Polytopes, dual graphs and line arrangements II

The algebraic point of view

ET'nA 2017

Catania, May 31 - June 4, 2017

Matteo Varbaro

Università degli Studi di Genova

Minimal graded free resolutions

Minimal graded free resolutions

- $S = \Bbbk[x_1, \dots, x_n]$ polynomial ring over a field $\Bbbk$;

Minimal graded free resolutions

- $S = \mathbb{k}[x_1, \ldots, x_n]$ polynomial ring over a field $\mathbb{k}$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \ldots, n$ (standard grading);

Minimal graded free resolutions

- $S = \mathbb{k}[x_1, \ldots, x_n]$ polynomial ring over a field $\mathbb{k}$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \ldots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \ldots, x_n)$ the irrelevant ideal;

Minimal graded free resolutions

- $S = \mathbb{k}[x_1, \ldots, x_n]$ polynomial ring over a field $\mathbb{k}$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \ldots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \ldots, x_n)$ the irrelevant ideal;
- $I \subset S$ homogeneous ideal.

Minimal graded free resolutions

- $S = \Bbbk[x_1, \ldots, x_n]$ polynomial ring over a field $\Bbbk$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \ldots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \ldots, x_n)$ the irrelevant ideal;
- $I \subset S$ homogeneous ideal.

A **minimal graded free resolution** of S/I is a complex of *free* S-modules

$$F_\bullet : \cdot \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \to 0$$

such that:

Minimal graded free resolutions

- $S = \Bbbk[x_1, \dots, x_n]$ polynomial ring over a field $\Bbbk$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \dots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \dots, x_n)$ the irrelevant ideal;
- $I \subset S$ homogeneous ideal.

A **minimal graded free resolution** of S/I is a complex of *free* S-modules

$$F_\bullet : \cdot \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \to 0$$

such that:

- $F_0 = S$, $d_1(F_1) = I$ and $H_i(F_\bullet) = 0 \ \forall \ i > 0$ ("resolution");

Minimal graded free resolutions

- $S = \Bbbk[x_1, \dots, x_n]$ polynomial ring over a field $\Bbbk$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \dots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \dots, x_n)$ the irrelevant ideal;
- $I \subset S$ homogeneous ideal.

A **minimal graded free resolution** of S/I is a complex of *free* S-modules

$$F_\bullet : \cdot \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \to 0$$

such that:

- $F_0 = S$, $d_1(F_1) = I$ and $H_i(F_\bullet) = 0 \ \forall \ i > 0$ ("resolution");
- for all $i > 0$ and $j \in \mathbb{Z}$, $d_i([F_i]_j) \subseteq [F_{i-1}]_j$ ("graded")

- $S = \mathbb{k}[x_1, \dots, x_n]$ polynomial ring over a field $\mathbb{k}$;
- $\deg(x_i) = 1 \ \forall \ i = 1, \dots, n$ (standard grading);
- $\mathfrak{m} = (x_1, \dots, x_n)$ the irrelevant ideal;
- $I \subset S$ homogeneous ideal.

A **minimal graded free resolution** of S/I is a complex of *free* S-modules

$$F_\bullet : \cdot \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \to 0$$

such that:

- $F_0 = S$, $d_1(F_1) = I$ and $H_i(F_\bullet) = 0 \ \forall \ i > 0$ ("resolution");
- for all $i > 0$ and $j \in \mathbb{Z}$, $d_i([F_i]_j) \subseteq [F_{i-1}]_j$ ("graded") and $d_i(F_i) \subset \mathfrak{m}F_{i-1}$ ("minimal").

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists,

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules.

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

The **projective dimension** of S/I is the length of the minimal graded free resolution $F_\bullet$ of S/I:

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

The **projective dimension** of S/I is the length of the minimal graded free resolution $F_\bullet$ of S/I:

$$\operatorname{projdim} S/I = \max\{k : F_k \neq 0\}.$$

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

The **projective dimension** of S/I is the length of the minimal graded free resolution $F_\bullet$ of S/I:

$$\operatorname{projdim} S/I = \max\{k : F_k \neq 0\}.$$

We always have $\operatorname{projdim} S/I \geq \operatorname{ht} I$ (recall that $\operatorname{ht} I$ equals the codimension of $V(I) \subset \mathbb{P}^{n-1}$).

