Commutative algebra of Stanley-Reisner rings

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The Upper Bound Conjecture (UBC)

Let $d < n$ be positive integers. Consider the *moment curve*

$$\{(t, t^2, \dots, t^d) : t \in \mathbb{R}\} \subseteq \mathbb{R}^d,$$

take n distinct points on it and let P be the convex hull of them. It is not difficult to show that the combinatorial type of the polytope P does not depend on the chosen points, so we denote P by $C(n, d)$ and call it a cyclic polytope.

Recall that the number of i-dimensional faces of a polytope P is denoted by $f_i(P)$. One can check that:

$$f_i(C(n,d)) = \binom{n}{i+1} \quad \forall\ 0 \leq i \leq \lfloor d/2 \rfloor - 1.$$

The Upper Bound Conjecture (UBC)

Upper Bound Conjecture (Motzkin, 1957)

For any d-polytope P on n vertices we have

$$f_i(P) \leq f_i(C(n,d)) \quad \forall\ i = 0, \ldots, d.$$

Clearly $f_i(P) \leq \binom{n}{i+1}$ for any polytope on n vertices and $i \in \mathbb{N}$, so:

$$f_i(P) \leq f_i(C(n,d)) \quad \forall\ i = 0, \ldots, \lfloor d/2 \rfloor - 1.$$

It is easy to reduce the UBC to *simplicial* polytopes (btw, $C(n,d)$ is simplicial). In 1964, **Klee** conjectured that the UBC is valid more generally for any triangulation Δ on n vertices of a $(d-1)$-sphere.

The Upper Bound Conjecture (UBC)

It is easy to see that the UBC for a $(d-1)$-sphere would follow by the inequalities:

$$h_i \leq \binom{n-d+i-1}{i} \quad \forall\ i = 0, \ldots, d,$$

where $h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{i-1}$. By the Stanley-Reisner correspondence, we can exploit tools from commutative algebra and infer the inequality above by the simple fact that the degree i monomials in $n-d$ variables are $\binom{n-d+i-1}{i}$ .....

Simplicial complexes I

Let n be a positive integer and $[n] := \{1, \ldots, n\}$.

A **simplicial complex** Δ on $[n]$ is a subset of $2^{[n]}$ such that:

$$\sigma \in \Delta, \ \tau \subseteq \sigma \implies \tau \in \Delta.$$

For simplicity, when $i \in [n]$, we write $i \in \Delta$ if $\{i\} \in \Delta$ and refer to i as a *vertex* of Δ.

Any element of Δ is called *face*, and a face maximal by inclusion is called *facet*. The set of facets is denoted by $\mathcal{F}(\Delta)$.

EXAMPLE: A matroid is a simplicial complex on its ground set: faces correspond to *independent sets*, and facets to *bases*.

Simplicial complexes I

Given a subset $A \subseteq 2^{[n]}$, $\langle A \rangle$ denotes the smallest simplicial complex containing A. In particular, notice that

$$\Delta = \langle \mathcal{F}(\Delta) \rangle.$$

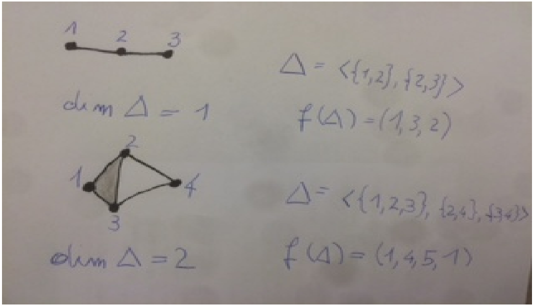
The dimension of a face σ is $\dim \sigma := |\sigma| - 1$, and the dimension of a simplicial complex Δ is

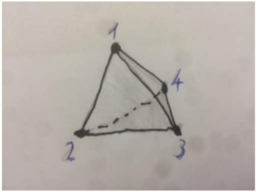
$$\dim \Delta := \sup\{\dim \sigma : \sigma \in \Delta\}.$$

The f-**vector** of a $(d-1)$-dimensional simplicial complex Δ is the vector $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ where

$$f_i := |\{\sigma \in \Delta : \dim \sigma = i\}|.$$

Examples





- $\Delta = \langle \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \rangle$.
- $\dim \Delta = 2$.
- $f(\Delta) = (1,4,6,4)$.

