CANONICAL COHEN-MACAULAY PROPERTY AND LYUBEZNIK NUMBERS UNDER GRÖBNER DEFORMATIONS

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ABSTRACT. In this note we draw some interesting consequences of the recent results on squarefree Gröbner degenerations obtained by Conca and the second author.

1. INTRODUCTION

Let $R = K[x_1, ..., x_n]$ be a positively graded polynomial ring over a field K, where x_i is homogeneous of degree $g_i \in \mathbb{N}_{>0}$, and $\mathfrak{m} = (x_1, \dots, x_n)$ denotes its homogeneous maximal ideal. Also denote the canonical module of R by $\omega_R = R(-|g|)$, where $|g| = g_1 + \dots + g_n$.

Definition 1.1. A graded finitely generated R-module M is called canonical Cohen-Macaulay (CCM for short) if $\operatorname{Ext}_R^{n-\dim M}(M, \omega_R)$ is Cohen-Macaulay.

This notion was introduced by Schenzel in [10], who proved in the same paper the following result that contributes to make it interesting: given a homogeneous prime ideal $I \subset R$, the ring R/I is CCM if and only if it admits a birational Macaulayfication (that is a birational extension $R/I \subset A \subset Q(R/I)$ such that A is a finitely generated Cohen-Macaulay R/I-module, where $Q(R/I)$ is the fraction field of R/I). In this case, furthermore, A is the endomorphism ring of $\operatorname{Ext}_R^{n-\dim R/I}(R/I, \omega_R)$.

In this note, we will derive by the recent result obtained by Conca and the second author in [4] the following: if a homogeneous ideal $I \subset R$ has a radical initial ideal $\operatorname{in}_\prec(I)$ for some monomial order $\prec$, then R/I is CCM whenever $R/\operatorname{in}_\prec(I)$ is CCM. In fact we prove something more general, from which we can also infer that, in positive characteristic, under the same assumptions the Lyubeznik numbers of R/I are bounded above from those of $R/\operatorname{in}_\prec(I)$. As a consequence of the latter result, we can infer that, also in characteristic 0 by reduction to positive characteristic, if $\operatorname{in}_\prec(I)$ is a radical monomial ideal the following are equivalent:

(1) The dual graph (a.k.a. Hochster-Huneke graph) of R/I is connected.
(2) The dual graph of $R/\operatorname{in}_\prec(I)$ is connected.

Motivated by these results, in the last section we study the CCM property for Stanley-Reisner rings $K[\Delta]$. We show that $K[\Delta]$ is CCM whenever Δ is a simply connected 2-dimensional simplicial complex.

2. CCM, Lyubeznik numbers and Gröbner deformations

Throughout this section, let us fix a monomial order $\prec$ on R. We start with the following crucial lemma:

Lemma 2.1. *Let I be a homogeneous ideal of R such that $\mathrm{in}_\prec(I)$ is radical. Then, for all $i,j,k \in \mathbb{Z}$, we have:*

$$\dim_K \operatorname{Ext}^i_R(\operatorname{Ext}^j_R(R/I,\omega_R),\omega_R)_k \leq \dim_K \operatorname{Ext}^i_R(\operatorname{Ext}^j_R(R/\operatorname{in}_\prec(I),\omega_R),\omega_R)_k$$

Proof. Let $w=(w_1,\dots,w_n)\in\mathbb{N}^n$ be a weight such that $\mathrm{in}_w(I)=\mathrm{in}_\prec(I)$. Let t be a new indeterminate over R. Set $P=R[t]$ and $S=P/\hom_w(I)$. By providing P with the graded structure given by $\deg(x_i)=g_i$ and $\deg(t)=0$, $\hom_w(I)$ is homogeneous. If $x\in\{t,t-1\}$, apply the functor $\operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),-)$ to the short exact sequence

$$0\to P\xrightarrow{\cdot x} P\to P/xP\to 0$$

getting the short exact sequences

$$0\to \mathrm{Coker}\mu_x^{i-1,j}\to \operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P/xP)\to \mathrm{Ker}\mu_x^{i,j}\to 0.$$

where $\mu_x^{i,j}$ is the multiplication by x on $\operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P)$. So, for all $k\in\mathbb{Z}$ we have exact sequences of K-vector spaces:

$$0\to [\mathrm{Coker}\mu_x^{i-1,j}]_k\to \operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P/xP)_k\to [\mathrm{Ker}\mu_x^{i,j}]_k\to 0.$$