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

The **projective dimension** of S/I is the length of the minimal graded free resolution $F_\bullet$ of S/I:

$$\operatorname{projdim} S/I = \max\{k : F_k \neq 0\}.$$

We always have $\operatorname{projdim} S/I \geq \operatorname{ht} I$ (recall that $\operatorname{ht} I$ equals the codimension of $V(I) \subset \mathbb{P}^{n-1}$). If equality holds, then S/I is called **Cohen-Macaulay**.

Minimal graded free resolutions

It is easy to check that a minimal graded free resolution of S/I always exists, and it is unique up to isomorphism of graded complexes of S-modules. Furthermore:

Hilbert syzygy theorem (1890)

If $F_\bullet$ is the minimal graded free resolution of S/I, then $F_i = 0$ whenever $i > n$ (n is the number of variables of S).

The **projective dimension** of S/I is the length of the minimal graded free resolution $F_\bullet$ of S/I:

$$\operatorname{projdim} S/I = \max\{k : F_k \neq 0\}.$$

We always have $\operatorname{projdim} S/I \geq \operatorname{ht} I$ (recall that $\operatorname{ht} I$ equals the codimension of $V(I) \subset \mathbb{P}^{n-1}$). If equality holds, then S/I is called **Cohen-Macaulay**. If furthermore $F_{\operatorname{ht} I}$ has rank 1, then S/I is said to be **Gorenstein**.

Being graded free S-modules, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, Being graded free S-modules, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$.

Being graded free S-modules, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$. So, it makes sense to define the **(Castelnuovo-Mumford) regularity** of S/I as:

$$\operatorname{reg} S/I = \max\{j - i : \beta_{ij} \neq 0\}.$$

Castelnuovo-Mumford regularity

Being graded free S-modules, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$. So, it makes sense to define the **(Castelnuovo-Mumford) regularity** of S/I as:

$$\operatorname{reg} S/I = \max\{j - i : \beta_{ij} \neq 0\}.$$

Example

Let $S = \Bbbk[x, y, z, u]$ and $I = (x^2,\ y^3 z^4,\ u^{30})$.

Being graded free S-modules, $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$. So, it makes sense to define the **(Castelnuovo-Mumford) regularity** of S/I as:

$$\operatorname{reg} S/I = \max\{j - i : \beta_{ij} \neq 0\}.$$

Example

Let $S = \mathbb{k}[x, y, z, u]$ and $I = (x^2,\ y^3z^4,\ u^{30})$. Being I a complete intersection (I is generated by as many polynomials as its height) the minimal graded free resolution of S/I is provided by the Koszul complex of $x^2,\ y^3z^4,\ u^{30}$,

Castelnuovo-Mumford regularity

Being graded free S-modules, $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$. So, it makes sense to define the **(Castelnuovo-Mumford) regularity** of S/I as:

$$\operatorname{reg} S/I = \max\{j - i : \beta_{ij} \neq 0\}.$$

Example

Let $S = \Bbbk[x, y, z, u]$ and $I = (x^2,\ y^3z^4,\ u^{30})$. Being I a complete intersection (I is generated by as many polynomials as its height) the minimal graded free resolution of S/I is provided by the Koszul complex of $x^2,\ y^3z^4,\ u^{30}$, and has the form:

$$0 \to S(-39) \longrightarrow S(-9) \oplus S(-32) \oplus S(-37) \longrightarrow$$
$$S(-2) \oplus S(-7) \oplus S(-30) \longrightarrow S \to 0$$

Being graded free S-modules, $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}}$, and it is simple to check that the **graded Betti numbers** β_{ij} are natural numbers and $\beta_{ij} = 0$ whenever $j < i$ or $j \gg 0$. So, it makes sense to define the **(Castelnuovo-Mumford) regularity** of S/I as:

$$\operatorname{reg} S/I = \max\{j - i : \beta_{ij} \neq 0\}.$$

Example

Let $S = \Bbbk[x, y, z, u]$ and $I = (x^2,\ y^3z^4,\ u^{30})$. Being I a complete intersection (I is generated by as many polynomials as its height) the minimal graded free resolution of S/I is provided by the Koszul complex of $x^2,\ y^3z^4,\ u^{30}$, and has the form:

$$0 \to S(-39) \longrightarrow S(-9) \oplus S(-32) \oplus S(-37) \longrightarrow$$
$$S(-2) \oplus S(-7) \oplus S(-30) \longrightarrow S \to 0$$

In particular, S/I is Gorenstein and $\operatorname{reg} S/I = 36$.