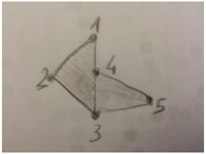
Any simplicial complex Δ has a *geometric realization* $\|\Delta\| \subseteq \mathbb{R}^N$:

A *k-simplex* $\sigma \subseteq \mathbb{R}^N$ is the convex hull of $k+1$ affinely independent points; a *face* τ of σ is the convex hull of any subset of the $k+1$ points above; to denote that τ is a face of σ we write $\tau \leq \sigma$.

A **geometric simplicial complex** $\mathcal{K}$ is a collection of simplices of $\mathbb{R}^N$ such that whenever $\sigma, \sigma' \in \mathcal{K}$:

(i) $\tau \leq \sigma \implies \tau \in \mathcal{K}$;

(ii) $\sigma \cap \sigma'$ is a face both of σ and of σ'.



Clearly to any geometric simplicial complex $\mathcal{K}$ can be associated an abstract simplicial complex Δ on the set of 0-simplices of $\mathcal{K}$, sending a simplex $\sigma \in \mathcal{K}$ to the set of 0-simplices of $\mathcal{K}$ contained in σ (note that they are forced to be faces of σ).

If $\mathcal{K}$ and $\mathcal{K}'$ are associated to the same Δ, then the topological spaces $\bigcup_{\sigma \in \mathcal{K}} \sigma$ and $\bigcup_{\sigma \in \mathcal{K}'} \sigma$ are homeomorphic. Such a topological space is called the **geometric realization** of Δ, and denoted by $\|\Delta\|$. It is not difficult to show that the image of the above function contains all the abstract simplicial complexes Δ such that $2\dim \Delta + 1 \leq N$. In particular every simplicial complex admits a geometric realization, just choose N large enough.

Standard graded algebras I

Let K be a field and $S := K[x_1, \dots, x_n]$ be the polynomial ring supplied with the *standard grading* $\deg(x_i) = 1 \ \ \forall\ i = 1, \dots, n$.

Given a homogeneous ideal $I \subseteq S$, consider the K-algebra $R = S/I$. The K-algebras arising this way are called **standard graded K-algebras**.

By definition the (Krull) dimension of R is n minus the minimum height of a minimal prime ideal of I. If K is algebraically closed, the dimension of R agrees with the dimension of

$$\mathcal{Z}(I) := \{P \in \mathbb{A}^n : f(P) = 0 \ \ \forall\ f \in I\} \subseteq \mathbb{A}^n,$$

i.e. the maximum dimension of an irreducible component of $\mathcal{Z}(I)$.

There are other various ways to define the dimension of a standard graded K-algebra $R = S/I$: if $\mathfrak{m} := (x_1, \ldots, x_n) \subseteq S$,

$$\dim R = \min\{d : \exists\ \theta_1, \ldots, \theta_d \in \mathfrak{m} \text{ such that } \frac{R}{(\overline{\theta_1}, \ldots, \overline{\theta_d})} = \frac{S}{I + (\theta_1, \ldots, \theta_d)} \text{ is a finite dimensional } K\text{-vector space}\}$$

Elements $\theta_1, \ldots, \theta_d$ attaining the minimum above are called a **system of parameters (s.o.p.)** for R. If K is infinite, then $\theta_1, \ldots, \theta_d$ can be chosen to be linear forms, and in this case they are called a **linear system of parameters (l.s.o.p.)** for R.

The subindex $\cdot_k$ means the degree k piece of the object $\cdot$ we are considering. For example, S_k denotes the set of homogeneous polynomials of S of degree k. Notice that S_k is a K-vector space of dimension $\binom{n+k-1}{k}$, so the numerical function:

$$k \mapsto \dim_K S_k = \frac{(k+n-1)(k+n-2)\cdots(k+1)}{(n-1)!}$$

is a polynomial of degree $n-1$. Let us consider the formal series

$$H_n(t) = \sum_{k\in\mathbb{N}} \dim_K S_k t^k \in \mathbb{Z}[[t]].$$

It is easy to check that:

(i) $H_1(t) = \sum_{k\in\mathbb{N}} t^k = 1/(1-t)$;

(ii) $H_n(t) = H_1(t)^n = 1/(1-t)^n$.

