_1$. Then*

$$\mathrm{e}_\lambda\Big(L_\mu \bigwedge^t V\Big) = \mathrm{e}_{\lambda^*}\big(L_\mu(\bigwedge^{\lambda_1 - t} V\big).$$

In particular, λ is of single $\bigwedge^t$-type if and only if λ^* is of single $\bigwedge^{\lambda_1-t}$-type.

Proof. Set $n = \dim V = \lambda_1$. Consider the $\mathrm{GL}(V)$-equivariant multiplication

$$\bigwedge^t V \otimes \bigwedge^{n-t} V \to \det V.$$

It induces an equivariant isomorphism

$$\bigwedge^t V \cong \mathrm{Hom}_K\Big(\bigwedge^{n-t} V, \det V\Big) = \Big(\bigwedge^{n-t} V\Big)^* \otimes \det V = \Big(\bigwedge^{n-t} V^*\Big) \otimes \det V.$$

Next we can pass to the d-th tensor power on the right and the left, and apply the Young symmetrizer $\mathbb{Y}_\mu$ (see Fulton and Harris [FH, p. 46] inverting rows and columns) to obtain a $\mathrm{GL}(V)$-equivariant isomorphism

$$\mathbb{Y}_\mu \bigotimes^d \bigwedge^t V \cong \mathbb{Y}_\mu \bigotimes^d \Big(\bigwedge^{n-t} V^* \otimes \det V\Big).$$

Next we can go from $\mathbb{Y}_\mu \bigotimes^d (\bigwedge^{n-t} V^* \otimes \det V)$ to $\mathbb{Y}_\mu \bigotimes^d \bigwedge^{n-t} V^*$, except that we have to subtract the weight of $\bigotimes^d \det V$ from each weight in $\mathbb{Y}_\mu \bigotimes^d (\bigwedge^{n-t} V^* \otimes \det V)$. Finally, if we replace $\mathrm{GL}(V)$ by $\mathrm{GL}(V^*)$ as the acting group, we see that every partition λ in $\mathbb{Y}_\mu \bigotimes^d \bigwedge^t V$ goes with equal multiplicity to the partition λ^* in $\mathbb{Y}_\mu \bigotimes^d \bigwedge^{n-t} V^*$. But the multiplicities depend only on the dimension of the basic vector space, and therefore we can replace $\bigwedge^{n-t} V^*$ by $\bigwedge^{n-t} V$. $\square$

Below we will use the obvious generalization of Proposition 2.2 to $\lambda^{*,n}$ that results from Proposition 2.1.

3. Partitions of single exterior type

The characterization of partitions of single exterior type is based on a recursive criterion established in [BCV]. For it and also for the characterization of the minimal relations of single exterior type we need the same terminology.

Let λ be a (t,d)-admissible diagram. Given $1 \le e \le d$, we say that α is a (t,e)-*predecessor* of λ if and only if α is a $(t,d-e)$-admissible diagram such that ${}^{\mathrm{t}}\alpha_i \le {}^{\mathrm{t}}\lambda_i \le {}^{\mathrm{t}}\alpha_i + e$ for all $i = 1,\dots,\lambda_1$ (we set ${}^{\mathrm{t}}\alpha_i = 0$ if $i > \alpha_1$). In such a case we also say that λ is a (t,e)-*successor* of α. If we just say that α is a t-*predecessor* of λ, we mean that α is a (t,e)-predecessor of λ for some e, and analogously for λ being a t-*successor* of α. (This terminology deviates slightly from [BCV] where a predecessor is necessarily a $(t,1)$-predecessor.) The Littlewood–Richardson rule implies at once that, for a (t,d)-admissible diagram λ and a $(t,d-e)$-admissible diagram α the following are equivalent:

(i) α is a (t,e)-predecessor of λ.
(ii) $L_\lambda V$ occurs in $(\bigotimes^e \bigwedge^t V) \otimes L_\alpha V$, where V is a K-vector space of dimension $\ge \lambda_1$.

Below we will use that dualization commutes with taking successors. More precisely, if λ is a successor of γ, then $\lambda^{*,n}$ is a successor of $\gamma^{*,n}$.