If $I = (f_1, \dots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, If $I = (f_1, \ldots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \ldots, f_h$ is the minimal graded free resolution of S/I.

Gorenstein rings

If $I = (f_1, \ldots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \ldots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

Gorenstein rings

If $I = (f_1, \dots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \dots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Gorenstein rings

If $I = (f_1, \ldots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \ldots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Example

Consider the monomial ideal $I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)$ of $S = \Bbbk[x_1, \ldots, x_5]$.

If $I = (f_1, \dots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \dots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Example

Consider the monomial ideal $I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)$ of $S = \Bbbk[x_1, \dots, x_5]$. Then S/I is Gorenstein (although $\operatorname{ht} I = 3$).

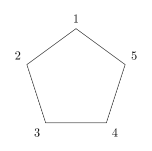
Gorenstein rings

If $I = (f_1, \ldots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \ldots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Example

Consider the monomial ideal $I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)$ of $S = \mathbb{k}[x_1, \ldots, x_5]$. Then S/I is Gorenstein (although $\operatorname{ht} I = 3$). In this case $I = I_\Delta$ is the Stanley-Reisner ideal of the pentagon Δ:

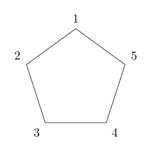


If $I = (f_1, \dots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \dots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Example

Consider the monomial ideal $I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)$ of $S = \Bbbk[x_1, \dots, x_5]$. Then S/I is Gorenstein (although $\mathrm{ht}\, I = 3$). In this case $I = I_\Delta$ is the Stanley-Reisner ideal of the pentagon Δ:

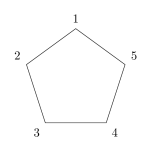


In general, if Δ is the triangulation of a sphere of dimension $d-1$ (e.g. the boundary of a simplicial d-polytope), If $I = (f_1, \ldots, f_h) \subset S$ is a complete intersection where the f_i's are homogeneous polynomials, then the Koszul complex of $f_1, \ldots, f_h$ is the minimal graded free resolution of S/I. From it, one sees that in this case S/I is Gorenstein and $\operatorname{reg} S/I = \sum_{i=1}^{h} \deg(f_i) - h$.

However, there are many more Gorenstein rings than these:

Example

Consider the monomial ideal $I = (x_1x_3,\ x_1x_4,\ x_2x_4,\ x_2x_5,\ x_3x_5)$ of $S = \mathbb{k}[x_1, \ldots, x_5]$. Then S/I is Gorenstein (although $\mathrm{ht}\, I = 3$). In this case $I = I_\Delta$ is the Stanley-Reisner ideal of the pentagon Δ:



In general, if Δ is the triangulation of a sphere of dimension $d-1$ (e.g. the boundary of a simplicial d-polytope), then S/I_Δ is Gorenstein and $\operatorname{reg} S/I_\Delta = d$.

For simplicity, from now on we will assume that I is radical. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal prime ideals of I.

Dual graphs

For simplicity, from now on we will assume that I is radical. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal prime ideals of I. The **dual graph** $G(I)$ of I is the simple graph with:

- $[s] := \{1, \ldots, s\}$ as vertex set;

Dual graphs

For simplicity, from now on we will assume that I is radical. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal prime ideals of I. The **dual graph** $G(I)$ of I is the simple graph with:

- $[s] := \{1, \ldots, s\}$ as vertex set;
- edges $\{i, j\}$ such that $\mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathrm{ht}\, I + 1$.

Dual graphs

For simplicity, from now on we will assume that I is radical. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal prime ideals of I. The **dual graph** $G(I)$ of I is the simple graph with:

- $[s] := \{1, \ldots, s\}$ as vertex set;
- edges $\{i, j\}$ such that $\mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathrm{ht}\, I + 1$.