Since $E_k^{i,j}=\operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P)_k$ is a finitely generated graded (w.r.t. the standard grading) $K[t]$-module, we can write $E_k^{i,j}=F_k^{i,j}+T_k^{i,j}$ where $F_k^{i,j}=K[t]^{f_k^{i,j}}$ and $T_k^{i,j}=\bigoplus_{r=1}^{g_k^{i,j}}K[t]/(t^{d_r})$ with $d_r\geq 1$. Therefore we have:

$$\dim_K[\mathrm{Coker}\mu_{t-1}^{i-1,j}]_k=f_k^{i-1,j}\leq f_k^{i-1,j}+g_k^{i-1,j}=\dim_K[\mathrm{Coker}\mu_t^{i-1,j}]_k$$

and

$$\dim_K[\mathrm{Ker}\mu_{t-1}^{i,j}]_k=0\leq g_k^{i,j}=\dim_K[\mathrm{Ker}\mu_t^{i-1,j}]_k.$$

So $\dim_K \operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P/(t-1)P)_k\leq \dim_K \operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P/tP)_k$.

Note that, by using [3, Proposition 1.1.5] one can infer the following: if A is a ring, M and N are A-modules, and $a\in\mathrm{Ann}(N)$ is A-regular and M-regular, then

$$\operatorname{Ext}^i_A(M,N)\cong \operatorname{Ext}^i_{A/aA}(M/aM,N)\quad \forall\ i\in\mathbb{N}.$$

Since by [4, Proposition 2.4] $\operatorname{Ext}^j_P(S,P)$ is a flat $K[t]$-module, the multiplication by x on it is injective: that is, x is $\operatorname{Ext}^j_P(S,P)$-regular. Therefore we have:

$$\operatorname{Ext}^i_P(\operatorname{Ext}^j_P(S,P),P/xP)\cong \operatorname{Ext}^i_{P/xP}(\operatorname{Ext}^j_P(S,P)/x\operatorname{Ext}^j_P(S,P),P/xP).$$

Again because the multiplication by x is injective on $\mathrm{Ext}^j_P(S,P)$ and by the property mentioned above, we have

$$\mathrm{Ext}^j_P(S,P)/x\mathrm{Ext}^j_P(S,P) \cong \mathrm{Ext}^j_P(S,P/xP) \cong \mathrm{Ext}^j_{P/xP}(S/xS,P/xP).$$

Putting all together we get:

$$\dim_K \mathrm{Ext}^i_{P/(t-1)P}(\mathrm{Ext}^j_{P/(t-1)P}(S/(t-1)S,P/(t-1)P),P/(t-1)P)_k \leq \\ \dim_K \mathrm{Ext}^i_{P/tP}(\mathrm{Ext}^j_{P/tP}(S/tS,P/tP),P/tP)_k,$$

that, because $\omega_R \cong R(-|g|)$, is what we wanted:

$$\dim_K \mathrm{Ext}^i_R(\mathrm{Ext}^j_R(R/I,R),R)_k \leq \dim_K \mathrm{Ext}^i_R(\mathrm{Ext}^j_R(R/\operatorname{in}_\prec(I),R),R)_k.$$

□

Corollary 2.2. *Let I be a homogeneous ideal of R such that $\operatorname{in}_\prec(I)$ is radical. Then, R/I is canonical Cohen-Macaulay whenever $R/\operatorname{in}_\prec(I)$ is so.*

Proof. For a homogeneous ideal $J \subset R$, R/J is CCM if and only if

$$\mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-\dim R/J}_R(R/J,\omega_R),\omega_R) = 0 \quad \forall\ i < \dim R/J,$$

so the result follows from Lemma 2.1. □

Remark 2.3. Corollary 2.2 fails without assuming that $\operatorname{in}_\prec(I)$ is radical. In fact, if $\prec$ is a degrevlex monomial order and I is in generic coordinates, by [8, Theorem 2.2] $R/\operatorname{in}_\prec(I)$ is sequentially Cohen-Macaulay, thus CCM (for example see [8, Theorem 1.4]). However, it is plenty of homogeneous ideals I such that R/I is not CCM.