Standard graded algebras I

The facts that we saw for S hold more in general: for a standard graded K-algebra $R = S/I$, the numerical function $\mathrm{HF}_R : \mathbb{N} \to \mathbb{N}$:

$$k \mapsto \dim_K R_k$$

is called the *Hilbert function* of R. The formal series $\mathrm{HS}_R(t) = \sum_{k\in\mathbb{N}} \mathrm{HF}_R(k)t^k \in \mathbb{Z}[[t]]$ is the *Hilbert series* of R.

Theorem (Hilbert)

There exists $\mathrm{HP}_R \in \mathbb{Q}[x]$ of degree $\dim R - 1$ such that $\mathrm{HF}_R(k) = \mathrm{HP}_R(k)$ for all $k \gg 0$. HP_R is called the *Hilbert polynomial* of R. There also exists $h(t) = h_0 + h_1 t + \ldots + h_s t^s \in \mathbb{Z}[t]$ such that $h(1) \neq 0$ and

$$\mathrm{HS}_R(t) = \frac{h(t)}{(1-t)^{\dim R}}.$$

h is called the *h-polynomial* of R, and $(h_0, h_1, \ldots, h_s)$ the *h-vector* of R.

Stanley-Reisner correspondence I

Let K be a field and $S := K[x_1, \ldots, x_n]$ be the polynomial ring.

To a simplicial complex Δ on $[n]$ we associate the ideal of S:

$$I_\Delta := (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \notin \Delta) \subseteq S.$$

I_Δ is a square-free monomial ideal, and conversely to any such ideal $I \subseteq S$ we associate the simplicial complex on $[n]$:

$$\Delta(I) := \{\{i_1, \ldots, i_k\} : x_{i_1} \cdots x_{i_k} \notin I\} \subseteq 2^{[n]}.$$

It is straightforward to check that the operations above yield a 1-1 correspondence:

$\{$simplicial complexes on $[n]\} \leftrightarrow \{$square-free monomial ideals of $S\}$

Stanley-Reisner correspondence I

For a simplicial complex Δ on $[n]$:

(i) $I_\Delta \subseteq S$ is called the **Stanley-Reisner ideal** of Δ;

(ii) $K[\Delta] := S/I_\Delta$ is called the **Stanley-Reisner ring** of Δ.

Lemma

$I_\Delta = \bigcap_{\sigma\in\Delta}(x_i : i \in [n] \setminus \sigma)$. In particular $\dim K[\Delta] = \dim \Delta + 1$.

Proof: For any $\sigma \subseteq [n]$, the ideal $(x_i : i \in \sigma)$ contains I_Δ if and only if $[n] \setminus \sigma \in \Delta$. Being I_Δ a monomial ideal, its minimal primes are monomial prime ideals, i.e. ideals generated by variables. So $\sqrt{I_\Delta} = \bigcap_{\sigma\in\Delta}(x_i : i \in [n] \setminus \sigma)$, and being I_Δ radical we conclude. $\square$

Stanley-Reisner correspondence I

Given a linear form $\ell = \sum_{i=1}^{n} \lambda_i x_i \in S$, for any subset $\sigma \subseteq [n]$ by $\ell_{|\sigma}$ we mean the linear form $\sum_{i \in \sigma} \lambda_i x_i$.

Lemma

Given linear forms $\ell_1, \ldots, \ell_d \in S$, the following are equivalent:

(i) $\ell_1, \ldots, \ell_d$ are a l.s.o.p. for $K[\Delta]$.

(ii) Δ has dimension $d-1$ and the K-vector space generated by $(\ell_1)_{|\sigma}, \ldots, (\ell_d)_{|\sigma}$ has dimension $|\sigma|$ for all $\sigma \in \mathcal{F}(\Delta)$.

Given a $(d-1)$-dimensional simplicial complex Δ, its f-vector and the h-polynomial of $K[\Delta]$ are related by a simple formula .....

Stanley-Reisner correspondence I

Let Δ be a $(d-1)$-dimensional simplicial complex, $(f_{-1},\ldots,f_{d-1})$ its f-vector and $h(t)=h_0+\ldots+h_s t^s$ the h-polynomial of $K[\Delta]$.