We quote the following criterion for single $\bigwedge^t$-type from [BCV, Prop. 1.22]. (Condition (iv) has been added here. It strengthens (iii), but follows from (iii) by induction.)

Proposition 3.1. *Let $\lambda \vdash dt$ and $\mu \vdash d$ be partitions such that $L_\lambda V$ occurs in $L_\mu(\bigwedge^t V)$. Then the following are equivalent:*

(i) *λ is of single $\bigwedge^t$-type;*
(ii) *the multiplicities of λ and of μ in $\bigotimes^d(\bigwedge^t V)$ coincide;*
(iii) *every $(t,1)$-predecessor λ' of λ is of single $\bigwedge^t$-type μ' where μ' is a $(1,1)$-predecessor of μ, and no two distinct $(t,1)$-predecessors of λ share the same $(1,1)$-predecessor μ' of μ;*
(iv) *every t-predecessor λ' of λ is of single $\bigwedge^t$-type μ' where μ' is a 1-predecessor of μ, and no two distinct t-predecessors of λ share the same 1-predecessor μ' of μ.*

As we will see in a moment, one class of single $\bigwedge^t$-type partitions is given by the hooks.

Definition 3.2. A diagram $\lambda = (\lambda_1,\dots,\lambda_k)$ with $\lambda_2 \le 1$ is called a *hook*.

A hook can be always written like $(a,1^b)$, where 1^b means b ones.

Lemma 3.3. *Let $d > 0$ and $k \in \{0,\dots,d-1\}$. Then $(td-k,1^k)$ is of single $\bigwedge^t$-type μ where:*

(i) *$\mu = (d-k,1^k)$ if t is odd.*
(ii) *$\mu = (k+1,1^{d-k-1})$ if t is even.*

Proof. Let us fix t and use induction on d. For $d = 2$ the statement is very easy to prove. For $d = 3$ [BCV, Prop. 1.18] implies that $L_{(3t-1,1,1)}V$ occurs in $L_{(2,1)}(\bigwedge^t V)$, so we are done in this case by Proposition 3.1 (ii). Therefore assume $d > 3$.

If t is odd, then $L_{(dt)}V$ occurs in $L_{(d)}(\bigwedge^t V)$: In fact, $L_{(dt)}V$ has multiplicity 1 in $\bigotimes \bigwedge^t V$, so it can occur in $L_\mu(\bigwedge^t V)$ only if $\mu = (d)$ (the dth exterior power) or $\mu = (1^d)$ (the d-th symmetric power). Furthermore $(2t)$ is a t-predecessor of (dt), and $\bigwedge^{2t} V$ occurs in $\bigwedge^2(\bigwedge^t V)$ (for instance, see [BCV, Lemma 2.1]). Therefore $L_{(dt)}V$ occurs in $\bigwedge^2(\bigwedge^t V) \otimes (\bigotimes^{d-2} \bigwedge^t V)$. In particular, it cannot occur in $L_{(1^d)}(\bigwedge^t V)$. In the same way, one sees that $L_{(dt)}V$ occurs in $L_{(1^d)}(\bigwedge^t V)$ whenever t is even.

From now on let us assume t odd; the even case is similar. If $0 < k < d-1$, then $(dt-k, 1^k)$ has two $(t,1)$-predecessors, namely,

$$((d-1)t-k, 1^k) \qquad \text{and} \qquad ((d-1)t-k+1, 1^{k-1}).$$

By induction, the respective Schur modules occur in

$$L_{(d-k-1,1^k)}\Big(\bigwedge^t V\Big) \qquad \text{and} \qquad L_{(d-k,1^{k-1})}\Big(\bigwedge^t V\Big).$$

