Bull. London Math. Soc. 46 (2014) 924–942
doi:10.1112/blms/bdu047

Generic and special constructions of pure O-sequences

Alexandru Constantinescu, Thomas Kahle and Matteo Varbaro

Abstract

It is shown that the h-vectors of Stanley–Reisner rings of three classes of matroids are pure O-sequences. The classes are (a) matroids that are truncations of matroids, or more generally of Cohen–Macaulay complexes, (b) matroids whose dual is $(\text{rank} + 2)$-partite, and (c) matroids of Cohen–Macaulay type at most 5. Consequences for the computational search for a counterexample to a conjecture of Stanley are discussed.

Introduction

The f-vector and h-vector are fundamental invariants of a simplicial complex, encoding the number of faces that the complex has in each dimension. What can be said in general about these vectors? Starting from Euler's polyhedron formula in the middle of the 18th century, different conditions and eventually characterizations have been found. It seems natural to ask for a description of the set of f- or equivalently h-vectors of all simplicial complexes or all pure simplicial complexes in a given dimension. The situations for these two classes are quite different. There is a precise characterization of the set of f-vectors of all simplicial complexes due to Schützenberger, Kruskal, and Katona [**40**, Theorem II.2.1]. The opposite is the case for pure simplicial complexes, a characterization is believed to be intractable. As Ziegler points out, it would *solve all basic problems in design theory* [**43**, Exercise 8.16]. The celebrated g-theorem characterizes h-vectors of simplicial polytopes [**4**, **5**, **39**] and it is conjectured that this characterization also applies to simplicial spheres (of which there are many more than boundaries of simplicial polytopes [**28**]). This indicates that subclasses of pure complexes, like Gorenstein, Cohen–Macaulay, or matroid complexes, may be feasible. It is known for a long time, essentially due to Macaulay, that the sets of vectors that arise as h-vectors of Cohen–Macaulay complexes consist exactly of *O-sequences*, Hilbert functions of Artinian algebras [**30**]. Although necessary conditions are known, characterizations for matroid or Gorenstein complexes are open and may be out of reach.

In this paper, we focus on matroids. They were originally introduced by Whitney as a way to study the concept of independence [**42**]. Subsequently, they appeared in a wide range of mathematical areas from linear algebra, (real) algebraic geometry, and combinatorial geometry to graph theory, optimization, and approximation theory. The new edition of Oxley's book [**37**] provides an excellent guide to the theory. Interest in algebraic properties of matroids is still growing as witnessed by recent work of DeConcini and Procesi [**14**], Holtz and Ron [**25**], Lenz [**29**], Moci [**35**], and Huh [**26**, **27**].

What properties should the h-vector of a matroid have? Since matroids are Cohen–Macaulay, their h-vectors must be O-sequences. In [**38**], Stanley shows that they are also Hilbert functions of Artinian algebras whose socle is concentrated in one degree. He conjectured that for any matroid one can even find a *monomial* algebra with this property. In this case, its Hilbert function is called a *pure O-sequence*.

Received 13 December 2012; revised 2 April 2014; published online 27 June 2014.

2010 *Mathematics Subject Classification* 52B05, 05E40 (primary), 13D40, 13E10, 13F55 (secondary).

The research for this project was carried out during stays of the authors at CIRM, Trento and MSRI, Berkeley.

CONJECTURE [**38**, p. 59]. The h-vector of a matroid complex is a pure O-sequence.

For an abstract simplicial complex Δ on $[n] := \{1, \ldots, n\}$, let $f_i(\Delta)$ be the number of faces of size i. Let $d = \max\{i : f_i \neq 0\}$ be the *rank* of Δ. The vector $f = (f_0, \ldots, f_d)$ is the f-*vector* of Δ. It encodes the same information as the h-*vector* $h(\Delta) = (h_0, \ldots, h_s)$ whose component h_i is the coefficient of x^{d-i} in the polynomial $\sum_{i=0}^{d} f_i(x-1)^{d-i}$. A central tool for the study of the h-vector is the *Stanley–Reisner ring* $\Bbbk[\Delta] := \Bbbk[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta = (\prod_{i \in G} x_i : G \notin \Delta)$ is the *Stanley–Reisner ideal*. In this setting, the h-vector appears as the coefficient vector of the numerator polynomial of the Hilbert series of $\Bbbk[\Delta]$ (see [**40**]). The field $\Bbbk$ in this definition is arbitrary, and homological properties of $\Bbbk[\Delta]$ may depend on the characteristic. However, Stanley's conjecture is field independent.

The problem raised by Stanley is extremely difficult and the authors are not strong believers in the validity of the conjecture. The complications are in part due to the strange properties of pure O-sequences. For instance, they need not be unimodal, and it is likely that they cannot be characterized well [**6**]. On the positive side, it is known that both pure O-sequences and h-vectors of matroid complexes satisfy a common set of inequalities [**10**, **24**]:

$$h_0 \leqslant h_1 \leqslant \ldots \leqslant h_{\lfloor s/2 \rfloor}, \quad h_i \leqslant h_{s-i} \quad \text{for } 0 \leqslant i \leqslant \left\lfloor \frac{s}{2} \right\rfloor.$$

In contrast, the Brown–Colbourn inequalities

$$\text{for any } b \geqslant 1 \quad (-1)^j \sum_{i=0}^{j} (-b)^i h_i \geqslant 0, \ 0 \leqslant j \leqslant s$$

hold for h-vectors of matroids, but not pure O-sequences [**7**]. Other than this our understanding is poor. Positive answers to Stanley's conjecture are known for short h-vectors [**15**, **21**], and for special classes of matroids [**32**, **33**, **36**]. In the present paper, we prove that Stanley's conjecture holds for matroids that are truncations of other matroids and for matroids whose h-vector $(1, h_1, \ldots, h_s)$ satisfies $h_s \leqslant 5$ (with no restriction on s). We employ two completely different methods of proof, both of which have potential for generalizations. As a consequence of our results, the search for counterexamples is pushed closer to today's computational limits.

Generic pure O-sequences

The Stanley–Reisner ring $\Bbbk[\Delta]$ of a matroid Δ is level. To produce a pure O-sequence which equals the h-vector of Δ it would suffice to pass to a *monomial* Artinian reduction. Unfortunately, a monomial ideal rarely has one. In this context, the generic initial ideal may come to mind. It has the same h-vector as the original ideal and (in characteristic zero) is strongly stable. Therefore, it possesses a regular sequence of variables and a monomial Artinian reduction. However, this does not prove Stanley's conjecture as typically the quotient modulo the generic initial ideal is not level. We envision an approach to Stanley's conjecture in which one interpolates between these two objectives with a less drastic version of the generic initial ideal (Remark 1.5). In Section 1, we study this genericity of matroids and show that a generalization of Stanley's conjecture holds for all simplicial complexes that are truncations (skeletons) of matroids (Theorem 1.10).