Lemma

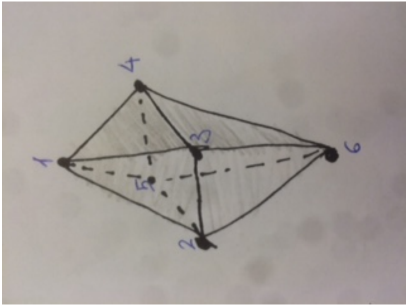
$h(t)=\sum_{i=0}^{d} f_{i-1}t^i(1-t)^{d-i}$. Therefore

$$h_j=\sum_{i=0}^{j}(-1)^{j-i}\binom{d-i}{j-i}f_{i-1} \quad \text{and} \quad f_{j-1}=\sum_{i=0}^{j}\binom{d-i}{j-i}h_i.$$

Proof: Let $\mathcal{M}(S)$ denote the set of monomials of S and $\mathcal{M}_k(S)=\mathcal{M}(S)\cap S_k$. The support of $u\in\mathcal{M}(S)$ is $\operatorname{supp} u := \{i\in[n] : x_i|u\}$. The class of u is nonzero in $K[\Delta]$ if and only if $\operatorname{supp} u$ is a face of Δ. So $\mathrm{HS}_{K[\Delta]}(t)=\sum_{k\in\mathbb{N}}|\{u\in\mathcal{M}_k(S) : \operatorname{supp} u\in\Delta\}|\cdot t^k$. To conclude, use that for a fixed face $\sigma\in\Delta$, we have the equality

$\sum_{k\in\mathbb{N}}|\{u\in\mathcal{M}_k(S) : \operatorname{supp} u=\sigma\}|\cdot t^k = t^{|\sigma|}/(1-t)^{|\sigma|}$. $\square$

Example



Octahedron

- $\Delta = \langle \{1,2,3\}, \{1,3,4\}, \{1,4,5\}, \{1,2,5\}, \{2,3,6\}, \{3,4,6\}, \{4,5,6\}, \{2,5,6\} \rangle$.
- $\dim \Delta = 2$.
- $f(\Delta) = (1, 6, 12, 8)$.
- $h(\Delta) = (1, 3, 3, 1)$

Let $R = S/I$ be a standard graded K-algebra. A homogeneous polynomial $f \in \mathfrak{m} = (x_1, \ldots, x_n)$ is R-regular if, for all $r \in R$:

$$fr = 0 \implies r = 0.$$

Homogeneous polynomials $f_1, \ldots, f_m \in \mathfrak{m}$ are called an R-regular sequence if:

$$f_{i+1} \text{ is } R/(\overline{f_1}, \ldots, \overline{f_i})\text{-regular for all } i = 0, \ldots, m-1.$$

It is easy to show that, if $f \in S$ is R-regular, then

$$\dim R/(\overline{f}) = \dim R - 1.$$

Standard graded algebras II

Let h_R be the h-polynomial of a standard graded K-algebra R.

Lemma

Let ℓ be a linear form of S such that $\overline{\ell} \neq 0$ in R. Then ℓ is R-regular if and only if $h_{R/(\overline{\ell})} = h_R$.

Proof: If $\dim R/(\overline{\ell}) = \dim R$, then $h_{R/(\overline{\ell})} \neq h_R$ because $\mathrm{HS}_{R/(\overline{\ell})} \neq \mathrm{HS}_R$. So assume $\dim R/(\overline{\ell}) = \dim R - 1$. Then

$$h_{R/(\overline{\ell})} = h_R \iff \mathrm{HS}_{R/(\overline{\ell})}(t) = (1-t)\,\mathrm{HS}_R(t) = \sum_{k \in \mathbb{N}} (\mathrm{HF}_R(k) - \mathrm{HF}_R(k-1))t^k.$$

So $h_{R/(\overline{\ell})} = h_R \iff \dim_K(R/(\overline{\ell}))_k = \dim_K R_k - \dim_K R_{k-1} \ \forall\ k \in \mathbb{N}.$

Standard graded algebras II

Consider the exact sequence of S-modules $R \xrightarrow{\cdot\ell} R \to R/(\bar{\ell}) \to 0$. For all $k \geq 1$, it yields the exact sequence of K-vector spaces

$$R_{k-1} \xrightarrow{\cdot\ell} R_k \to (R/(\bar{\ell}))_k \to 0.$$

So $\dim_K(R/(\bar{\ell}))_k \geq \dim_K R_k - \dim_K R_{k-1}$, with equality if and only if the map $R_{k-1} \xrightarrow{\cdot\ell} R_k$ is injective. Finally, such a map is injective for any $k \geq 1$ if and only if ℓ is R-regular. $\square$

Definition

Let R be a standard graded K-algebra. Then the **depth** of R is:

$$\text{depth } R = \max\{s : \exists\ R\text{-regular sequence of length } s\}$$

Notice that $\text{depth } R \leq \dim R$.