We do not know whether the implication of Corollary 2.2 can be reversed. Without assuming that $\operatorname{in}_\prec(I)$ is radical, we already noticed that Corollary 2.2 fails in Remark 2.3. The following example shows that in general R/I CCM but $R/\operatorname{in}_\prec(I)$ not CCM can also happen:

Example 2.4. Let $R = K[x_1,...,x_9]$ and

$$I = (x_1^3 + x_2^3, x_5^2x_9 + x_4^2x_8, x_5^3x_7 + x_6^3x_9, x_7^2x_1 + x_6^2x_5, x_3x_9 - x_4x_8).$$

Since I is a complete intersection, R/I is CCM. However one can check that, if $\prec$ is the lexicographic order extending $x_1 > \ldots > x_9$, $R/\operatorname{in}_\prec(I)$ is not CCM.

2.1. **Lyubeznik numbers and connectedness.** Let $I \subset R = K[x_1, \dots, x_n]$. In [9] Lyubeznik introduced the following invariants of $A = R/I$:

$$\lambda_{i,j}(A) = \dim_K \mathrm{Ext}^i_R(K, H^{n-j}_I(R)) \ \ \forall\, i, j \in \mathbb{N}.$$

It turns out that these numbers, later named *Lyubeznik numbers*, depend only on A, i and j, in the sense that if $A \cong S/J$ where $J \subset S = K[y_1, \dots, y_m]$,

$$\lambda_{i,j}(A) = \dim_K \mathrm{Ext}^i_S(K, H^{m-j}_J(S)) \ \ \forall\, i, j \in \mathbb{N}.$$

Also, $\lambda_{i,j}(A) = 0$ whenever $i > j$ or $j > \dim A$, and $\lambda_{d,d}(A)$ is the number of connected components of the *dual graph* (also known as the *Hochster-Huneke graph*) of $A \otimes_K \overline{K}$, [17]. (We recall that the dual graph of a Noetherian ring A of dimension d is the graph with the minimal primes of A as vertices and such that $\{\mathfrak{p}, \mathfrak{q}\}$ is an edge if and only if $\dim A/(\mathfrak{p} + \mathfrak{q}) = d - 1$). We will refer to the upper triangular matrix $\Lambda(A) = (\lambda_{i,j}(A))$ of size $(\dim A + 1) \times (\dim A + 1)$ as the *Lyubeznik table* of A. By *trivial Lyubeznik table* we mean that $\lambda_{\dim A, \dim A}(A) = 1$ and $\lambda_{i,j}(A) = 0$ otherwise.

Corollary 2.5. *Let I be a homogeneous ideal of R such that* $\mathrm{in}_{\prec}(I)$ *is radical. If K has positive characteristic,*

$$\lambda_{i,j}(R/I) \le \lambda_{i,j}(R/\operatorname{in}_{\prec}(I)) \ \ \forall\, i, j \in \mathbb{N}.$$

Proof. By [18, Theorem 1.2], if $J \subset R$ is a homogeneous ideal,

$$\lambda_{i,j}(R/J) = \dim_K (\mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-j}_R(R/J, \omega_R), \omega_R)_0)_s,$$

where the subscript $(-)_s$ stands for the stable part under the natural Frobenius action. In particular

$$\lambda_{i,j}(R/J) \le \dim_K \mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-j}_R(R/J, \omega_R), \omega_R)_0.$$

On the other hand, if $J \subset R$ is a radical monomial ideal, Yanagawa proved in [16, Corollary 3.10] (independently of the characteristic of K) that:

$$\lambda_{i,j}(R/J) = \dim_K \mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-j}_R(R/J, \omega_R), \omega_R)_0.$$

So the result follows from Lemma 2.1. □

The following two examples show that Corolarry 2.5 is false without assuming both that $\mathrm{in}_{\prec}(I)$ is radical and that K has positive characteristic:

Example 2.6. [5, Example 4.11] Let $R = K[x_1, ..., x_6]$ and $\mathrm{char}(K) = 5$. Let

$$\begin{aligned} I = (&x_4^3 + x_5^3 + x_6^3, x_4^2 x_1 + x_5^2 x_2 + x_6^2 x_3, x_1^2 x_4 + x_2^2 x_5 + x_3^2 x_6, \\ &x_1^3 + x_2^3 + x_3^3, x_5 x_3 - x_6 x_2, x_6 x_1 - x_4 x_3, x_4 x_2 - x_5 x_1). \end{aligned}$$

Then

$$\Lambda(R/I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{bmatrix}.$$

If $\prec$ is the degree reverse lexicographic term order extending $x_1 > \ldots > x_6$ one has:

$$\mathrm{in}_\prec(I) = (x_3x_5, x_3x_4, x_2x_4, x_4^3, x_1x_4^2, x_1^2x_4, x_1^3).$$

One can check that $R/\mathrm{in}_\prec(I)$ has a trivial Lyubeznik table.