Special pure O-sequences

In matroid theory, duality is central. If Δ is a matroid, then the complex Δ^c whose facets are the complements of facets of Δ is the *dual matroid*. Directly from the definitions, its Stanley–Reisner ideal I_{Δ^c} equals the *cover ideal* $J(\Delta)$ of Δ. In this paper, h_Δ is the h-vector of (the quotient by) I_Δ and h^Δ that of (the quotient by) $J(\Delta)$. By matroid duality, it suffices to prove Stanley's conjecture for either of the classes. Several known results on matroid complexes are stated in terms of the dual matroid [15, 32, 36], which may be taken as an indication that the cover ideal is a natural object. This perspective permeates the work of the first and third authors and also our Section 2, where we aim at a generalization of the construction of pure O-sequences in [13]. This construction is recursive and relies on finding pure O-sequences for links and deletions in the matroid. When trying to generalize the construction, we require a *compatibility condition* (Lemma 2.1 and Definition 2.6) the checking of which remains an obstacle. Carefully keeping track of the contributions in the recursion allows us to prove Stanley's conjecture for duals of matroids with at most rank + 2 parallel classes (Theorem 2.18). Exploiting the constraints on the h-vectors of matroids whose dual has a fixed number of parallel classes, proved in [13], we can show Stanley's conjecture when the type is at most 5 (Theorem 3.3).

The search for a counterexample

Matroids on nine or fewer elements have been enumerated by Mayhew and Royle [31] and Stanley's conjecture has been confirmed for all of them in [15]. Beyond nine vertices, mostly due to the lack of a good list of candidates, only sporadic experiments have been carried out. Our results have implications for the search for a counterexample. By Theorem 3.3, any candidate counterexample must be of Cohen–Macaulay type at least 6. To confirm such a counterexample in silico would include enumeration of all $\binom{N}{6}$ socles where N is a binomial coefficient (see Example 4.1). The methods of Section 2, in particular Lemma 2.1, imply faster searches for pure O-sequences realizing the h-vector of the cover ideal of a given matroid. In Section 4, we discuss our computational efforts. As part of this project, we developed a small `C++`-library which can be used to enumerate pure O-sequences The source code is available at https://github.com/tom111/GraphBinomials and is licensed under the GNU general public license. We also made intensive use of CoCoA [11], Macaulay2 [20], and Sage [41].

1. *Linear resolutions and the generic initial ideal*

Let $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring over a field $\Bbbk$. For any ideal $I \subseteq S$, we denote $\mathrm{gin}(I)$ the *generic initial ideal* with respect to the graded reverse lexicographic term order. Any graded S-module M has a minimal graded free resolution:

$$0 \longrightarrow F_p \xrightarrow{\delta_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0,$$

in which $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. Let $Z_i(M) = \ker \delta_i$ be the ith syzygy module of M. The module M has a *k-linear resolution* if $\beta_{i,j}(M) = 0$ whenever $j \neq i + k$. It is *componentwise linear* if $M_{\langle k\rangle}$ has k-linear resolution for all $k \in \mathbb{Z}$, where $M_{\langle k\rangle}$ is the submodule of M generated by all homogeneous elements of degree k. It is not difficult to show that, if M has a linear resolution, then it is componentwise linear, for example, using [12, Corollary 2.5]. Linearity of the free resolution is a genericity condition. This intuition is justified by the following theorem.

THEOREM 1.1 [1, Theorem 1.1]. *Let char$(\Bbbk) = 0$. An ideal $I \subset S$ is componentwise linear if and only if $\beta_{i,j}(S/I) = \beta_{i,j}(S/\mathrm{gin}(I))$ for all i, j.*

Since $I = Z_0(S/I)$, one may ask which conclusions are implied if $Z_i(S/I)$ is componentwise linear. The following result gives one direction.

PROPOSITION 1.2 [9, Theorem 5.7]. *Let $I \subset S$ be a graded ideal such that $\beta_{i,j}(S/I) = \beta_{i,j}(S/\mathrm{gin}(I))$ for all $i > s + 1$ and $j \in \mathbb{Z}$. Then $Z_s(S/I)$ is componentwise linear.*

In general, the other implication in Proposition 1.2 does not hold (Example 1.4). In fact, it would imply Stanley's conjecture for cover ideals of simple matroids. To see this, let $I \subset S$ be an ideal such that $S/\operatorname{gin}(I)$ is level. In characteristic zero, the generic initial ideal is strongly stable and thus $x_n, x_{n-1}, \ldots, x_{d+1}$ is a regular sequence in $S/\operatorname{gin}(I)$. The Artinian reduction $S/(\operatorname{gin}(I) + (x_n, x_{n-1}, \ldots, x_{d+1}))$ is an Artinian level monomial algebra with the same h-vector as S/I. In fact, having a binomial regular sequence would suffice to ensure monomiality of the quotient (see Remark 1.5). Consequently, the h-vector of S/I is a pure O-sequence. If the converse of Proposition 1.2 were true, then the h-vector of any level algebra whose second to last syzygy module is componentwise linear would be a pure O-sequence. This is the case for cover ideals of *simple* matroids, that is, matroids without parallel elements.

PROPOSITION 1.3. *Let Δ be a rank d simple matroid on n vertices. Then $\beta_{d-1,j}(S/J(\Delta)) \neq 0$ only for $j = n-1$. In particular, $Z_{d-2}(S/J(\Delta))$ is componentwise linear.*

Proof. Let $\Gamma = \Delta^c$. Hochster's formula implies:

$$\beta_{d-1,j}(S/J(\Delta)) = \beta_{d-1,j}(S/I_\Gamma) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_{\Bbbk} \tilde{H}_{j-d}(\Gamma_W, \Bbbk),$$

where Γ_W denotes the restriction of Γ to the vertex subset W. If $j > n-1$, then the only summand that could occur is $\dim_{\Bbbk} \tilde{H}_{n-d}(\Gamma, \Bbbk) = 0$ in the case $j = n$. If $j < n-1$, then we can find two distinct vertices outside of W. Since Δ is simple, they must be contained in a facet F of Δ. Therefore, $G := [n] \setminus F$ is a facet of Γ, and $|G \cap ([n] \setminus W)| \leqslant n - j - 2$. Thus, $\dim(\Gamma_W) \geqslant j - d + 1$. By Reisner's criterion, $\widetilde{H}_{j-d}(\Gamma_W, \Bbbk) = 0$ since Γ_W is a matroid and can thus have only top-dimensional homology. $\square$

EXAMPLE 1.4. Let Δ be the rank 3 simple matroid on $\{1, \ldots, 7\}$ with the following facets:

$$123, 124, 125, 127, 135, 136, 137, 145, 146, 147, 156, 167, 234, 235,$$
$$236, 246, 247, 256, 257, 267, 345, 346, 347, 357, 367, 456, 457, 567,$$

commonly known as the Fano matroid. A quick computation with Macaulay2 shows that the cover ideal is level of Cohen–Macaulay type 8 while its generic initial ideal is not level ($\beta_3(S/\operatorname{gin}(J(\Delta)) = 10$). Since Δ is simple, Proposition 1.3 shows that $Z_1(S/J(\Delta))$ is componentwise linear.