Standard graded algebras II

Definition

A standard graded K-algebra R is **Cohen-Macaulay** if

$$\operatorname{depth} R = \dim R.$$

Theorem

For a standard graded K-algebra R the following are equivalent:

(i) R is Cohen-Macaulay.

(ii) Every s.o.p. of R is an R-regular sequence.

If K is infinite, then the above facts are equivalent to:

(iii) for any l.s.o.p. $\theta_1, \ldots, \theta_d$ of R we have $h_R = h_{R/(\overline{\theta_1}, \ldots, \overline{\theta_d})}$.

(iv) there exists one l.s.o.p. $\theta_1, \ldots, \theta_d$ of R s.t. $h_R = h_{R/(\overline{\theta_1}, \ldots, \overline{\theta_d})}$.

Standard graded algebras II

From the theorem above one sees that, if R is Cohen-Macaulay and $(h_0, h_1, \ldots, h_s)$ its h-vector, then

$$h_i = \dim_K(R/(\overline{\theta_1}, \ldots, \overline{\theta_d})) \geq 0 \quad \forall\ i.$$

Definition

A standard graded K-algebra R is **Gorenstein** if it is Cohen-Macaulay and for some (equivalently any) s.o.p. $f_1, \ldots, f_d$, the algebra $A = R/(\overline{f_1}, \ldots, \overline{f_d})$ has the property that

$$\operatorname{soc} A = \{a \in A : \mathfrak{m} \cdot a = 0\}$$

is a 1-dimensional K-vector space.

With the above notation, if $s = \max\{i : A_i \neq 0\}$, then $A_s \subseteq \operatorname{soc} A$. This implies that if R is Gorenstein and $(h_0, h_1, \ldots, h_s)$ its h-vector (with $h_s \neq 0$), then $h_s = 1 (= h_0)$.

Theorem

If R is Gorenstein, then its h-vector is symmetric $(h_i = h_{s-i} \ \forall \ i)$.

Simple examples of Gorenstein rings are $R = S/I$ where I is generated by $n - \dim R$ polynomials. Such rings are called *complete intersections*.

Stanley-Reisner correspondence II

For which simplicial complexes is the Stanly-Reisner ring Cohen-Macaulay? And Gorenstein?

For example, one can show that if Δ is a matroid, then $K[\Delta]$ is Cohen-Macaulay. Furthermore, if Δ is a matroid then $K[\Delta]$ is Gorenstein $\iff$ it is a complete intersection. However matroids are a very special case of Cohen-Macaulay simplicial complexes:

Theorem (_ 2011, Minh-Trung 2011)

A simplicial complex Δ on $[n]$ is a matroid if and only if $S/I_\Delta^{(k)}$ is Cohen-Macaulay for any $k \geq 1$, where $I_\Delta^{(k)} = \bigcap_{\sigma \in \mathcal{F}(\Delta)} (x_i : i \notin \sigma)^k$.

Simplicial complexes II

Let $\sigma \subseteq [n]$, $v \in \sigma$ and $q = |\{i \in \sigma : i < v\}|$. We set

$$\text{sign}(v, \sigma) := (-1)^q.$$

Let Δ be a $(d-1)$-simplicial complex on $[n]$, and R a ring with a unit. For any $i = -1, 0, \ldots, d-1$, let

$$C_i(\Delta; R) := \bigoplus_{\substack{\sigma \in \Delta \\ \dim \sigma = i}} R$$

be the free R-module with basis $\{e_\sigma : \sigma \ i\text{-dimensional face of } \Delta\}$.

The **reduced simplicial homology** with coefficients in R of Δ is the homology of the complex of free R-modules $(C_i(\Delta;R),\partial_i)$:

$$\begin{aligned}\partial_i: \quad C_i(\Delta;R) \to &\quad C_{i-1}(\Delta;R)\\ e_\sigma \mapsto &\quad \sum_{v\in\sigma}\mathrm{sign}(v,\sigma)\cdot e_{\sigma\setminus\{v\}}\end{aligned}$$

We denote such an ith reduced simplicial homology by $\widetilde{H_i}(\Delta;R)$.