Note that "$G(I)$ connected $\Rightarrow$ I height-unmixed ($\mathrm{ht}\, \mathfrak{p}_i = \mathrm{ht}\, I\ \forall i$)".

For simplicity, from now on we will assume that I is radical. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal prime ideals of I. The **dual graph** $G(I)$ of I is the simple graph with:

- $[s] := \{1, \ldots, s\}$ as vertex set;
- edges $\{i, j\}$ such that $\mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_j) = \mathrm{ht}\, I + 1$.

Note that "$G(I)$ connected $\Rightarrow$ I height-unmixed ($\mathrm{ht}\,\mathfrak{p}_i = \mathrm{ht}\, I \ \forall i$)". On the other hand, in 1962 Hartshorne proved that:

$$S/I \text{ is Cohen-Macaulay } \Rightarrow G(I) \text{ is connected.}$$

Given a subset $A \subseteq [s]$, by $G(I)_A$ we mean the subgraph of $G(I)$ induced on the vertices in A.

Given a subset $A \subseteq [s]$, by $G(I)_A$ we mean the subgraph of $G(I)$ induced on the vertices in A. Equivalently, $G(I)_A$ is the dual graph of the ideal $\cap_{i \in A} \mathfrak{p}_i$.

Given a subset $A \subseteq [s]$, by $G(I)_A$ we mean the subgraph of $G(I)$ induced on the vertices in A. Equivalently, $G(I)_A$ is the dual graph of the ideal $\cap_{i \in A} \mathfrak{p}_i$. Using ideas coming from liaison theory and some properties of local cohomology, we managed to prove the following:

Given a subset $A \subseteq [s]$, by $G(I)_A$ we mean the subgraph of $G(I)$ induced on the vertices in A. Equivalently, $G(I)_A$ is the dual graph of the ideal $\cap_{i \in A} \mathfrak{p}_i$. Using ideas coming from liaison theory and some properties of local cohomology, we managed to prove the following:

Theorem (Benedetti-V, 2015)

If S/I is Gorenstein, then $G(I)_A$ is connected whenever $A \subseteq [s]$ is such that $\operatorname{reg} \cap_{i \in [s] \setminus A} \mathfrak{p}_i < \operatorname{reg} S/I$.

Given a subset $A \subseteq [s]$, by $G(I)_A$ we mean the subgraph of $G(I)$ induced on the vertices in A. Equivalently, $G(I)_A$ is the dual graph of the ideal $\cap_{i \in A} \mathfrak{p}_i$. Using ideas coming from liaison theory and some properties of local cohomology, we managed to prove the following:

Theorem (Benedetti-V, 2015)

If S/I is Gorenstein, then $G(I)_A$ is connected whenever $A \subseteq [s]$ is such that $\operatorname{reg} \cap_{i \in [s] \setminus A} \mathfrak{p}_i < \operatorname{reg} S/I$.

The above result allows us to say something on the dual graph of an ideal defining a Gorenstein ring.

If Δ is a simplicial complex on n vertices, then the minimal prime ideals of I_Δ are $P_F = (x_i : i \notin F)$, for any facet $F \in \Delta$.

If Δ is a simplicial complex on n vertices, then the minimal prime ideals of I_Δ are $P_F = (x_i : i \notin F)$, for any facet $F \in \Delta$. So, the dual graph of Δ and the one of I_Δ are the same object !

If Δ is a simplicial complex on n vertices, then the minimal prime ideals of I_Δ are $P_F = (x_i : i \notin F)$, for any facet $F \in \Delta$. So, the dual graph of Δ and the one of I_Δ are the same object !

As explained by Bruno, Balinski's theorem states that, if Δ is the boundary of a simplicial d-polytope, then the dual graph of Δ is d-connected.

Connectedness properties of dual graphs

If Δ is a simplicial complex on n vertices, then the minimal prime ideals of I_Δ are $P_F = (x_i : i \notin F)$, for any facet $F \in \Delta$. So, the dual graph of Δ and the one of I_Δ are the same object !

As explained by Bruno, Balinski's theorem states that, if Δ is the boundary of a simplicial d-polytope, then the dual graph of Δ is d-connected. In this case, S/I_Δ is Gorenstein and $\operatorname{reg} S/I_\Delta = d$ ...