REMARK 1.5. Propositions 1.2 and 1.3 inspired the search for a less generic initial ideal in which the coordinate transform has block structure. The hope was to find a construction that balances between preserving the last Betti number, yielding a *level* quotient, and maintaining the existence of a binomial regular sequence, needed to have a *monomial* quotient. However, we did not find a definition that realizes just the right balance.

If the generic initial ideal of I_Δ is level, then h_Δ is a pure O-sequence since it equals the Hilbert function of the Artinian reduction of $\operatorname{gin}(I_\Delta)$ by variables. To implement this strategy, we employ the following two general lemmas. Following [**22**], let $I_{<k}$ denote the subideal of a homogeneous ideal I generated by the homogeneous elements of I of degree less than k.

LEMMA 1.6. *Let $I \subset S$ be a homogeneous ideal of projective dimension p and regularity k. If $\operatorname{pd}(I_{<k}) < p$ and char$(\Bbbk) = 0$, then $\beta_p(I) = \beta_{p,p+k}(I) = \beta_{p,p+k}(\operatorname{gin}(I)) = \beta_p(\operatorname{gin}(I))$.*

Proof. Let $J_1 = \text{gin}(I)_{<k}$ and $J_2 = \text{gin}(I_{<k})_{<k}$. It is easy to see that $J_1 = J_2$. In characteristic zero, the generic initial ideal is strongly stable and [**3**, Theorem 2.4(a)] shows $\text{pd}(J_2) < p$. Using the Eliahou–Kervaire resolution [**18**, Theorem 2.1], we get that no monomial $x_{p+1}u$ with $u \in \Bbbk[x_1, \dots, x_{p+1}]$ is a minimal generator of $J_2 = J_1$. Therefore, any minimal generator of $\text{gin}(I)$ of the form $x_{p+1}u$ with $u \in \Bbbk[x_1, \dots, x_{p+1}]$ must be of degree at least k. Since, by [**3**, Theorem 2.4(b)], we have $\text{reg}(I) = \text{reg}(\text{gin}(I))$ it must be of degree exactly k.

The Eliahou–Kervaire formula [**23**, Corollary 7.2.3] gives one of the equations: $\beta_p(\text{gin}(I)) = \beta_{p,p+k}(\text{gin}(I))$. Since $\beta_{p,p+k}(I)$ is an extremal Betti number, we have $\beta_{p,p+k}(\text{gin}(I)) = \beta_{p,p+k}(I)$ by [**2**, Corollary 1.3]. Finally, it is a general fact (see, for example, [**34**, Theorem 8.29]) that $\beta_{p,p+j}(I) \leqslant \beta_{p,p+j}(\text{gin}(I))$ for any j, so actually $\beta_p(I) = \beta_{p,p+k}(I) = \beta_{p,p+k}(\text{gin}(I)) = \beta_p(\text{gin}(I))$. □

LEMMA 1.7. *Let Δ be a Cohen–Macaulay complex of dimension d, and F be a minimal non-face of cardinality $d+1$. Then $\Delta \cup F$ is Cohen–Macaulay.*

Proof. Let $\langle F \rangle$ denote the complex on $[n]$ with one facet F. By construction, $\langle F \rangle \cap \Delta$ is the boundary of a d-simplex. In particular, $\Bbbk[\langle F \rangle \cap \Delta]$ is a d-dimensional Cohen–Macaulay ring. So the statement follows at once from the exact sequence

$$0 \longrightarrow \Bbbk[\Delta \cup F] \longrightarrow \Bbbk[\Delta] \oplus \Bbbk[\langle F \rangle] \longrightarrow \Bbbk[\langle F \rangle \cap \Delta] \longrightarrow 0,$$

and the depth inequalities. □

The following theorem is the main result of this section. We state it for Stanley–Reisner ideals.

THEOREM 1.8. *Let Δ be the $(d-1)$-skeleton of a d-dimensional Cohen–Macaulay complex. Then h_Δ is a pure O-sequence. Furthermore, if char$(\Bbbk) = 0$, then $\Bbbk[\Delta]$ is level.*

Proof. By Hochster's formula, $\text{reg}(\Bbbk[\Delta]) \leqslant d$ and since I_Δ has a generator of degree $d+1$, $\text{reg}(\Bbbk[\Delta]) = d$. Write $J = (I_\Delta)_{<d+1}$ and let Γ be the corresponding simplicial complex. The result follows from Lemma 1.6 once we show $\text{depth}(\Bbbk[\Gamma]) > d$, which, in turn, is equivalent to the d-skeleton of Γ being Cohen–Macaulay. The d-skeleton of Γ is the complex Γ_d that arises from Δ by turning all non-faces of size $d+1$ into facets. Now, Δ is the $(d-1)$-skeleton of a d-dimensional Cohen–Macaulay complex Ω. There are two kinds of facets of Γ_d: those that are facets of Ω and those that are not. Those that are not are minimal non-faces in Ω. By Lemma 1.7, Γ_d is Cohen–Macaulay. The statement about the h-vector is characteristic-free because the h-vector of a simplicial complex does not depend on the coefficient field. □

It is equivalent to say that a vector is the Hilbert function of an Artinian monomial algebra and that it is the f-vector of an *order ideal of monomials*, also known as a *multicomplex*. In this language, pure O-sequences are f-vectors of pure multicomplexes. Similar to simplicial complexes, there are theories of shellability of multicomplexes (such as M-shellability) and the work of Chari suggests that a characterization of f-vectors of shellable multicomplexes may be possible [**10**]. He also conjectures that the h-vector of any coloop-free matroid is a *shellable O-sequence* [**10**, Conjecture 3] which would imply Stanley's conjecture.

REMARK 1.9. Let $I \subset S = \Bbbk[x_1, \dots, x_r]$ be a strongly stable ideal such that S/I is an Artinian level ring. In this case, the h-vector of S/I is the f-vector of an M-shellable multicomplex.