It is a classical fact that simplicial and singular homology agree:

$$\widetilde{H_i}(\Delta;R) \cong \widetilde{H_i}(\|\Delta\|;R).$$

Simplicial complexes II

The **link** of a face $\sigma \in \Delta$ is the simplicial complex:

$$\mathrm{lk}_\Delta \sigma := \{\tau \subseteq [n] : \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset\}.$$

Notice that $\Delta = \mathrm{lk}_\Delta \emptyset$.

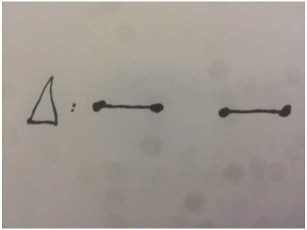
Theorem (Reisner, 1976)

For a simplicial complex Δ on $[n]$ the following are equivalent:

- $K[\Delta]$ is Cohen-Macaulay.
- $\widetilde{H}_i(\mathrm{lk}_\Delta \sigma; K) = 0$ for all $\sigma \in \Delta$ and $i < \dim \mathrm{lk}_\Delta \sigma$.
- If $X = \|\Delta\|$, $\widetilde{H}_i(X; K) = \widetilde{H}_i(X, X \setminus P; K) = 0$ for all $P \in X$ and $i < \dim X$.

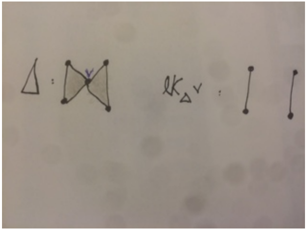
In particular, *the Cohen-Macaulay property of $K[\Delta]$ is topological !*

Examples



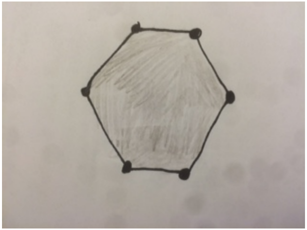
$K[\Delta]$ not CM

$\dim \Delta = 1$,
but $\widetilde{H_0}(\Delta; K) = K$.
$\mathrm{HS}_{K[\Delta]} = \frac{1+2t-t^2}{(1-t)^2}$

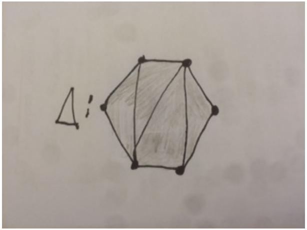


$K[\Delta]$ not CM

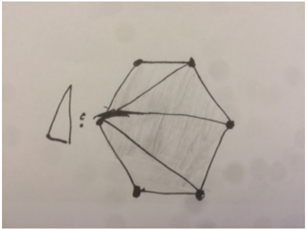
$\dim \Delta = 2$ and
$\widetilde{H_0}(\Delta; K) = \widetilde{H_1}(\Delta; K) = 0$,
but $\dim \mathrm{lk}_\Delta v = 1$ and
$\widetilde{H_0}(\mathrm{lk}_\Delta v; K) = 0$
$\mathrm{HS}_{K[\Delta]} = \frac{1+2t-t^2}{(1-t)^2}$



2-ball

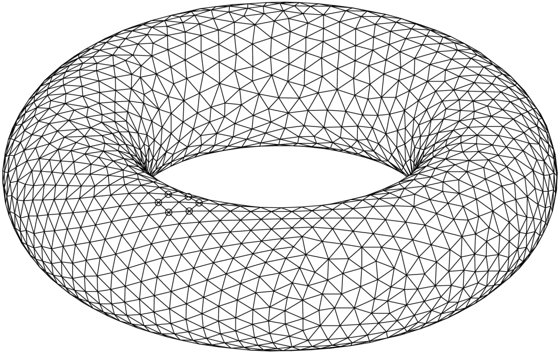


$K[\Delta]$ CM



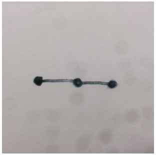
$K[\Delta]$ CM

Examples

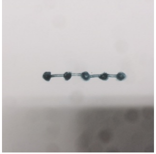


$K[\Delta]$ not CM: $\dim \Delta = 2$ but $\widetilde{H_1}(\Delta; K) = K^2$

We saw that the Cohen-Macaulay property of $K[\Delta]$ is topological, what about the Gorenstein property? Unfortunately ...



Gorenstein



not Gorenstein

However, being Gorenstein is almost a topological property...

