ON A CONJECTURE BY KALAI

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Abstract. We show that monomial ideals generated in degree two satisfy a conjecture by Eisenbud, Green and Harris. In particular we give a partial answer to a conjecture of Kalai by proving that h-vectors of flag Cohen-Macaulay simplicial complexes are h-vectors of Cohen-Macaulay balanced simplicial complex.

1. Introduction

An unpublished conjecture of Gil Kalai, recently verified by Frohmader [Fr], states that for any flag simplicial complex Δ there exists a balanced simplicial complex Γ with the same f-vector. Here a $(d-1)$-simplicial complex is balanced if you can use d colors to label its vertices so that no face contains two vertices of the same colour. Kalai's conjecture has also a second part which is still open: If Δ happens to be Cohen-Macaulay (CM), then Γ is required to be CM as well.

In this note we show that for any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector. Such a result has been proved in [CV, Theorem 3.3] under the additional assumption that Δ is vertex decomposable. Other recent developments concerning Kalai's conjecture and related topics can be found in [CN, BV]. To this purpose we will show a stronger statement, Theorem 2.1, namely that the *Eisenbud-Green-Harris conjecture* (**EGH**) holds for quadratic monomial ideals.

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. The **EGH** conjecture, in the general form, states that for every homogeneous ideal I of S containing a regular sequence $f_1, \dots, f_r$ of degrees $d_1 \leq \dots \leq d_r$ there exists a homogeneous ideal $J \subseteq S$, with the same Hilbert function as I (i.e. $\mathrm{HF}_I = \mathrm{HF}_J$) and containing $x_1^{d_1}, \dots, x_r^{d_r}$. Furthermore, by a theorem of Clements and Lindstöm [CL], the ideal J, when it exists, can be chosen to be the sum of the ideal $(x_1^{d_1}, \dots, x_r^{d_r})$ and a lex-segment ideal of S. We refer to [EGH1] and [EGH2] for the original formulation of this conjecture. The only large classes for which the **EGH** conjecture is known are: when $f_1, \dots, f_r$ are Gröbner basis (by a deformation argument), when $d_i > \sum_{j<i}(d_j - 1)$ for all $i = 2, \dots, r$ ([CM]) and when each f_i factors as product of linear forms ([Ab, Corollary 4.3] for the case $r = n$, and [Ab] together with [CM, Proposition 10] for the general case).

2. The result

Theorem 2.1. *Let $I \subseteq S = K[x_1, \dots, x_n]$ be a monomial ideal generated in degree 2, of $\mathrm{ht} I = g$. There exists a monomial ideal $J \in S$, such that $(x_1^2, \dots, x_g^2) \subseteq J$ and*

$$\mathrm{HF}_I = \mathrm{HF}_J.$$

Furthermore J can be chosen with the same projective dimension as I.

Proof. Since the Hilbert function and the projective dimension are invariant with respect to field extension, we can assume without loss of generality that K is infinite We will prove that

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I contains a regular sequence of the form $x_1\ell_1,\dots,x_g\ell_g$, where ℓ_i is a linear form for every $i\in[g]=\{1,\dots,g\}$. Then we will infer the theorem by a result in [Ab].

Without loss of generality we may assume that $(x_1,\dots,x_g)$ is a minimal prime of I. Thus, we may decompose the degree 2 component of I as

$$I_2 = x_1V_1\oplus\ldots\oplus x_gV_g,$$

where each V_i is a linear space generated by indeterminates. Our goal is to find g linear forms $\ell_i\in V_i$, such that:

(*) for all $A\subseteq[g]$, the K-vector space $\langle x_i : i\in A\rangle+\langle \ell_i : i\in[g]\setminus A\rangle$ has dimension g.

To see that (*) is equivalent to $x_1\ell_1,\dots,x_g\ell_g$ being a S-regular sequence (from now on we will just write regular sequence for S-regular sequence), consider the following short exact sequence (where $C=(x_1\ell_1,\dots,x_g\ell_g)$):

$$0\to K[x_1,\dots,x_n]/(C:\ell_g)(-1)\to K[x_1,\dots,x_n]/C\to K[x_1,\dots,x_n]/(C+(\ell_g))\to 0.$$

By induction on the number of variables, both the extremes of the above exact sequence are graded modules of Krull dimension $\leq n-g$. By looking at the Hilbert polynomials, $K[x_1,\dots,x_n]/C$ has dimension $\leq n-g$ too, that is possible only if C is a complete intersection.