Proof. By the Eliahou–Kervaire resolution, the variable x_r appears only in the minimal generators of I of maximal degree. Let k be this maximal degree, and let $u_1, \ldots, u_t$ be the degree k minimal generators of I divisible by x_r. Write $u_i = v_i x_r$ for all $i = 1, \ldots, t$. One easily checks that $v_1, \ldots, v_t$ generate the order ideal of S/I. Let $\prec$ be the graded revlex order induced by $x_r > \ldots > x_1$. We can assume $v_1 \prec \ldots \prec v_t$. Now write $v_i = v_i' x_r^{e_i}$, where e_i is the maximum power of x_r dividing v_i, and let $V_i = \{v_i' x_r^j : j = 0, \ldots, e_i\}$. We claim that $V_t, \ldots, V_1$ is a shelling of the multicomplex S/I. It remains to show that, if u is a monomial of degree e dividing v_i, then there exists $j \geqslant i$ such that $v_j = u x_r^{k-e-1}$. Let m be the monomial of degree $k-e-1$ such that $v_i = um$. If no such j existed, then $u x_r^{k-e-1}$ would be in I, so there would exist a minimal generator u' of I, say of degree a, such that $u = u'u''$ for some u''. Then $u' x_r^{e-a}$ would be in I as well. Since I is strongly stable, $u = u' x_r^{e-a} / x_r^{e-a} \cdot u'' \in I$. This is a contradiction to u_i being a minimal generator. $\square$

In matroid theory, passing from a matroid of rank d to its k-skeleton for $k < d-1$ is called a *truncation*. The rank function of the truncation is $A \mapsto \min\{\mathrm{rk}(A), k+1\}$. The shift of one arises because the k-skeleton is of dimension k which means rank $k+1$. All together, we have the following theorem.

THEOREM 1.10. *Any truncation of a matroid satisfies Chari's conjecture and consequently also Stanley's conjecture.*

Proof. If $I \subset S$ is a strongly stable ideal such that S/I is level, then the h-vector of S/I is the f-vector of an M-shellable multicomplex by Remark 1.9. By Theorem 1.8, the h-vectors of truncated matroids satisfy Chari's and consequently also Stanley's conjecture. $\square$

Evidently the next question is: Which matroids are truncations? Certainly not all of them.

EXAMPLE 1.11. Any complete bipartite graph is a rank 2 matroid that is not the truncation of a matroid. More generally, any matroid that becomes a simplex after identifying parallel elements is not a truncation.

REMARK 1.12. If a rank d matroid Γ is a truncation, then it is a truncation of a rank $d+1$ matroid Δ. In this case, any facet of Δ is a *spanning circuit* of Γ, that is, a minimal non-face of size $d+1$. In particular, the facets of Δ are contained in the spanning circuits of Γ. Moreover, if Γ has no spanning circuit, then it is not the truncation of a matroid.

EXAMPLE 1.13. The dual of the Fano matroid from Example 1.4 has no spanning circuit.

REMARK 1.14. Let Δ be a matroid which has a spanning circuit. In [**8**], Brylawski gives an algorithm that decides if there exists a matroid Γ such that Δ is the truncation of Γ, and constructs the *freest* such matroid whenever possible.

In the remainder of the section, we discuss Schubert matroids (also known as shifted matroids, PI-matroids, and generalized Catalan matroids [**19**]). They play an important role in the study of Hopf algebras of (poly)matroids [**16**].

DEFINITION 1.15. Let $1 \leqslant s_1 < s_2 < \ldots < s_d \leqslant n$ be a sequence of strictly ascending integers. The *Schubert matroid* $\mathrm{SM}_n(s_1, \ldots, s_d)$ is the rank d matroid on $[n]$ with facets:

$$\{\{i_1, \ldots, i_d\} : \ i_j \leqslant s_j\}. \tag{1}$$

Remark 1.16. For any simplicial complex Δ, the ideal $(I_\Delta)_{\langle k\rangle}$, generated by the degree k part of I_Δ, is generated by all monomials corresponding to non-faces of size k.

Lemma 1.17. *If $\Delta = \mathrm{SM}_n(s_1, \ldots, s_d)$ is a Schubert matroid of rank d and $s_1 \geqslant 2$, then for any $k < d+1$, $(I_\Delta)_{\langle k\rangle}$ is the ideal generated by the degree k part of the Stanley–Reisner ideal of $\mathrm{SM}_n(s_1 - 1, s_1, \ldots, s_d)$.*

Proof. If $\{j_1, \ldots, j_{d+1}\}$ is a facet of $\mathrm{SM}_n(s_1 - 1, s_1, \ldots, s_d)$, then it is a minimal non-face of $\mathrm{SM}_n(s_1, \ldots, s_d)$ since any $\{j_1, \ldots, \widehat{j_l}, \ldots, j_{d+1}\}$ satisfies (1). On the other hand, if $\{j_1, \ldots, j_{d+1}\}$ is a non-face of $\mathrm{SM}_n(s_1 - 1, s_1, \ldots, s_d)$, then $\{j_2, \ldots, j_{d+1}\}$ is a non-face of $\mathrm{SM}_n(s_1, \ldots, s_d)$, assuming without loss of generality that $j_1 < j_2 < \ldots < j_{d+1}$. By Remark 1.16, the statement holds for any $k < d+1$. □

Theorem 1.18. *Schubert matroids have componentwise linear Stanley–Reisner ideals and in particular satisfy Chari's (and thus Stanley's) conjecture.*

Proof. If $s_1 = 1$, then $\mathrm{SM}_n(s_1, \ldots, s_d) \cong \mathrm{SM}_{n-1}(s_1 - 1, \ldots, s_d - 1) * \{v\}$. The Stanley–Reisner ideal of $\mathrm{SM}_{n-1}(s_1 - 1, \ldots, s_d - 1) * \{v\}$ does not use the variable of v and is componentwise linear if and only if the Stanley–Reisner ideal of $\mathrm{SM}_n(s_1, \ldots, s_d)$ is componentwise linear. If $s_d < n$, then $\mathrm{SM}_n(s_1, \ldots, s_d) \cong \mathrm{SM}_{s_d}(s_1, \ldots, s_d)$. The Stanley–Reisner ideal of $\mathrm{SM}_{s_d}(s_1, \ldots, s_d)$ equals that of $\mathrm{SM}_n(s_1, \ldots, s_d)$ plus variables. One is componentwise linear if and only if the other is. Consequently, assume $1 < s_1 < s_2 < \ldots < s_d = n$. We proceed by induction on the corank $n - d$. The base case is $n - d = 1$ in which I_Δ is principal. To check that I is componentwise linear, it suffices to check $I_{\langle k\rangle}$ for any k in which I has minimal generators [**22**], and I_Δ has minimal generators in degrees at most $d+1$. Since $\mathrm{reg}(I_\Delta) = d+1$, the ideal $(I_\Delta)_{\langle d+1\rangle}$ has a linear resolution [**17**, Proposition 1.1]. By Lemma 1.17 and the induction hypothesis, we conclude. □

2. *Matroids with $d+2$ parallel classes*

In the remainder of the paper, we focus on duals of matroids, or equivalently, h-vectors of cover ideals. If Δ is a matroid, then $h^\Delta = h_{\Delta^c}$ is the h-vector of $S/J(\Delta)$, the quotient by the cover ideal of Δ. The one-dimensional skeleton of a matroid is a complete p-partite graph whose groups of vertices correspond to the partition of the vertex set of the matroid set into *parallel classes* [**13**, Corollary 2.3]. The main result of this section (Theorem 2.18) says that Stanley's conjecture holds for cover ideals of matroids whose number of parallel classes is at most 2 more than the rank. Due to the technical nature of the proof, we divide it into several smaller results, give various examples along the way, and state the general theorem at the very end.

