Dual graphs of projective schemes

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August 26th, Haeundae, Busan, KOREA

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One way to make precise the concept of "combinatorial configuration of its irreducible components" is by meaning of the dual graph of X ....

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Hartshorne's connectedness theorem

Given $X \subseteq \mathbb{P}^n$ and the unique saturated homogeneous ideal $I_X \subseteq S = K[x_0, \ldots, x_n]$ s.t. $X = \mathrm{Proj}(S/I_X)$, let us recall that $X \subseteq \mathbb{P}^n$ is **arithmetically Cohen-Macaulay** (resp. **arithmetically Gorenstein**) if S/I_X is Cohen–Macaulay (resp. Gorenstein).

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On the other hand ......

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- There is a curve $C \subseteq \mathbb{P}^n$ such that no 3 of its irreducible components meet at one point, $\text{reg}(C) = 2$, and $G(C) = G$.

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- G is a tree.

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Connectivity of graphs

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Menger theorem (Max-flow-min-cut).

A graph is d-connected iff between any 2 vertices one can find d vertex-disjoint paths.

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Let $X \subseteq \mathbb{P}^n$ be an arithmetically Gorenstein projective scheme such that $\mathrm{reg}(X) = \mathrm{reg}(I_X) = r + 1$. If $\mathrm{reg}(\mathfrak{q}) \leq \delta$ for all primary components $\mathfrak{q}$ of I_X, then $G(X)$ is $\lfloor (r + \delta - 1)/\delta \rfloor$-connected.

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Benedetti-V. 2014

Let $X \subseteq \mathbb{P}^n$ be an arithmetically Gorenstein subspace arrangement such that $\mathrm{reg}(X) = \mathrm{reg}(I_X) = r + 1$. Then $G(X)$ is r-connected.

Example: 27 lines

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1. $G(C)$ is 10-connected.
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3. There is a partition $V_1, \ldots, V_9$ of the nodes of $G(C)$ such that the induced subgraph of $G(C)$ on each V_i is a triangle.

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$$C \subseteq \left(\bigcup_{i=1}^{9} H_i \right) \bigcap X \quad \text{and} \quad (f, g) \subseteq I_C,$$

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In particular $C \subseteq \mathbb{P}^3$ is an an arithmetically Gorenstein subspace arrangement of regularity $\deg(f) + \deg(g) - 1 = 3 + 9 - 1 = \mathbf{11}$. Thus our result confirms that $G(C)$ is **10**-connected.

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Question

Is $\mathrm{diam}(G(C)) \leq 2$ for any aCM line arrangement $C \subseteq \mathbb{P}^3$?

Hirsch embeddings

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Be careful:

- There exist nonreduced complete intersections $C \subseteq \mathbb{P}^3$ such that $C_{\text{red}} \subseteq \mathbb{P}^3$ is a line arrangement and $\operatorname{diam}(G(C))$ is arbitrarily large.
- For large n, there are arithmetically Gorenstein line arrangements that are not Hirsch (Santos).

Hirsch embeddings

Many projective embeddings, however, are Hirsch:

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Conjecture: Benedetti–V. 2014

If $X \subseteq \mathbb{P}^n$ is a (reduced) aCM scheme and I_X is generated by quadrics, then $X \subseteq \mathbb{P}^n$ is Hirsch.

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Let $I_X = \cap_{i=1}^{s} \mathfrak{q}_i$ be the primary decomposition of I_X, choose $A \subseteq \{1, \ldots, s\}$ of cardinality less than $\lfloor (r + \delta - 1)/\delta \rfloor$ and let $B = \{1, \ldots, s\} \setminus A$. Let $I_A = \cap_{i \in A} \mathfrak{q}_i$, $I_B = \cap_{i \in B} \mathfrak{q}_i$ and

$$X_A = \mathrm{Proj}(S/I_A) \quad X_B = \mathrm{Proj}(S/I_B).$$

1. X_A and X_B are geometrically linked by X which is aGorenstein; so by a result of Hartshorne and Schenzel, we have a graded isomorphism

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5. But then the dual graph of X_B, which is the same as the dual graph of X with the vertices of A removed, is connected.

An 'Eisenbud-Goto style' question

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This implies that the question above would admit a positive answer in dimension 2 if the EG conjecture was true in dimension 2 in its full generality (not only for irreducible surfaces).

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