The ideal $I \subset S$ defines a **subspace arrangement** if I is the intersection of ideals generated by linear forms.

The ideal $I \subset S$ defines a **subspace arrangement** if I is the intersection of ideals generated by linear forms.

Being the intersection of ideals generated by variables, Stanley-Reisner ideals define subspace arrangements.

Subspace arrangements

The ideal $I \subset S$ defines a **subspace arrangement** if I is the intersection of ideals generated by linear forms.

Being the intersection of ideals generated by variables, Stanley-Reisner ideals define subspace arrangements.

Theorem (Derksen-Sidman, 2002)

If $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ where each $\mathfrak{p}_i$ is generated by linear forms, then

$$\operatorname{reg} I \leq t.$$

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

Subspace arrangements

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms).

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS,

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS, so $G(I)$ has at least $d+1$ vertices.

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS, so $G(I)$ has at least $d+1$ vertices.

Now pick a set of vertices $A \subseteq [s]$ of cardinality less than d.

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS, so $G(I)$ has at least $d+1$ vertices.

Now pick a set of vertices $A \subseteq [s]$ of cardinality less than d. Then $\operatorname{reg} \cap_{i \in A} \mathfrak{p}_i < d$ again by DS,

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS, so $G(I)$ has at least $d+1$ vertices.

Now pick a set of vertices $A \subseteq [s]$ of cardinality less than d. Then $\operatorname{reg} \cap_{i \in A} \mathfrak{p}_i < d$ again by DS, so $G(I)_{[s] \setminus A}$ is connected by BV.

Corollary (Benedetti-V, 2015)

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein and $\operatorname{reg} S/I = d$, then $G(I)$ is d-connected.

To prove it, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the minimal prime ideals of I (which by the assumption are generated by linear forms). Note that $d = \operatorname{reg} S/I < s$ by DS, so $G(I)$ has at least $d+1$ vertices.

Now pick a set of vertices $A \subseteq [s]$ of cardinality less than d. Then $\operatorname{reg} \cap_{i \in A} \mathfrak{p}_i < d$ again by DS, so $G(I)_{[s] \setminus A}$ is connected by BV. $\square$

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected.

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere),

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet.

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors;

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**,

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected.

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$.

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$.

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$. If $I = (f, g) \subset S = \mathbb{k}[x, y, z, w]$, $V(I) \subset \mathbb{P}^3$ consists of ab lines passing through the point $[0; 0; 0; 1]$.

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$. If $I = (f, g) \subset S = \mathbb{k}[x, y, z, w]$, $V(I) \subset \mathbb{P}^3$ consists of ab lines passing through the point $[0; 0; 0; 1]$. So $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity $a + b - 2$,

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$. If $I = (f, g) \subset S = \mathbb{k}[x, y, z, w]$, $V(I) \subset \mathbb{P}^3$ consists of ab lines passing through the point $[0; 0; 0; 1]$. So $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity $a + b - 2$, but $G(I)$ is the complete graph on ab vertices,

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\mathbb{k}[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$. If $I = (f, g) \subset S = \mathbb{k}[x, y, z, w]$, $V(I) \subset \mathbb{P}^3$ consists of ab lines passing through the point $[0; 0; 0; 1]$. So $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity $a + b - 2$, but $G(I)$ is the complete graph on ab vertices, in particular it is $(ab - 1)$-connected,

Subspace arrangements

If $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity d, then $G(I)$ is d-connected. Can $G(I)$ be r-connected for $r > d$?

If Δ is the boundary of a simplicial d-polytope (or more generally a triangulation of a $(d-1)$-sphere), then each facet shares each of its codimension 1 faces with exactly one other facet. On the dual graph, this translates into the fact that each vertex has exactly d neighbors; In other words, the dual graph of a triangulation of a $(d-1)$-sphere is d-**regular**, in particular it is not $(d+1)$-connected. On the other hand:

Example

Take f and g homogeneous polynomials of degrees a and b in $\Bbbk[x, y, z]$. If they generate a radical ideal J of height 2, $V(J)$ will consist in ab points in $\mathbb{P}^2$. If $I = (f, g) \subset S = \Bbbk[x, y, z, w]$, $V(I) \subset \mathbb{P}^3$ consists of ab lines passing through the point $[0; 0; 0; 1]$. So $I \subset S$ defines a subspace arrangement, S/I is Gorenstein of regularity $a + b - 2$, but $G(I)$ is the complete graph on ab vertices, in particular it is $(ab - 1)$-connected, although $ab - 1 > a + b - 2$ whenever $\min\{a, b\} \geq 2$.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$.