So, the Schur modules corresponding to the $(t,1)$-successors of $((d-1)t-k, 1^k)$ can occur in $L_{(d-k,1^k)}(\bigwedge^t V)$ or in $L_{(d-k+1,1^{k-1})}(\bigwedge^t V)$, and the ones corresponding to the $(t,1)$-successors of $((d-1)t-k+1, 1^{k-1})$ can occur in $L_{(d-k+1,1^{k-1})}(\bigwedge^t V)$ or in $L_{(d-k,1^k)}(\bigwedge^t V)$. By counting multiplicities and using $d > 3$, one can check that the only possibility is that $L_{(dt-k,1^k)}V$ occurs in $L_{(d-k,1^k)}(\bigwedge^t V)$. Notice that the multiplicity of $L_{(dt-k,1^k)}V$ is the same as the one of $L_{(d-k,1^k)}(\bigwedge^t V)$ in $\bigotimes \bigwedge^t V$, i.e., $\binom{d-1}{k}$, so Proposition 3.1 (ii) lets us conclude. □

We must pay particular attention to the duals of hooks: The dual of the hook $(dt-k, 1^k) \vdash dt$ is the diagram $((dt-k)^{d-k-1}, (dt-k-1)^k) \vdash d(dt-k-t)$. Notice that is the unique partition of $d(dt-k-1)$ with $\lambda_d = 0$ and $\lambda_{d-1} \geq \lambda_1 - 1$.

Before stating the main theorem it is useful to remark the following:

Lemma 3.4. *A diagram $(a,b,c) \vdash 3t$ (where $c = 0$ is not excluded) is of single $\bigwedge^t$-type if and only if*

$$\min\{a-b, b-c\} \leq 1.$$

Since all partitions $\lambda \vdash 2t$ are of single $\bigwedge^t$-type, one must find exactly those partitions $(a,b,c) \vdash 3t$ that have no two predecessors in the second symmetric or second exterior power. Since the latter are easily characterized (for example, see [BCV, Lemma 2.1]), the proof of Lemma 3.4 is an easy exercise. Because of Proposition 2.1 one may assume $c = 0$, and Proposition 2.2 helps to further reduce the number of cases.

For the proof of the next theorem we will abbreviate "single $\bigwedge^t$-type" by "ST" and "not of single $\bigwedge^t$-type" by "NST".

Theorem 3.5. *A t-admissible diagram $\lambda = (\lambda_1, \dots, \lambda_k) \vdash dt$ is of single $\bigwedge^t$-type $\mu \vdash d$ if and only if it satisfies one (or more) of the following:*

(i) $\lambda_d \geq t-1$, *in which case* $\mu = (\lambda_1 - t + 1, \dots, \lambda_d - t + 1)$.
(ii) $\lambda_1 \leq t+1$, *in which case* $\mu = \lambda^{*,t+1}$.

(iii) $\lambda_d \geq \lambda_2 - 1$. *If* $\lambda = (t^d)$, *then* $\mu = (1^d)$. *Otherwise put* $k = \max\{i : \lambda_i > \lambda_d\}$*: according with* $t - \lambda_d$ *being odd or even,* $\mu = (d-k,\ 1^k)$ *or* $\mu = (k+1,\ 1^{d-k-1})$.

(iv) $\lambda_{d-1} \geq \lambda_1 - 1$. *If* $\lambda = (t^d)$, *then* $\mu = (1^d)$. *Otherwise put* $k = \min\{i : \lambda_i < \lambda_1\}$*: according with* $\lambda_1 - t$ *being odd or even,* $\mu = (k,\ 1^{d-k})$ *or* $\mu = (d-k+1,\ 1^{k-1})$.

If λ is in one of the four classes above, then we know that it is of single $\bigwedge^t$-type from what was done until now: (i) If $\lambda_d \geq t-1$, then it is a trivial extension of $\mu = (\lambda_1 - t + 1, \lambda_2 - t + 1, \ldots, \lambda_d - t + 1) \vdash d$, that is obviously of single $\bigwedge^1$-type; (ii) if $\lambda_1 \leq t+1$, then $\mu = \lambda^{*,t+1} \vdash d$ is of single $\bigwedge^1$-type, so Proposition 2.2 lets us conclude; (iii) If $\lambda_d \geq \lambda_2 - 1$, then λ is a trivial extension of a hook. The shape of μ follows from Proposition 2.1 and Lemma 3.3; (iv) if $\lambda_{d-1} \geq \lambda_1 - 1$, then λ^* is a hook. From this, combining Lemma 3.3 and Proposition 2.2, we get the shape of μ.