Our notation follows closely that of [**13**]. Let Δ be a matroid of rank d, with parallel classes $A_1, \ldots, A_p$, of cardinalities $a_1, \ldots, a_p$. Such matroids are *p-partite*. The *simplification* ${}^{\mathrm{si}}\Delta$ of Δ is the matroid that arises from Δ by replacing each parallel class by a single vertex. We begin with a technical condition to be used in many inductive constructions.

Lemma 2.1. *Let $\Gamma' = \langle N_1, \ldots, N_u\rangle$ be a pure order ideal in variables $y_1, \ldots, y_d$, and let $\Gamma'' = \langle M_1, \ldots, M_v\rangle$ be a pure order ideal in the variables $y_1, \ldots, \widehat{y_r}, \ldots, y_d$, that is, not using y_r. Assume that $h^{\Delta\setminus A_p} = f(\Gamma')$ and that $h^{\mathrm{link}_\Delta A_p} = f(\Gamma'')$. Suppose that $\forall i \in [u]$, $\exists j \in [v]$ such that*

$$\frac{N_i}{y_r^{n_i}} \mid M_j \quad \text{where } n_i = \max\{m : y_r^m \mid N_i\}. \tag{2}$$

Then h^Δ equals the f-vector of the pure order ideal

$$\Gamma = \langle y_r^{a_p} N_1, \ldots, y_r^{a_p} N_u, y_r^{a_p-1} M_1, \ldots, y_r^{a_p-1} M_v \rangle.$$

Proof. By [**13**], we have for any $i \geqslant 0$ that

$$h_i^\Delta = h_{i-a_p}^{\Delta \setminus A_p} + \sum_{j=0}^{a_p-1} h_{i-j}^{\mathrm{link}_\Delta A_p}.$$

It suffices to show the corresponding formula for Γ:

$$f_i(\Gamma) = f_{i-a_p}(\Gamma') + \sum_{j=0}^{a_p-1} f_{i-j}(\Gamma'').$$

Fix an index i and write $\Gamma_i = \{M \in \Gamma : \deg M = i\}$. We write Γ_i as the disjoint union $G_{\geqslant a_p} \sqcup G_{a_p-1} \sqcup \ldots \sqcup G_0$, where $G_j = \{M \in \Gamma_i : y_r^j \mid M \text{ but } y_r^{j+1} \nmid M\}$, and $G_{\geqslant a_p} = \{M \in \Gamma_i : y_r^{a_p} \mid M\}$. If a generator of Γ is divisible by $y_r^{a_p}$, then it cannot come from generators of Γ''. Hence, $f_{i-a_p}(\Gamma') = |G_{\geqslant a_p}|$, and it suffices to check that $f_{i-j}(\Gamma'') = |G_{a_p-j-1}|$. The inequality $f_{i-j}(\Gamma'') \leqslant |G_{a_p-j-1}|$ follows from the definition of Γ. To obtain equality, we confirm that each monomial in G_{a_p-j-1} divides some generator $y_r^{a_p-1} M_l$. Assume that there exists a monomial $M = y_r^{a_p-j-1} M' \in \Gamma$ (with $y_r \nmid M'$), such that $M \mid y_r^{a_p} N_k$, for some k. By (2), there exists l such that

$$\frac{N_k}{y_r^{n_k}} \mid M_l.$$

This implies that $M' \mid M_l$, and as $a_p - j - 1 \leqslant a_p - 1$ we conclude. □

In our inductive proofs, the matroids Γ' are special simplicial complexes for which Stanley's conjecture is known by [**13**]. They are defined as follows.

DEFINITION 2.2. Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a vector of positive integers. Fix integers $2 \leqslant d \leqslant p$ and $0 \leqslant t \leqslant d-2$. Let $A_1, \ldots, A_p$ be disjoint sets of vertices with $|A_i| = a_i$ for any i. The matroid $\Delta_t(d, p, \mathbf{a})$ is the rank d matroid on $\sum_i a_i$ vertices with facets

$$A_{i_1} \cdots A_{i_{d-t}} A_{p-t+1} \cdots A_p \quad \text{where } 1 \leqslant i_1 < \ldots < i_{d-t} \leqslant p-t.$$

Here $A_{j_1} \cdots A_{j_k}$ stands for all sets $\{v_{j_1}, \ldots, v_{j_k}\}$ such that $v_{j_i} \in A_{j_i}$. The matroid $\Delta_0(d, p, \mathbf{a})$ is the *complete matroid* of rank d with p parallel classes of sizes $a_1, \ldots, a_p$.

The simplification of $\Delta_t(d, p, \mathbf{a})$ is isomorphic to $\Delta_t(d, p, \mathbf{1})$, which in turn equals the simplicial join of the uniform matroid $U_{d-t,p-t}$ of rank $d-t$ on $p-t$ vertices, with a simplex on t vertices. The matroids Δ_t appear in [**13**] with a different numbering of the parallel classes, but here we find this convention more natural. The h-vector of the cover ideal of $\Delta_t(d, p, \mathbf{a})$ is a pure O-sequence by [**13**, Theorem 3.7] and we give its order ideal in Example 2.4, after setting up a useful notation.

NOTATION 2.3. Fix positive integers $(a_1, \ldots, a_p)$. For any set partition $\mathcal{P} = P_1 \sqcup \ldots \sqcup P_d$ of $[p]$, denote by $[\mathcal{P}] = [P_1 | P_2 | \cdots | P_d]$ the monomial in d variables:

$$y_1^{-1+\sum_{j \in P_1} a_j} \cdots y_d^{-1+\sum_{j \in P_d} a_j}.$$

When no confusion may arise, we will use this notation for the corresponding partition as well.