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

2. For line arrangements, being a hypersurface in codimension 1 means that if three or more lines in the arrangement meet at the same point, they must be coplanar.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

2. For line arrangements, being a hypersurface in codimension 1 means that if three or more lines in the arrangement meet at the same point, they must be coplanar. This notion is also known has having **planar singularities**.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

2. For line arrangements, being a hypersurface in codimension 1 means that if three or more lines in the arrangement meet at the same point, they must be coplanar. This notion is also known has having **planar singularities**.

3. Stanley-Reisner ideals defining Gorenstein rings are as in 1.,

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

2. For line arrangements, being a hypersurface in codimension 1 means that if three or more lines in the arrangement meet at the same point, they must be coplanar. This notion is also known has having **planar singularities**.

3. Stanley-Reisner ideals defining Gorenstein rings are as in 1., so the subspace arrangement they define is a hypersurface in codimension 1.

Subspace arrangements hypersurfaces in codimension 1

Let $I \subset S$ define a subspace arrangement $V = V(I) \subset \mathbb{P}^{n-1}$. So $V = \cup_{i=1}^{s} V_i$, where the V_i's are linear subspaces of $\mathbb{P}^{n-1}$. We say that V is a **hypersurface in codimension 1** if, whenever $\cap_{i \in A} V_i$ has codimension 1 in V, $\cup_{i \in A} V_i$ is a *hyperplane* arrangement.

Remarks

1. If $\cap_{i \in A} V_i$ has codimension more than 1 whenever $|A| > 2$ (that is the usual case), then V is a hypersurface in codimension 1.

2. For line arrangements, being a hypersurface in codimension 1 means that if three or more lines in the arrangement meet at the same point, they must be coplanar. This notion is also known has having **planar singularities**.

3. Stanley-Reisner ideals defining Gorenstein rings are as in 1., so the subspace arrangement they define is a hypersurface in codimension 1.

Theorem (Benedetti-Di Marca-V, 2016)

Let $I \subset S$ define an subspace arrangement such that S/I is Gorenstein.

Subspace arrangements hypersurfaces in codimension 1

Theorem (Benedetti-Di Marca-V, 2016)

Let $I \subset S$ define an subspace arrangement such that S/I is Gorenstein. If $V(I)$ is a *hypersurface in codimension 1*, then the following are equivalent:

Subspace arrangements hypersurfaces in codimension 1

Theorem (Benedetti-Di Marca-V, 2016)

Let $I \subset S$ define an subspace arrangement such that S/I is Gorenstein. If $V(I)$ is a *hypersurface in codimension 1*, then the following are equivalent:

- $\operatorname{reg} S/I = d$;
- $G(I)$ is d-connected but not $(d+1)$-connected;

Theorem (Benedetti-Di Marca-V, 2016)

Let $I \subset S$ define an subspace arrangement such that S/I is Gorenstein. If $V(I)$ is a *hypersurface in codimension 1*, then the following are equivalent:

- $\operatorname{reg} S/I = d$;
- $G(I)$ is d-connected but not $(d+1)$-connected;
- $G(I)$ is d-regular.

Subspace arrangements hypersurfaces in codimension 1

Theorem (Benedetti-Di Marca-V, 2016)

Let $I \subset S$ define an subspace arrangement such that S/I is Gorenstein. If $V(I)$ is a *hypersurface in codimension 1*, then the following are equivalent:

- $\operatorname{reg} S/I = d$;
- $G(I)$ is d-connected but not $(d+1)$-connected;
- $G(I)$ is d-regular.

Note that, if a line arrangement lies on a smooth surface of $\mathbb{P}^3$, then it automatically has planar singularities ...