Simplicial complexes II

Definition

A simplicial complex Δ is a **homology sphere** if one of the following two equivalent conditions holds:

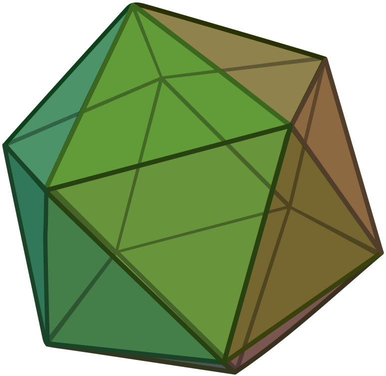
- For all $\sigma \in \Delta$, one has $\widetilde{H}_i(\mathrm{lk}_\Delta \sigma; K) = 0$ for all $i < \dim \mathrm{lk}_\Delta \sigma$ and $\widetilde{H}_{\dim \mathrm{lk}_\Delta \sigma}(\mathrm{lk}_\Delta \sigma; K) \cong K$.
- If $X = \|\Delta\|$, for all $P \in X$ one has $\widetilde{H}_i(X; K) = \widetilde{H}_i(X, X \setminus P; K) = 0$ for all $i < \dim X$ and $\widetilde{H}_{\dim X}(X; K) \cong \widetilde{H}_{\dim X}(X, X \setminus P; K) \cong K$.

Theorem (Stanley, 1977)

For a simplicial complex Δ, set $U = \bigcap_{\sigma \in \mathcal{F}(\Delta)} \sigma$. Then the following two conditions are equivalent:

- $K[\Delta]$ is Gorenstein.
- The simplicial complex $\langle \sigma \setminus U : \sigma \in \mathcal{F}(\Delta) \rangle$ is a homology sphere.

Example



$K[\Delta]$ is Gorenstein, since all the links of Δ are spheres

Proof of UBC

Let Δ be a $(d-1)$-sphere on $[n]$. It is harmless to assume that K is infinite, so let $\theta_1, \ldots, \theta_d \in S$ be a l.s.o.p. of $K[\Delta]$. Let

$$A := K[\Delta]/(\overline{\theta_1}, \ldots, \overline{\theta_d}) \cong S'/\overline{I_\Delta},$$

where $S' = S/(\theta_1, \ldots, \theta_d)$ is a polynomial ring in $n-d$ variables over K. Therefore

$$\dim_K A_i \leq \dim_K S'_i = \binom{n-d+i-1}{i}.$$

Since $K[\Delta]$ is Cohen-Macaulay (indeed even Gorenstein) the h-vector $(h_1, \ldots, h_d)$ of Δ satisfies

$$h_i = \dim_K A_i \qquad \square$$

The g-conjecture

Definition

A vector $h = (h_0, \ldots, h_d)$ is an M-**vector** if there is a 0-dimensional standard graded algebra R such that $\text{HF}_R(i) = h_i$. The g-**vector** of h is $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$.

For example, we saw that an h-vector of a Cohen-Macaulay simplicial complex is an M-vector.

Let Δ be the boundary of a simplicial d-polytope. Then $\|\Delta\|$ is a $(d-1)$-sphere, so the h-vector of Δ (i.e. of $K[\Delta]$) satisfies $h_i = h_{d-i}$.

The g-conjecture for polytopes (McMullen, 1971)

A vector $h = (h_0, \ldots, h_d)$ is the h-vector of the boundary of a simplicial d-polytope if and only if $h_i = h_{d-i}$ and its g-vector is an M-vector.

In particular, the h-vector of the boundary of a d-polytope would be unimodal, that is:

$$h_0 \leq h_1 \leq \ldots \leq h_{\lfloor d/2 \rfloor} = h_{\lceil d/2 \rceil} \geq \ldots \geq h_d.$$

The g-conjecture

Definition

A 0-dimensional standard graded algebra R satisfies the **hard Lefschetz theorem** if there is a linear form $\ell \in R$ such that

$$\times \ell^{d-2i} : R_i \longrightarrow R_{d-i}, \quad \text{where } d = \max\{j : R_j \neq 0\}$$

is an isomorphism for all $i \leq \lfloor d/2 \rfloor$.

Lemma

Let R be a 0-dimensional standard graded algebra satisfying the hard Lefschetz theorem, with h-vector $h = (h_0, \ldots, h_d)$. Then the g-vector of h is an M-vector.