EXAMPLE 2.4. Fix integers t, d, p such that $0 \leqslant t \leqslant d-2 \leqslant p-2$, and an integer vector $\mathbf{a} = (a_1, \ldots, a_p)$. For any ascending sequence $1 = l_0 < l_1 < \ldots < l_d = p+1$ of integers, let $\mathcal{P}(l_0, \ldots, l_d)$ be the d-partition into sets $P_i = \{l_{i-1}, \ldots, l_i - 1\}$. We define the following pure order ideal:

$$\Gamma_t(d, p, \mathbf{a}) := \langle [\mathcal{P}(l_0, \ldots, l_{d-t})|p-t+1|\cdots|p] : \text{ for all } 1 = l_0 < l_1 < \ldots < l_{d-t} = p-t+1 \rangle.$$

In particular, when $t = 0$ we have

$$\Gamma_0(d, p, \mathbf{a}) := \langle [\mathcal{P}(l_0, \ldots, l_d)] : \text{ for all } 1 = l_0 < l_1 < \ldots < l_d = p+1 \rangle.$$

By [**13**, Theorem 3.7], the vector $h^{\Delta_t(d,p,\mathbf{a})}$ equals the f-vector of $\Gamma_t(d, p, \mathbf{a})$. This equality is not easy to check in general. One may prove it by induction for complete matroids, then note that

$$\Delta_t(d, p, (a_1, \ldots, a_p)) = \Delta_0(d-t, p-t, (a_1, \ldots, a_{d-t})) * \Delta_0(t, t, (a_{p-t+1}, \ldots, a_p)),$$

check that a similar equality holds for the pure order ideals (viewed as multicomplexes), and finally use the behavior of h-vectors and f-vectors over star products. In this section, we are mainly interested in the case $p = d+1$, where $\Gamma_t(d, d+1, \mathbf{a})$ is generated by

$$\begin{array}{lccccccccl}
[& 1 & | \; 2 & | \; \cdots & | \; d-t+1 & | \; d-t, d-t+1 & | \; d-t+2 & | \; \cdots & | \; d+1 &] \\
[& 1 & | \; 2 & | \; \cdots & | \; d-t+1, d-t & | \; d-t+1 & | \; d-t+2 & | \; \cdots & | \; d+1 &] \\
 & & & & & \vdots & & & & \\
[& 1 & | \; 2,3 & | \; \cdots & | \; d-t & | \; d-t+1 & | \; d-t+2 & | \; \cdots & | \; d+1 &] \\
[& 1,2 & | \; 3 & | \; \cdots & | \; d-t & | \; d-t+1 & | \; d-t+2 & | \; \cdots & | \; d+1 &].
\end{array}$$

In particular, for $t = 1, d = 3, p = 4$ and some $\mathbf{a}$ we obtain:

$$\begin{aligned}
\Gamma_1(3, 4, \mathbf{a}) &= \langle [\mathcal{P}(l_0, l_1, l_2)|4] : \text{ for all } 1 = l_0 < l_1 < l_2 = 4 \rangle \\
&= \langle [\mathcal{P}(1, 2, 4)|4], [\mathcal{P}(1, 3, 4)|4] \rangle \\
&= \langle [1|2, 3|4], [1, 2|3|4] \rangle \\
&= \langle y_1^{a_1-1} y_2^{a_2+a_3-1} y_3^{a_4-1}, y_1^{a_1+a_2-1} y_2^{a_3-1} y_3^{a_4-1} \rangle.
\end{aligned}$$

Plugging in various values for $\mathbf{a}$ one can directly check $h^{\Delta_1(3,4,\mathbf{a})} = f_{\Gamma_1(3,4,\mathbf{a})}$.

DEFINITION 2.5. Let $[P_1|\cdots|P_d]$, $[Q_1|\cdots|Q_d]$ be d-partitions of subsets of $[p]$. For every vector of positive integers $\mathbf{a} = (a_1, \ldots, a_p)$, let $\leqslant_{\mathbf{a}}$ be the partial order defined by

$$[P_1|\cdots|P_d] \leqslant_{\mathbf{a}} [Q_1|\cdots|Q_d] \iff \sum_{j \in P_i} a_j \leqslant \sum_{j \in Q_i} a_j \quad \text{for all } i = 1, \ldots, d.$$

For any $(d-1)$-partition $[Q'_1|\cdots|Q'_{d-1}]$ of $[p]$ and integer $r \in [d]$, a partial order $\leqslant^r_{\mathbf{a}}$ is defined by

$$[P_1|\cdots|P_d] \leqslant^r_{\mathbf{a}} [Q'_1|\cdots|Q'_{d-1}] \iff [P_1|\cdots|\widehat{P_r}|\cdots|P_d] \leqslant_{\mathbf{a}} [Q'_1|\cdots|Q'_{d-1}].$$

The compatibility condition (2) in Lemma 2.1 can be rewritten using the new notation.

DEFINITION 2.6. Let $\mathrm{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ be a set of d-partitions of $[p]$, $\mathrm{Q} = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_r\}$ be a set of $(d-1)$-partitions of $[p]$. For every $r \in [d]$, we say that the sets P, Q satisfy the *r-compatibility condition* if for each $\mathcal{P} \in \mathrm{P}$ there exists a $\mathcal{Q} \in \mathrm{Q}$ such that $\mathcal{P} \leqslant^r_{\mathbf{a}} \mathcal{Q}$.

EXAMPLE 2.7. The sets of partitions $\mathrm{P} = \{[1|2|3,4], [1|2,3|4], [1,2|3|4]\}$ and $\mathrm{Q} = \{[1,2|3,4]\}$ are 3-compatible if and only if $a_2 \leqslant a_4$, while the collections $\mathrm{P}' = \{[1|2,3|4,5], [1,2|3|4,5]\}$ and $\mathrm{Q}' = \{[1|2,3,4,5], [1,2|3,4,5], [1,2,3|4,5], [1,2,3,4|5]\}$ are i-compatible for any $\mathbf{a}$ and any $i = 1,2,3$.

In the new notation, the *gluing* in Lemma 2.1 takes two sets Γ' and Γ'' of partitions of $[p-1]$ and produces a set of d-partitions of $[p]$. The procedure consists of

(i) adding the element p to each rth set of a partition in Γ';
(ii) inserting the set $\{p\}$ into each partition of Γ'' as the rth set, shifting the index of the last $d-r$ sets by 1.

Here is an example of how Lemma 2.1 can be applied. It is one of the base cases in the proof of Proposition 2.17.

EXAMPLE 2.8. Let Δ be the rank 3 matroid with five parallel classes and facets:

$$A_1A_2A_3,\ A_1A_2A_4,\ A_1A_3A_4,\ A_2A_3A_4,\ A_1A_3A_5,\ A_1A_4A_5,\ A_2A_3A_5,\ A_2A_4A_5.$$

As $\Delta \setminus A_5 = \Delta_0(3,4,(a_1,a_2,a_3,a_4))$, it holds that $h^{\Delta\setminus A_5} = f(\Gamma_0(3,4,(a_1,a_2,a_3,a_4)))$, corresponding to

$$\mathrm{P} = \{[1|2|3,4], [1|2,3|,4], [1,2|3|4]\}.$$

The rank 2 matroid $\mathrm{link}_\Delta A_5$ is the complete bipartite graph $\Delta_0(2,2,(a_1+a_2, a_3+a_4))$, and thus its h-vector is obtained from the order ideal generated by $\mathrm{Q} = \{[1,2|3,4]\}$. Example 2.7 shows that P and Q are 3-compatible if and only if $a_2 \leqslant a_4$. Switching the pairs (A_1,A_2) and (A_3,A_4) in Δ gives an isomorphic matroid, therefore we may assume without loss of generality that $a_2 \leqslant a_4$, and obtain by Lemma 2.1 that $h^\Delta = f(\Gamma)$ for

$$\Gamma = \langle [1|2|3,4,5], [1|2,3|4,5], [1,2|3|4,5], [1,2|3,4|5] \rangle.$$

A crucial property of $(d+2)$-partite matroids is that they possess a dual graph, which together with the vector $(a_1,\dots,a_p)$ completely encodes their structure.