27 lines

27 lines

Let $Z = V(f) \subseteq \mathbb{P}^3$ be a smooth cubic,

27 lines

Let $Z = V(f) \subseteq \mathbb{P}^3$ be a smooth cubic, and $X = V(I) = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z.

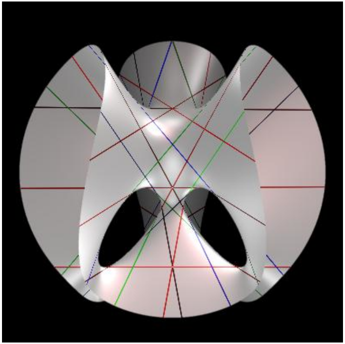
27 lines

Let $Z = V(f) \subseteq \mathbb{P}^3$ be a smooth cubic, and $X = V(I) = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z. Below is a representation of the Clebsch's cubic, given by:

$$f = x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3.$$

Let $Z = V(f) \subseteq \mathbb{P}^3$ be a smooth cubic, and $X = V(I) = \bigcup_{i=1}^{27} X_i$ be the union of all the lines on Z. Below is a representation of the Clebsch's cubic, given by:

$$f = x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3.$$



The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i.

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^6 P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

One easily checks that:

- $\{i, jk\}$ is an edge of $G(I) \Leftrightarrow i \in \{j, k\}$;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^6 P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

One easily checks that:

- $\{i, jk\}$ is an edge of $G(I) \Leftrightarrow i \in \{j, k\}$;
- $\{i, j\}$ is an edge of $G(I) \Leftrightarrow i \neq j$;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

One easily checks that:

- $\{i, jk\}$ is an edge of $G(I) \Leftrightarrow i \in \{j, k\}$;
- $\{i, j\}$ is an edge of $G(I) \Leftrightarrow i \neq j$;
- $\{ij, k\}$ is an edge of $G(I) \Leftrightarrow k \in \{i, j\}$;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let i be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let i be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

One easily checks that:

- $\{i, jk\}$ is an edge of $G(I) \Leftrightarrow i \in \{j, k\}$;
- $\{i, j\}$ is an edge of $G(I) \Leftrightarrow i \neq j$;
- $\{ij, k\}$ is an edge of $G(I) \Leftrightarrow k \in \{i, j\}$;
- $\{ij, hk\}$ is an edge of $G(I) \Leftrightarrow \{i, j\} \cap \{h, k\} = \emptyset$;

27 lines

The cubic Z is the blow-up of $\mathbb{P}^2$ along $\cup_{i=1}^{6} P_i$; let E_i denote the exceptional divisor corresponding to P_i. Let us describe $G(I)$:

- let $\color{blue}{i}$ be the vertex corresponding to E_i;
- let ij be the vertex corresponding to the strict transform of the line passing through P_i and P_j;
- let $\color{red}{i}$ be the vertex corresponding to the strict transform of the unique conic passing through all P_j with $j \neq i$;

One easily checks that:

- $\{\color{blue}{i}, jk\}$ is an edge of $G(I) \Leftrightarrow i \in \{j, k\}$;
- $\{\color{blue}{i}, \color{red}{j}\}$ is an edge of $G(I) \Leftrightarrow i \neq j$;
- $\{ij, \color{red}{k}\}$ is an edge of $G(I) \Leftrightarrow k \in \{i, j\}$;
- $\{ij, hk\}$ is an edge of $G(I) \Leftrightarrow \{i, j\} \cap \{h, k\} = \emptyset$;
- $\{\color{blue}{i}, \color{blue}{j}\}$ and $\{\color{red}{i}, \color{red}{j}\}$ are never edges of $G(I)$.

One can realize that $I = (f, g)$, where f is the polynomial defining the cubic and g is the product of nine linear forms.

One can realize that $I = (f, g)$, where f is the polynomial defining the cubic and g is the product of nine linear forms. So S/I is Gorenstein of regularity $3 + 9 - 2 = 10$;

One can realize that $I = (f, g)$, where f is the polynomial defining the cubic and g is the product of nine linear forms. So S/I is Gorenstein of regularity $3 + 9 - 2 = 10$; one can verify that:

- $G(I)$ is 10-connected;
- $G(I)$ is 10-regular,

27 lines

One can realize that $I = (f, g)$, where f is the polynomial defining the cubic and g is the product of nine linear forms. So S/I is Gorenstein of regularity $3 + 9 - 2 = 10$; one can verify that:

- $G(I)$ is 10-connected;
- $G(I)$ is 10-regular,

which confirms our theorem.