Example 2.7. Let K be a field of characteristic 0 and $R = K[x_1, ..., x_6]$. Let $I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)$. By [1, Example 2.2], Lyubeznik table of R/I is

$$\Lambda(R/I) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 1 \end{bmatrix}.$$

If $\prec$ is the degree reverse lexicographic term order extending $x_1 > \ldots > x_6$ we have

$$\mathrm{in}_\prec(I) = (x_2x_4, x_3x_4, x_3x_5).$$

So $\mathrm{in}_\prec(I)$ is a radical monomial ideal, however $\Lambda(R/\mathrm{in}_\prec(I))$ is trivial.

In Corollary 2.5 we have an equality when R/I is generalized Cohen-Macaulay:

Corollary 2.8. *Let I be a homogeneous ideal of R such that $\mathrm{in}_\prec(I)$ is radical. If K has positive characteristic and R/I is generalized Cohen-Macaulay,*

$$\lambda_{i,j}(R/I) = \lambda_{i,j}(R/\mathrm{in}_\prec(I)) \quad \forall\ i, j \in \mathbb{N}.$$

Proof. Since R/I is generalized Cohen-Macaulay so is $R/\mathrm{in}_\prec(I)$ by [4, Corollary 2.11]. Therefore it is enough to show that $\lambda_{0,j}(R/I) = \lambda_{0,j}(R/\mathrm{in}_\prec(I))$ for all j (see [1, Subsection 4.3]). By [4, Proposition 3.3], both $R/\mathrm{in}_\prec(I)$ and R/I are cohomologically full. So from [6, Proposition 4.11]:

$$\lambda_{0,j}(R/I) = \dim_K[H^j_{\mathfrak{m}}(R/I)]_0,$$
$$\lambda_{0,j}(R/\mathrm{in}_\prec(I)) = \dim_K[H^j_{\mathfrak{m}}(R/\mathrm{in}_\prec(I))]_0.$$

Now by [4, Theorem 1.3] we get the result.

□

Remark 2.9. Let I be an ideal of $R = K[x_1, \dots, x_n]$ such that $\mathrm{in}_\prec(I)$ is generated by monomials $u_1, \dots, u_r$. Suppose that K has characteristic 0. Since I is finitely generated, there exists a finitely generated $\mathbb{Z}$-algebra $A \subset K$ such that I is defined over A, i.e. $I'R = I$ if $I' = I \cap A[x_1, \dots, x_n]$. Given a prime number p and a prime ideal $\mathfrak{p} \in \mathrm{Spec} A$ minimal over (p), let $Q(\mathfrak{p})$ denote the field of fractions of $A/\mathfrak{p}$ (note that $Q(\mathfrak{p})$ has characteristic p), $R(\mathfrak{p}) = Q(\mathfrak{p})[x_1, \dots, x_n]$ and $I(\mathfrak{p}) = I'R(\mathfrak{p})$. We call the objects $R(\mathfrak{p}), I(\mathfrak{p}), R(\mathfrak{p})/I(\mathfrak{p})$ reductions mod p of $R, I, R/I$, and by abusing notation we denote them by $R_p, I_p, R_p/I_p$.

Seccia proved in [11] that

$$\mathrm{in}_\prec(I_p) = \mathrm{in}_\prec(I)_p$$

for any reduction mod p if p is a large enough prime number, i.e. $\mathrm{in}_\prec(I_p)$ is generated by $u_1, \dots, u_r$.

Remark 2.10. Let A be a Noetherian ring of dimension d. The ring A is said to be connected in codimension 1 if $\mathrm{Spec}\, A \setminus V(\mathfrak{a})$ is connected whenever $\dim A/\mathfrak{a} < d-1$ (here $V(\mathfrak{a})$ denotes the set of primes containing $\mathfrak{a}$). A result of Hartshorne [7, Proposition 1.1] implies that the dual graph of A is connected if and only if A is connected in codimension 1.