DEFINITION 2.9. Let Δ be a matroid of rank d with $d+2$ parallel classes and let ${}^{\mathrm{si}}\Delta$ be its simplification. The graph G_Δ is the rank 2 matroid $({}^{\mathrm{si}}\Delta)^c$.

By construction, G_Δ is a complete q-partite graph on $[d+2]$, for some $q \in \{2,\dots,d+2\}$. If G_Δ is a complete graph on $d+2$ vertices (that is, if $q = d+2$), then its dual is the complete $(d+2)$-partite matroid, for which Stanley's conjecture holds by [**13**, Theorem 3.5]. However, not all complete q-partite graphs have simple matroids as their duals.

REMARK 2.10. For every $d \geqslant 2$, the bipartite graph with partition $\{1,2\} \cup \{3,\dots,d+2\}$ and the tripartite graph with partition $\{1\} \cup \{2\} \cup \{3,\dots,d+2\}$ have duals in which 1 and 2 are parallel and these are the only n-partite graphs with this property.

Proof. The set $\{1,2\}$ is a minimal non-face in the dual of a complete n-partite graph G if and only if every edge of G has at least one of 1 and 2 as a vertex. $\square$

REMARK 2.11. The $[(d-1)+2]$-partite matroid $\operatorname{link}_\Delta A_i$ of rank $d-1$ corresponds to the deletion of i in G_Δ, that is, $G_{\operatorname{link}_\Delta A_i} = G_\Delta \setminus i$. The $(d+1)$-partite matroid $\Delta \setminus A_i$ of rank d corresponds to $\operatorname{link}_{G_\Delta} i$ viewed as a matroid on $[d+2] \setminus \{i\}$. That is, if j is parallel to i in G_Δ, then it is a loop in the rank 1 matroid $\operatorname{link}_{G_\Delta} i$. If the parallel class in G_Δ of $d+2$ (the vertex corresponding to the parallel class A_{d+2} in Δ) has cardinality s, then

$$\Delta \setminus A_{d+2} \cong \Delta_{s-1}(d, d+1, (a_1, \ldots, a_{d+1})).$$

Similar isomorphisms hold for the deletions of the other parallel class A_i and each one is determined by which vertices of G_Δ are parallel to i.

Our proof of Theorem 2.18 is an induction on the number of vertices of G_Δ. Remark 2.10 implies that there are three different bases of induction to consider, dividing the proof into three cases:

(1) G_Δ has at most one parallel class of cardinality at least 2;
(2) G_Δ is bipartite;
(3) G_Δ is r-partite for $r \geqslant 3$, and has at least two parallel classes of cardinality at least 2.

PROPOSITION 2.12. *If G_Δ is complete n-partite on $\{1, \ldots, r\} \cup \{r+1\} \cup \ldots \cup \{d+2\}$, for some $r \geqslant 1$, then h^Δ is a pure O-sequence.*

Proof. The proof is by induction on the number $d+2-r$ of singleton classes. By Remark 2.10, the base case is $d+2-r = 3$, since for larger r the graph G_Δ is not the dual of a simple matroid. Decompose Δ into deletion and link at A_{d+2}. By Remark 2.11, it holds that $\Delta \setminus A_{d+2} = \Delta_0(d, d+1, (a_1, \ldots, a_{d+1}))$, thus its h-vector is realized by $\Gamma' = \Gamma_0(d, d+1, (a_1, \ldots, a_{d+1}))$, which is generated by

$$\mathrm{P} = \{[1|2|\cdots|d-1|d, d+1], [1|2|\cdots|d-1, d|d+1], \ldots, [1, 2|3|\cdots|d|d+1]\}.$$

By Remark 2.10, A_d and A_{d+1} are parallel in $\operatorname{link}_\Delta A_{d+2}$, so by Remark 2.11 we have that $\operatorname{link}_\Delta A_{d+2}$ is the matroid $\Delta_0(d-1, d, (a_1, \ldots, a_{d-1}, a_d + a_{d+1}))$. Thus, $h^{\operatorname{link}_\Delta A_{d+2}} = f(\Gamma'')$, with Γ'' generated by

$$\mathrm{Q} = \{[1|2|\cdots|d-1, d, d+1], [1|2|\cdots|d, d+1], \ldots, [1, 2|3|\cdots|d, d+1]\}.$$

It is easy to check that P and Q are d-compatible.

In the induction step, Γ' is as above and Γ'' is given by the inductive hypothesis. That is to say, we may assume that we applied Lemma 2.1 $(d-r-1)$ times already, and thus, from the last application we have that

$$\Gamma'' \supseteq \langle [1|2|\cdots|d-1, d, d+1], [1|2|\cdots|d, d+1], \ldots, [1, 2|3|\cdots|d, d+1] \rangle.$$

Compatibility is again straightforward and we conclude. □

The second case, when G_Δ is bipartite, follows from a general fact about the join of simplicial complexes (or multicomplexes). Let Δ and Δ' be two simplicial (multi)complexes on disjoint vertex sets. Their *join* is the (multi)complex $\Delta * \Delta' = \{\sigma \cup \sigma' : \sigma \in \Delta \text{ and } \sigma' \in \Delta'\}$. The join operation commutes with duals: $(\Delta * \Delta')^c = \Delta^c * \Delta'^c$. The tensor product of the Stanley–Reisner rings is the Stanley–Reisner ring of their join, and by duality, the same statement holds for tensor product of the quotients by their cover ideals. In the following remark, the simplicial join of two order ideals is computed by viewing them as multicomplexes.

Remark 2.13. Let Δ and Δ' be two matroids, and let Γ and Γ' be two order ideals. If $h^{\Delta} = f(\Gamma)$ and $h^{\Delta'} = f(\Gamma')$, then $h^{\Delta * \Delta'} = f(\Gamma * \Gamma')$.

In the next proposition, we allow also bipartite graphs with partitions of cardinality two (that is, Δ is $(d+1)$ partite). This turns out useful in the third case.