Schläfli double six

Schläfli double six

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors

Schläfli double six

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics,

Schläfli double six

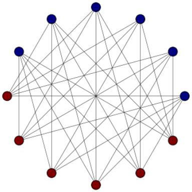
If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X = V(I) \subseteq \mathbb{P}^3$ known as **Schläfli double six**.

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X = V(I) \subseteq \mathbb{P}^3$ known as **Schläfli double six**. One can check that $I = (f, g)$, where f is the polynomial defining the cubic and g is a polynomial of degree 4 coprime with f.

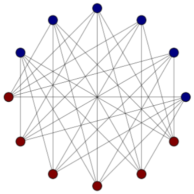
Schläfli double six

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X = V(I) \subseteq \mathbb{P}^3$ known as **Schläfli double six**. One can check that $I = (f, g)$, where f is the polynomial defining the cubic and g is a polynomial of degree 4 coprime with f. So S/I is Gorenstein and $\operatorname{reg} S/I = 3 + 4 - 2 = 5$.

If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X = V(I) \subseteq \mathbb{P}^3$ known as **Schläfli double six**. One can check that $I = (f, g)$, where f is the polynomial defining the cubic and g is a polynomial of degree 4 coprime with f. So S/I is Gorenstein and $\operatorname{reg} S/I = 3 + 4 - 2 = 5$. $G(I)$ is:



If, among the 27 lines on a smooth cubic, we take only the 6 corresponding to the exceptional divisors and the 6 corresponding to the strict transforms of the conics, we get a line arrangement $X = V(I) \subseteq \mathbb{P}^3$ known as **Schläfli double six**. One can check that $I = (f, g)$, where f is the polynomial defining the cubic and g is a polynomial of degree 4 coprime with f. So S/I is Gorenstein and $\operatorname{reg} S/I = 3 + 4 - 2 = 5$. $G(I)$ is:



- $G(I)$ is 5-connected.
- $G(I)$ is 5-regular.

Stanley-Reisner ideals vs line arrangements

Stanley-Reisner ideals vs line arrangements

Given a d-dimensional simplicial complex Δ, by taking $d-1$ general hyperplane sections of $V(I_\Delta)$ we get a line arrangement with same dual graph as Δ.

Stanley-Reisner ideals vs line arrangements

Given a d-dimensional simplicial complex Δ, by taking $d - 1$ general hyperplane sections of $V(I_\Delta)$ we get a line arrangement with same dual graph as Δ. The 27 lines and Schläfli double six arrangements do not arise in this way:

Stanley-Reisner ideals vs line arrangements

Given a d-dimensional simplicial complex Δ, by taking $d-1$ general hyperplane sections of $V(I_\Delta)$ we get a line arrangement with same dual graph as Δ. The 27 lines and Schläfli double six arrangements do not arise in this way:

Concerning Schläfli's double six, for example, the dual graph has diameter 3,

Stanley-Reisner ideals vs line arrangements

Given a d-dimensional simplicial complex Δ, by taking $d-1$ general hyperplane sections of $V(I_\Delta)$ we get a line arrangement with same dual graph as Δ. The 27 lines and Schläfli double six arrangements do not arise in this way:

Concerning Schläfli's double six, for example, the dual graph has diameter 3, while the dual graph of any normal simplicial complex of codimension 2 has diameter at most 2.

Stanley-Reisner ideals vs line arrangements

Given a d-dimensional simplicial complex Δ, by taking $d-1$ general hyperplane sections of $V(I_\Delta)$ we get a line arrangement with same dual graph as Δ. The 27 lines and Schläfli double six arrangements do not arise in this way:

Concerning Schläfli's double six, for example, the dual graph has diameter 3, while the dual graph of any normal simplicial complex of codimension 2 has diameter at most 2.

For the moment, we are not able to find a family of complete intersection line arrangements in $\mathbb{P}^3$ with dual graph of arbitrarily large diameter (not even > 3) ...

THANKS !!