As we have just seen, the four classes can be described as follows: (i) consists of the trivial extensions of 1-admissible partitions, (ii) is dual to (i) in the sense of Proposition 2.2, (iii) contains the hooks and their trivial extensions, and (iv) is dual to (iii).

The classification in the theorem completely covers the cases $d = 1$ and $d = 2$, in which all shapes are of single $\bigwedge^t$-type, and also the case $d = 3$ done in Lemma 3.4. Therefore we may assume that $d \geq 4$. Then the theorem follows from the next lemma and Proposition 3.1. In its proof we will use the theorem inductively.

Lemma 3.6. *If* $d \geq 4$ *and* λ *is not one of the types in the theorem, then it has an NST* $(t,1)$*-predecessor.*

The lemma shows that the critical degree is $d = 3$ in which the condition that the predecessors of λ occur in pairwise different predecessors of μ must be used.

Proof. If $t = 1$ all partitions λ fall into the class (i) and are certainly ST. So we can assume $t \geq 2$.

Suppose first that λ is itself a trivial extension. Then we pass to its trivial reduction λ'. It is enough to find an NST predecessor for λ'. It yields an NST predecessor of λ after trivial extension. From now on we can assume that λ has at most $d-1$ rows.

Suppose first that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a successor of a hook. Let $k' = \max\{2, k-1\}$. We choose $\gamma = ((d-1)t - k', 2, 1^{k'-2}) \vdash (d-1)t$. Then γ does not fall into one of the classes (i)–(iv), provided $\gamma_1 \geq t+2$. Using $k' \leq d-2$, one derives this immediately from $d \geq 4$ and $t \geq 2$. The inequality $\gamma_1 \geq t+1$ is sufficient to make γ a predecessor of λ.

Next suppose $\lambda = (\lambda_1, \lambda_2, 1^{k-2}) \vdash dt$. If λ has a hook predecessor, then we are done by the previous case. Therefore we can assume that $\lambda_2 \geq t+2$. If $k = 2$, we pass to $\gamma = (\lambda_1, \lambda_2 - t)$, and if $k \geq 3$, we choose $\gamma = (\lambda_1, \lambda_2 - (t-1), 1^{k-3})$. Then γ is not of types (i)–(iv). (We are dealing with this case separately since the duals will come up below.)

In the remaining case we choose the predecessor γ of λ with the lexicographic smallest set of indices for the columns in which γ and λ differ by 1. If γ is a hook, then we are done as above. So we can assume that γ is not a hook.

Suppose that $\gamma_1 < \lambda_1$. Then $\lambda_2 \leq t-1$, and γ is not a trivial extension since the bottom row of λ has been removed completely, and γ has at most $d-2$ rows. On the other hand, $\lambda_1 + (d-2)\lambda_2 \geq dt$ implies $\lambda_1 \geq 2t+2$, and so $\gamma_1 \geq t+2$. It follows that $\gamma_{d-2} \leq \lambda_{d-2} < \gamma_1 - 1$, and γ is not of type (i)–(iv).

The case $\gamma_1 = \lambda_1 > t+1$ remains. We can assume that γ is ST. This is only possible if (1) $\gamma_{d-2} \geq \gamma_1 - 1$, or (2) γ is the trivial extension of a hook, or (3) $\gamma_{d-1} \geq t-1$.

(1) If $\gamma_{d-2} \geq \gamma_1 - 1$, then $\lambda_{d-2} \geq \lambda_1 - 1$, and λ^* is of the second type discussed. We find an NST predecessor of λ^* and dualize back.

(2) If γ is a trivial extension of a hook, then $\gamma_2 \leq \gamma_{d-1}+1$ and $\lambda_{d-1} \geq t+1$. In particular $\gamma_2 = \lambda_2$, and $\gamma_{d-1} = \lambda_{d-1} - t \leq \lambda_2 - t = \gamma_2 - t$, which is a contradiction since $t \geq 2$.