So we have to seek $\ell_i\in V_i$ satisfying (*). Let $A=\{i_1,\dots,i_a\}$ be a subset of $[g]$. We define $U_A\subseteq\prod_{i\in A}V_i$ to be the following set:

$$U_A=\{(v_{i_1},\dots,v_{i_a})\in\prod_{i\in A}V_i \ : \ \langle v_{i_1},\dots,v_{i_a}\rangle+\langle x_j : j\in[g]\setminus A\rangle \text{ has dimension } g\}.$$

As the condition of linear dependence is obtained by imposing certain determinantal relations to be zero, U_A is a Zariski open set of $\prod_{i\in A}V_i$. Thus the $\widetilde{U}_A$ below is a Zariski open set of $\prod_{i=1}^g V_i$

$$\widetilde{U}_A=U_A\times\prod_{i\in[g]\setminus A}V_i\subseteq\prod_{i=1}^g V_i.$$

This construction can be done for every $A\subseteq[g]$, and thus we can define the open set

$$U=\bigcap_{A\subseteq[g]}\widetilde{U}_A\subset\prod_{i=1}^g V_i.$$

Any element $(\ell_1,\dots,\ell_g)\in U$ will automatically satisfy (*), so our goal is to show that $U\neq\emptyset$. As $\prod_{i=1}^g V_i$ is irreducible, it is enough to show that all the open sets U_A's are nonempty. For any $A\subseteq[g]$ we have

$$\dim_K\Big(\sum_{i\in A}V_i+\sum_{j\in[g]\setminus A}\langle x_j\rangle\Big)\geq g, \tag{1}$$

otherwise $(\sum_{i\in A}V_i+\sum_{j\in[g]\setminus A}\langle x_j\rangle)$ would be a prime ideal containing I of height $<g$.

Given $A\subseteq[g]$, we define a bipartite graph G_A. The vertex set of G_A has the following partition: $V(G_A)=\{x_1,\dots,x_n\}\cup\{1,\dots,g\}$, and the edge set of G_A is given by:

$$\{x_i,j\}\in E(G_A)\iff\begin{cases} x_i\in V_j & , \text{ if } j\in A\\ i=j & , \text{ if } j\notin A\end{cases}$$

We fix A and prove that G_A satisfies the hypothesis of the Marriage Theorem. For a subset $B\subseteq V(G_A)$, we denote by $N(B)$ the set of vertices adjacent to some vertex in B. Choose now $B\subseteq\{1,\dots,g\}$. By applying (1) to the set $A\cap B\subseteq[g]$ we can deduce that

$$\dim_K\Big(\sum_{i\in A\cap B}V_i+\sum_{j\in([g]\setminus A)\cap B}\langle x_j\rangle\Big)\geq|B|,$$

Furthermore notice that the dimension of the above vector space is $|N(B)|$, thus we can apply the Marriage Theorem and infer the existence of a matching in G_A of the form $\{x_{i_j}, j\}_{j\in[g]}$. Therefore U_A is nonempty as it contains $(x_{i_j} : j \in A)$.

So we found a regular sequence of quadrics $f_1, \dots, f_g$ in I consisting of products of linear forms.

Let $\mathrm{pd}(I)$ be the projective dimension of I and assume that $\mathrm{pd}(I) = p - 1$. By applying a linear change of coordinates, we may assume that $x_{p+1}, \dots, x_n$ is a S/I-regular sequence. Going modulo $(x_{p+1}, \dots, x_n)$, the image $I' \subseteq K[x_1, \dots, x_p]$ of I may not be monomial, but still contains a regular sequence of quadrics which are products of linear forms, namely the image of $f_1, \dots, f_g$. So we find $J' \subseteq K[x_1, \dots, x_p]$ containing $(x_1^2, \dots, x_g^2)$ with the same Hilbert function of I' by [Ab, Corollary 4.3] and [CM, Proposition 10]. Clearly $\mathrm{pd}(J') \le p - 1$, but we can actually choose J' such that $\mathrm{pd}(J') = p-1$ by [CS, Theorem 4.4]. Defining $J = J'K[x_1, \dots, x_n]$ we have $(x_1^2, \dots, x_g^2) \subseteq J$, $\mathrm{pd}(J) = \mathrm{pd}(I)$ and $\mathrm{HF}_I = \mathrm{HF}_J$. □

The following example shows that the above proof cannot be extended to prove EGH for all monomial ideals.

Example 2.2. The ideal $I = (x_1^2x_2, x_2^2x_3, x_1x_3^2) \subseteq K[x_1, x_2, x_3]$ does not contain a regular sequence of the form $\ell_1\ell_2\ell_3, q_1q_2q_3$, where all ℓ_i and q_j are linear forms. Elementary direct computations allow one to see that the generators are the only products of three linear forms, which are contained in I. Clearly any choice of two of them does not produce a regular sequence.

The following corollary is the main motivation for this note.

Corollary 2.3. *For any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector.*

Proof. Let g be the height of the Stanley-Reisner ideal I_Δ. By Theorem 2.1, there exists a CM ideal $J \subseteq S$, containing $(x_1^2, \dots, x_g^2)$ and with the same Hilbert function as I_Δ. Since J is unmixed, it has to be the extension to S of a monomial ideal $J' \subseteq K[x_1, \dots, x_g]$. Hence $\mathrm{HF}_{K[x_1,\dots,x_g]/J'}$ equals the h-vector of Δ. The CM balanced Γ is the simplicial complex associated to the polarization of J'. □

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