Proposition 2.11. *Let I be a homogeneous ideal of R such that $\mathrm{in}_\prec(I)$ is radical. Then:*

1. $\mathrm{Proj} R/I$ *is connected if and only if* $\mathrm{Proj} R/\mathrm{in}_\prec(I)$ *is connected.*
2. *The dual graph of R/I is connected if and only if the dual graph of $R/\mathrm{in}_\prec(I)$ is connected.*

Proof. The "only if" parts hold without the assumption that $\mathrm{in}_\prec(I)$ is radical and they have been proved in [14]. So we will concentrate on the "if" parts.

Since computing initial ideal, as well as the connectedness properties concerning $R/\mathrm{in}_\prec(I)$, are not affected extending the field, while the connectedness properties concerning R/I follow from the corresponding connectedness properties of $R/I \otimes_K \overline{K}$, it is harmless to assume that K is algebraically closed. Under this assumption, if $J \subset R$ is a homogeneous radical ideal, we have that:

(a) $\mathrm{Proj} R/J$ is connected if and only if $H^1_{\mathfrak{m}}(R/J)_0 = 0$.
(b) The dual graph of R/J is connected if and only if $\lambda_{\dim R/J, \dim R/J}(R/J) = 1$ by the main theorem of [17].

Under our hypothesis I is radical, so (1) follows at once from (a) and the fact that the Hilbert function of the local cohomology modules of R/I is bounded above by that of the ones of $R/\mathrm{in}_\prec(I)$ (in this case we even have equality by [4]). Concerning the "if-part" of (2), since $\lambda_{\dim R/I, \dim R/I}(R/I) \neq 0$ in any case, if K has positive characteristic it follows from (b) and Corollary 2.5. So, assume that K has characteristic 0. If, by contradiction, R/I were not connected in codimension 1, there would be two ideals $H \supsetneq I$ and $J \supsetneq I$ such that $H \cap J = I$ and $\dim R/(H+J) < \dim R/I - 1$ (see [2, Lemma 19.1.15]). By Remark 2.9, it is not difficult to check that we can choose a prime number $p \gg 0$ such that $H_p \supsetneq I_p$ and $J_p \supsetneq I_p$, $H_p \cap J_p = I_p$, $\dim R_p/(H_p+J_p) < \dim R_p/I_p - 1$ and $\operatorname{in}_\prec(I_p) = \operatorname{in}_\prec(I)_p$ (for instance, to compute the intersection of two ideals amounts to perform a Gröbner basis calculation). Clearly the dual graph of a Stanley-Reisner ring does not depend on the characteristic of the base field. So the dual graph of $R_p/\operatorname{in}_\prec(I_p)$ would be connected but that of R_p/I_p would be not, and this contradicts the fact that we already proved the result in positive characteristic. □

3. CCM Simplicial Complexes

Let Δ be a simplicial complex on the vertex set $[n] = \{1, ..., n\}$. We denote the Stanley-Reisner ring R/I_Δ by $K[\Delta]$. See [12] for generalities on these objects. The aim of this section is to examine the CCM property for the Stanley-Reisner rings $K[\Delta]$, especially when Δ has dimension 2.

Recall that a $\mathbb{N}^n$-graded R-module M is *squarefree* if, for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the multiplication by x_j from M_α to $M_{\alpha+e_j}$ is bijective whenever $\alpha_j \neq 0$. It turns out that $K[\Delta]$, I_Δ and $\operatorname{Ext}^i_R(K[\Delta], \omega_R)$ are squarefree modules by [15].

Lemma 3.1. *Let M be a nonzero squarefree module. If $M_0 = 0$, then* $\operatorname{depth} M > 0$.