(3) In this case we must have $\lambda_{d-1} \geq 2t-1$ since we remove $\min\{\lambda_{d-1}, t\}$ boxes from row $d-1$ of λ. This is evidently impossible (because $t \geq 2$ and $d \geq 4$). $\square$

4. Minimal relations of single exterior type

In this last section we are going to prove the result which motivated us to produce this note. We will adopt here the notation given in the introduction.

Let us first recall a result of [BCV]. As already mentioned, a decomposition of $S_t = \mathrm{Sym}(E \otimes F^*)$ in irreducible H-representations is provided by the Cauchy formula (1.2), namely,

$$S_t = \bigoplus_{\mu} L_\mu E \otimes L_\mu F^*,$$

where μ ranges among all the partitions. So, because G is a subgroup of H whose action is the restriction of that of H, the irreducible G-representation $L_\gamma V \otimes L_\lambda W^*$ occurs in the G-decomposition of S_t if and only if there exists $\mu \vdash d$ such that $L_\gamma V$ occurs in the $\mathrm{GL}(V)$-decomposition of $L_\mu(\bigwedge^t V)$ and $L_\lambda W^*$ occurs in the $\mathrm{GL}(W)$-decomposition of $L_\mu(\bigwedge^t W^*)$. Moreover, if such a $\mu \vdash d$ exists and γ and λ are both of single $\bigwedge^t$-type, then $L_\gamma V \otimes L_\lambda W^*$ is a direct summand of $J_t \otimes_{S_t} K$ if and only if $\gamma \neq \lambda$ and the predecessors of γ and of λ coincide [BCV, Prop. 1.21 and Thm. 1.23(iv)]. This is the fact on which the proof of the next theorem is based.

Theorem 4.1. *Let $L_\gamma V \otimes L_\lambda W^*$ be a direct summand of $J_t \otimes_{S_t} K$ such that both γ and λ are diagrams of single $\bigwedge^t$-type. Then $(\gamma|\lambda) \in A \cup B$.*

Proof. For $t = 1$ there is nothing to prove because $J_1 = (0)$. So assume $t \geq 2$.

From what was said above γ and λ must be t-admissible partitions of the same number dt. If $d = 1$ then $\gamma = \lambda = (t)$; if $d = 2$, then $(J_t)_2 \cong \bigoplus_{(\gamma|\lambda)\in A} L_\gamma V \otimes L_\lambda W^*$ by [BCV, Lemma 2.1]; if $d = 3$ then [BCV, Prop. 3.16] does the job.

So from now on we will focus on $d \geq 4$. Recall that in Theorem 3.5 there have been identified 4 (not disjoint) sets of diagrams, say $E_1^t = \{\text{diagrams as in (i)}\}$, $E_2^t = \{\text{diagrams as in (ii)}\}$, and so on, such that:

$$\{\text{diagrams of single } {\textstyle\bigwedge^t}\text{-type}\} = E_1^t \cup E_2^t \cup E_3^t \cup E_4^t.$$

We start by showing the following:

Lemma 4.2. *Let $L_\gamma V \otimes L_\lambda W^*$ be a direct summand of $J_t \otimes_{S_t} K$ such that both γ and λ are diagrams of single $\bigwedge^t$-type belonging to the same E_i^t for some $i \in \{1,2,3,4\}$. Then $(\gamma|\lambda) \in A \cup B$.*

Proof of Lemma 4.2. We know that γ and λ are different t-admissible partitions of dt sharing the same $\mu \vdash d$. This excludes $i \in \{1,2\}$, because in these cases Theorem 3.5 says that γ and λ cannot share the same μ if they are different.

Suppose $i = 3$. We must have $\gamma_d \neq \lambda_d$ if λ and γ belong to the same μ. Assume $\lambda_d > \gamma_d$. The diagram γ has a predecessor γ' with $\gamma'_{d-1} = \gamma_d$. This cannot be a predecessor of λ, and so γ and λ do not have the same predecessors.