Proof: If $i < \lfloor d/2 \rfloor$, notice that the isomorphism $\times \ell^{d-2i} : R_i \to R_{d-i}$ is

$$R_i \xrightarrow{\times \ell} R_{i+1} \xrightarrow{\times \ell^{d-2i-1}} R_{d-i},$$

so $\times \ell : R_i \to R_{i+1}$ is *injective* for all $i < \lfloor d/2 \rfloor$.

Consider the 0-dimensional standard graded algebra $A = R/(\ell)$, and note:

$$\dim_K A_i = \dim_K R_i - \dim_K \ell R_{i-1} = h_i - h_{i-1} \quad \forall\, i \leq \lfloor d/2 \rfloor \quad \square.$$

The g-conjecture for polytopes has been proven in 1980 by **Billera-Lee** (if - part) and **Stanley** (only if - part). In the next slides follows a description of the idea of the original Stanley's proof .....

The g-conjecture

Given a d-dimensional projective variety $X \subseteq \mathbb{P}^n$ over $\mathbb{C}$, let

$$H^{2*}(X;\mathbb{C}) := \bigoplus_{i=0}^{d} H^{2i}(X;\mathbb{C})$$

be its **even cohomology ring**. The product is given by the *cup-product*, which makes of it a commutative ring.

To any subvariety $Y \subseteq X$ of codimension k can be associated a *cohomology class* $[Y] \in H^{2k}(X;\mathbb{C})$.

In particular, taking as Y a general hyperplane section of X, we get an element of $H^2(X;\mathbb{C})$. Such an element is known as the **fundamental class**, and usually denoted by $\omega \in H^2(X;\mathbb{C})$.

The classical hard Lefschetz theorem, first proved completely by **Hodge**, states that if X is nonsingular, then the multiplication map

$$\times\omega^{d-k} : H^k(X, \mathbb{C}) \to H^{2d-k}(X; \mathbb{C})$$

is an isomorphism of $\mathbb{C}$-vector spaces. .

Now, to any complete fan Σ (collection of cones with certain gluing properties), one can associate a scheme X_Σ. Given a d-polytope $P \subseteq \mathbb{Q}^d$ containing the origin in its interior, for any proper face $F \in P$ consider the cone σ_F consisting of the points belonging to some ray through the origin and F. As F varies, we get a complete fan Σ, the so-called *normal fan* of P, and the corresponding scheme X_Σ, that we denote by $X(P)$. It turns out that $X(P)$ is projective, i.e. it admits an embedding $X(P) \subseteq \mathbb{P}^n$.

Unfortunately $X(P)$ may have singularities... However, if P is simplicial, then it has finite (Abelian) quotient singularities (this means that $X(P)$ is not smooth but it is very close to!). By a result of **Steenbrink**, $H^{2*}(X, \mathbb{C})$ satisfies the hard Lefschetz theorem for any projective variety with finite quotient singularities. So $H^{2*}(X(P), \mathbb{C})$ satisfies the hard Lefschetz theorem.

If Δ is the boundary of P, then **Danilov** proved in 1978 that there exists a l.s.o.p. $\theta_1, \ldots, \theta_d$ of $\mathbb{C}[\Delta]$ such that there is a graded isomorphism

$$H^{2*}(X(P), \mathbb{C}) \cong \mathbb{C}[\Delta]/(\overline{\theta_1}, \ldots, \overline{\theta_d}),$$

where the degree i piece of $H^{2*}(X(P), \mathbb{C})$ is $H^{2i}(X(P), \mathbb{C})$.

Therefore $\mathbb{C}[\Delta]/(\overline{\theta_1}, \ldots, \overline{\theta_d})$ satisfies the hard Lefschetz theorem, so the g-vector of its h-vector is an M-vector. Since the h-vector of $\mathbb{C}[\Delta]$ agrees with that of $\mathbb{C}[\Delta]/(\overline{\theta_1}, \ldots, \overline{\theta_d})$ (because $K[\Delta]$ is CM) we conclude. $\square$

The following is still open:

The g-conjecture for spheres

If a vector $h = (h_0, \ldots, h_d)$ is the h-vector of a $(d-1)$-sphere, then $h_i = h_{d-i}$ and its g-vector is an M-vector.

Also, **what about Gorenstein simplicial complexes ???**

Note: Standard graded Gorenstein K-algebras in general have not unimodal h-vector (example of **Stanley** with h-vector $(1, 13, 12, 13, 1)$). So far it seems there are no examples among Stanley-Reisner rings ...