Proof. Assume, by way of contradiction, that $\operatorname{depth} M = 0$. Then $\mathfrak{m} \in \operatorname{Ass} M$. So there exist $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $0 \neq u \in M_\alpha$ such that $\mathfrak{m} = \operatorname{Ann}(u)$. So for $j = 1, ..., n$, $x_j \cdot u = 0$. It follows that the multiplication map on M_α by x_j is not injective for all j. So, because M is a squarefree module, $\alpha = 0$ and $u \in M_0 = 0$, a contradiction. Hence $\operatorname{depth} M > 0$. □

Lemma 3.2. *For any homogeneous ideal $I \subset R$, for all $i < 3$ the R-module $\operatorname{Ext}^{n-i}_R(\operatorname{Ext}^{n-\dim R/I}_R(R/I, R), R)$ has finite length.*

Proof. If $\left(\cap_{i=1}^r \mathfrak{q}_i\right) \cap \left(\cap_{j=1}^s \mathfrak{q}'_j\right)$ is an irredundant primary decomposition of I with $\dim R/\mathfrak{q}_i = \dim R/I$ and $\dim R/\mathfrak{q}'_j > \dim R/I$, one has

$$\operatorname{Ext}^{n-\dim R/I}_R(R/I, R) \cong \operatorname{Ext}^{n-\dim R/I}_R(R/\cap_{i=1}^r \mathfrak{q}_i, R).$$

So we can assume that $\dim R/\mathfrak{p} = \dim R/I$ for all $\mathfrak{p} \in \operatorname{Ass} R/I$.

Let $\mathfrak{p} \neq \mathfrak{m}$ be a homogeneous prime ideal of R containing I, and set $M_i = \operatorname{Ext}^{n-i}_R(\operatorname{Ext}^{n-\dim R/I}_R(R/I, R), R)$. We have:

$$(M_i)_\mathfrak{p} = \operatorname{Ext}^{\operatorname{ht}(\mathfrak{p})-(i-n+\operatorname{ht}(\mathfrak{p}))}_{R_\mathfrak{p}}(\operatorname{Ext}^{\operatorname{ht}(\mathfrak{p})-(\dim R_\mathfrak{p}/IR_\mathfrak{p})}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, R_\mathfrak{p}), R_\mathfrak{p}).$$

Since $i-n+\mathrm{ht}(\mathfrak{p}) \leq 1$ by the assumptions and $\mathrm{Ext}_{R_\mathfrak{p}}^{\mathrm{ht}(\mathfrak{p})-(\dim R_\mathfrak{p}/IR_\mathfrak{p})}(R_\mathfrak{p}/IR_\mathfrak{p}, R_\mathfrak{p}), R_\mathfrak{p})$ has depth at least 2 by [10, Proposition 2.3] we have $(M_i)_\mathfrak{p} = 0$.

□

Corollary 3.3. *Let Δ be a 2-dimensional simplicial complex. Then $K[\Delta]$ is CCM if and only if $\lambda_{2,3}(K[\Delta]) = 0$.*

Proof. Since $\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R)$ satisfy Serre's condition (S_2) by [10, Proposition 2.3], it is enough to show that $\mathrm{Ext}_R^{n-2}(\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R) = 0$. By Lemma 3.2 $\mathrm{Ext}_R^{n-2}(\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)$ has finite length; so, since it is a squarefree module,

$$\mathrm{Ext}_R^{n-2}(\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R) = 0 \iff \mathrm{Ext}_R^{n-2}(\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)_0 = 0.$$

We conclude because $\lambda_{2,3}(K[\Delta]) = \mathrm{Ext}_R^{n-2}(\mathrm{Ext}_R^{n-3}(K[\Delta], \omega_R), \omega_R)_0$ by [16, Corollary 3.10]. □

Remark 3.4. If Δ is a $(d-1)$-dimensional simplicial complex, it is still true that if $K[\Delta]$ is CCM, then $\lambda_{j,d}(K[\Delta]) = 0$ for all $j < d$. The converse, however, is not true as soon as $\dim(\Delta) > 2$:

Let $R = K[x_1, ..., x_6]$ and I be the monomial ideal of R generated by

$$x_1x_2x_3x_4, x_1x_3x_4x_5, x_1x_2x_3x_6, x_1x_2x_5x_6, x_1x_4x_5x_6 \text{ and } x_3x_4x_5x_6.$$

The ring R/I has a trivial Lyubeznik table but it is not CCM. Here I is the Stanley-Reisner ring of a 3-dimensional simplicial complex.