So only the case $i = 4$ remains. Let $s = \max\{\gamma_1, \lambda_1\}$. If $s = t$, then $\gamma = \lambda = (t^d)$, so we can assume $s > t$. If γ and λ share the same μ, by combining Propositions 2.2 and 2.1, $\gamma^{*,s}$ and $\lambda^{*,s}$ share μ as well. Of course $\gamma^{*,s}$ and $\lambda^{*,s}$ belong to E_3^{s-t}, and they are different if γ and λ are different. In this case, we know by the previous case that $\gamma^{*,s}$ and $\lambda^{*,s}$ have different predecessors, and by dualizing we infer the same for γ and λ. □

Let us go ahead with the proof of Theorem 4.1. Set $\gamma = (\gamma_1, \ldots, \gamma_h)$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $h, k \le d$. If $h = k = d$ we can use induction on t since both γ and λ are trivial extensions.

If h and k are both less than d, then neither γ nor λ belong to E_1^t. Assume that $\gamma \in E_3^t$. Then μ is a hook. By the lemma, $\lambda \notin E_3^t$. Since μ is a hook, $k < d$ and $\lambda \notin E_3^t$, it follows that $\lambda \in E_4^t$ (recall that E_2^t and E_4^t are not disjoint). Because γ is a t-admissible hook and $h < d$, we get $\gamma_1 > dt - d + 1 \ge 4t - 3$. Then $\lambda_1 > 3t - 3$; otherwise γ and λ would have different predecessors. Therefore $\lambda \vdash dt > (d-1)(3t-3)$, which is impossible whenever $d \ge 3$ (recall that $t \ge 2$). So, by symmetry, we can assume that neither γ nor λ is in E_3^t. Therefore γ and λ belong to $E_2^t \cup E_4^t$. However, γ and λ share the same μ and, in such a situation, μ is a hook if and only if γ and λ both belong to E_4^t, a case already excluded in the lemma.

So, we can assume by symmetry that $h < d$ and $k = d$. Notice that $h = d - 1$, because all the predecessors of λ will have $d - 1$ rows. For the same reason we can even infer that $\gamma_{d-1} > t$, otherwise we could entirely remove γ_{d-1}, getting a predecessor of γ with $d-2$ rows. Since $d \ge 4$, we have $\gamma'_2 > t$ for all γ' predecessors of γ. So λ does not belong to E_3^t, since in this case $\lambda_2 \le t$. Since $\gamma_{d-1} > t$, Theorem 3.5 tells us that $\gamma \in E_4^t$ (once again, recall that E_2^t and E_4^t are not disjoint): so μ must be a hook. If $\lambda \in E_1^t$, then $\gamma_{d-1} \ge 2t - 1$ (otherwise γ would have a predecessor γ' with $\gamma'_{d-1} < t - 1$, that cannot be a predecessor of λ). This is evidently impossible if $d \ge 4$. So Theorem 3.5 implies that $\lambda \in E_4^t$, and the lemma lets us conclude.

Remark 4.3. Luke Oeding noticed that the shapes $\gamma_u \vdash 3t$ and $\lambda_u \vdash 3t$ in every $(\gamma_u|\lambda_u) \in B$ are dual to each other, namely $\lambda_u = \gamma_u^{*,2t}$, and that the same relationship holds within the bishapes $(\rho_v|\sigma_v) \in B$.

We have no a priori argument explaining this fact, but it is at least plausible. Both γ_u and λ_u have the same predecessors $\alpha \vdash 2t$, and α is automatically self-dual: $\alpha^{*,2t} = \alpha$. Therefore the set of successors of α is closed under dualization, and since single $\bigwedge^t$-type is preserved by it, the encounter of λ_u and γ_u is not surprising. All this holds for $(\rho_v|\sigma_v)$ as well. $\square$

References

[BCV] W. Bruns, A. Conca, M. Varbaro, *Relations among the minors of a generic matrix*, Adv. Math. **244** (2013), 171–206.

[DEP] C. De Concini, D. Eisenbud, C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. **56** (1980), 129–165.

[FH] W. Fulton, J. Harris, *Representation Theory. A First Course*, Graduate Texts in Mathematics, Vol. 129, Readings in Mathematics, Springer-Verlag, New York, 1991.

[We] J. M. Weyman, *Cohomology of Vector Bundles and Syzygies*, Cambridge Tracts in Mathematics, Vol. 149, Cambridge University Press, Cambridge, 2003.