Proposition 3.5. *Let Δ be a 2-dimensional simplicial complex such that $H_1(\Delta; K)$ vanishes. Then $K[\Delta]$ is CCM.*

Proof. Since $H_1(\Delta; K) = 0$, by Hochster formula we get $\mathrm{Ext}_R^{n-2}(K[\Delta], \omega_R)_0 = 0$. If $\mathrm{Ext}_R^{n-2}(K[\Delta], \omega_R) \neq 0$, since it is a squarefree module it has positive depth by Lemma 3.1.

So, in any case, $\mathrm{Ext}_R^{n}(\mathrm{Ext}_R^{n-2}(K[\Delta], \omega_R), \omega_R) = 0$, and hence

$$\lambda_{0,2}(K[\Delta]) = \mathrm{Ext}_R^{n}(\mathrm{Ext}_R^{n-2}(K[\Delta], \omega_R)), \omega_R)_0 = 0.$$

By [1, Remark 2.3], $\lambda_{2,3}(K[\Delta]) = \lambda_{0,2}(K[\Delta]) = 0$. Now by Corollary 3.3 $K[\Delta]$ is CCM. □

The converse of this corollary does not hold in general:

Example 3.6. Let Δ be the simplicial complex on 6 vertices with facets $\{1, 2, 3\}$, $\{1, 4, 5\}$ and $\{3, 4, 6\}$. Then $K[\Delta]$ is CCM but $H_1(\Delta; K) \neq 0$

Proposition 3.7. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex. The ring $K[\Delta]$ is CCM if and only if $H_i(\Delta; K) = 0$ for all $1 \leq i < d-1$.*

Proof. Let $K[\Delta]$ be CCM and fix $i \in \{1, \dots, d-2\}$. Since Δ is Buchsbaum, $K[\Delta]$ behaves cohomologically like an isolated singularity, hence:

$$\lambda_{0,i+1}(K[\Delta]) = \lambda_{d-i,d}(K[\Delta])$$

(see [1, Subsection 4.3]). On the other hand, since the canonical module of $K[\Delta]$ is a d-dimensional Cohen-Macaulay module, $\lambda_{d-i,d}(K[\Delta]) = 0$ by [16, Corollary 3.10]. So

$$\lambda_{0,i+1}(K[\Delta]) = \dim_K \mathrm{Ext}^n_R(\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R), \omega_R)_0 = 0.$$

By local duality $H^0_{\mathfrak{m}}(\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R))_0 = 0$. Since $\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R)$ is of finite length

$$H^0_{\mathfrak{m}}(\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R))_0 = \mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R)_0 = 0.$$

Therefore Hochster formula tells us that $H_i(\Delta; K) = 0$.

Conversely, assume that $H_i(\Delta; K) = 0$ for all $1 \leq i < d-1$. Then we have that $\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R)_0 = 0$ by Hochster formula. As Δ is Buchsbaum, $\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R)$ is of finite length, so

$$\mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R) = \mathrm{Ext}^{n-i-1}_R(K[\Delta], \omega_R)_0 = 0 \quad \forall\, 1 \leq i < d-1.$$

Now [13, Theorem 4.9] and local duality follow that for $1 \leq i < d-1$,

$$H^{i+1}_{\mathfrak{m}}(\mathrm{Ext}^{n-d}_R(K[\Delta], \omega_R) \cong \mathrm{Ext}^{n-d+i}_R(K[\Delta], \omega_R) = 0.$$

Thus $K[\Delta]$ is CCM. □

Example 3.8. Propositions 3.5 and 3.7 provide the following situation concerning CCM 2-dimensional simplicial complexes:

(i) $H_1(\Delta; K) = 0 \implies K[\Delta]$ is CCM.
(ii) If Δ is Buchsbaum, $H_1(\Delta; K) = 0 \iff K[\Delta]$ is CCM.

Item (ii) above yields many examples of Buchsbaum 2-dimensional nonCCM simplicial complexes. We conclude this note with an example of a 2-dimensional simplicial complex which is neither Buchsbaum nor CCM:

Let $R = K[x_1, ..., x_8]$ and Δ be the simplicial complex with facets $\{x_1, x_2, x_6\}$, $\{x_2, x_6, x_4\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_3, x_5\}$, $\{x_3, x_5, x_6\}$, $\{x_1, x_3, x_6\}$, $\{x_1, x_7, x_8\}$. One can check that Δ is not Buchsbaum and $K[\Delta]$ is not CCM. Accordingly with Proposition 3.5, $H_1(\Delta; K) \neq 0$.

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