Proposition 2.14. *If G_{Δ} is bipartite with partition $\{1,\ldots,s\} \cup \{s+1,\ldots,d+2\}$, then the h-vector of the cover ideal of Δ is a pure O-sequence.*

Proof. From the bipartition of G_{Δ}, we obtain

$$\Delta = \Delta_0(s-1, s, \mathbf{a}') * \Delta_0(d+1-s, d+2-s, \mathbf{a}''),$$

where $\mathbf{a}' = (a_1,\ldots,a_s)$ and $\mathbf{a}'' = (a_{s+1},\ldots,a_{d+2})$. Thus, [**13**, Theorem 3.5] and Remark 2.13 show that Δ satisfies Stanley's conjecture. $\square$

Example 2.15. If $h^{\Delta} = f(\Gamma_0(s-1, s, \mathbf{a}') * \Gamma_0(d+1-s, d+2-s, \mathbf{a}''))$, then an explicit description of the order ideal generators follows from Example 2.4:

$$\begin{array}{lcccccccccccccccr}
[& 1 & | & \cdots & | & s-2 & | & s-1,s & | & s+1 & | & \cdots & | & d & | & d+1,d+2 &] \\
[& 1 & | & \cdots & | & s-2 & | & s-1,s & | & s+1 & | & \cdots & | & d,d+1 & | & d+2 &] \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1 & | & \cdots & | & s-2 & | & s-1,s & | & s+1,s+2 & | & \cdots & | & d+1 & | & d+2 &] \\
[& 1 & | & \cdots & | & s-2,s-1 & | & s & | & s+1 & | & \cdots & | & d & | & d+1,d+2 &] \\
 & & & & & & & & \vdots & & & & & & & & \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1,2 & | & \cdots & | & s-1 & | & s & | & s+1,s+2 & | & \cdots & | & d+1 & | & d+2 &].
\end{array}$$

Lemma 2.16. *If G_{Δ} is tripartite, with partition $\{1,\ldots,s\} \cup \{s+1,\ldots,d+1\} \cup \{d+2\}$, where $s \geqslant 2$ and $d \geqslant 4$, then h^{Δ} is a pure O-sequence. It equals $f(\Gamma)$, where Γ is the pure order ideal obtained by applying Lemma 2.1 to*

$$\Gamma' = \Gamma_0(d, d+1, (a_1,\ldots,a_{d+1}))$$

and

$$\Gamma'' = \Gamma_0(s-1, s, (a_1,\ldots,a_s)) * \Gamma_0(d+1-s, d+2-s, (a_{s+1},\ldots,a_{d+2})).$$

Proof. Without loss of generality, assume $a_s \leqslant a_{s+1} \leqslant \ldots, a_{d+1}$. The matroid $\Delta \setminus A_{d+2}$ equals $\Delta_0(d, d+1, (a_1,\ldots,a_{d+1}))$, so $\Gamma' = \Gamma_0(d, d+1, (a_1,\ldots,a_{d+1}))$. The matroid $\mathrm{link}_{\Delta_0} A_{d+2}$ corresponds to the bipartite graph from Proposition 2.14, thus Γ'' can be chosen as in the statement and Example 2.15. To apply Lemma 2.1, we check d-compatibility of the generators of Γ' and Γ''. Let $\mathrm{P} = [1|\cdots|i,i+1|\cdots|d|d+1]$ be a generator of Γ'.

(i) If $i \leqslant s-1$, then choose $\mathrm{Q} = [1|\cdots|i,i+1|\cdots|s|s+1|\cdots|d,d+1]$ and $\mathrm{P} \leqslant^d_{\mathbf{a}} \mathrm{Q}$ for any $\mathbf{a}$.

(ii) If $s \leqslant i \leqslant d$, then choose $\mathrm{Q} = [1|\cdots|s-1,s|s+1|\cdots|i+1,i+2|\cdots|d+1]$. For $j < s$, the inequality of the jth entries is clear. For $j \geqslant s$, and $j \neq i$ the a_j are again ordered, because we assume that $a_j \leqslant a_{j+1}$ whenever $j \geqslant s$. Their ith entries correspond to $\{i,i+1\}$ and $\{i+1,i+2\}$, thus as also $a_i \leqslant a_{i+2}$ we conclude.

(iii) If $i = d+1$, then $[1|2|\cdots|d-1|\widehat{d,d+1}] \leqslant_{\mathbf{a}} [1|2|\cdots|d-1,d|\widehat{d+1}]$ for any $\mathbf{a}$ and we conclude by the previous case. $\square$

Example 2.8 reproduced the above construction in the case $d = s = 2$. We are now ready to prove the third and most complicated case.

PROPOSITION 2.17. *If G_Δ is q-partite with $q \geqslant 3$ and has at least two parallel classes of cardinality at least 2, then the h-vector h^Δ is a pure O-sequence.*

Proof. The proof is a repeated application of Lemma 2.1 with the tripartite graph of Lemma 2.16 as the base case. This is possible because of the two parallel classes of cardinality at least 2. Order the vertices of G_Δ such that each parallel class contains consecutive vertices. With this convention, there are only two cases to consider:

Case 1: $d+2$ is parallel to $d+1$ in G_Δ.
Case 2: $d+2$ is not parallel to any vertex is G_Δ.

We use the notation of Lemma 2.1 for Γ' and Γ''.

Case 1. Let $\{r,\ldots,d+1,d+2\}$ be the parallel class of $d+1$ in G_Δ. By Remark 2.11, $\Delta \setminus A_{d+2} = \Delta_{d+2-r}(d,d+1,(a_1,\ldots,a_{d+1}))$, we can choose $\Gamma' = \Gamma_{d+2-r}(d,d+1,(a_1,\ldots,a_{d+1}))$. The matroid $\mathrm{link}_\Delta A_{d+2}$ corresponds to $G_\Delta \setminus \{d+2\}$, thus by the inductive hypothesis there exists an order ideal Γ'' such that $h^{\mathrm{link}_\Delta A_{d+2}} = f(\Gamma'')$. We may also assume that Γ'' was obtained by a repeated application of Lemma 2.1, and thus among its generators has:

$$[1|2|\cdots|r-2,r-1|r|\cdots|d,d+1],\ldots,[1,2|3|\cdots|r-1|r|\cdots|d,d+1].$$

These generators appear from generators of the Γ' at the previous step because $\mathrm{link}_{G_\Delta}(d+1)$ is isomorphic to $\mathrm{link}_{G_\Delta}(d+2)$. Compatibility is easy to confirm.

Case 2. Let $\{r,\ldots,d+1\}$ be the parallel class of $d+1$ in G_Δ. Define a permutation σ of the vertices of $G_\Delta \setminus \{d+2\}$. In order to not complicate notation more than necessary, do this inductively on the parallel classes. The first two parallel classes remain unchanged. For every other parallel, reverse the order of its vertices. More precisely, assume for every $i < r$ that σ is already defined. For every $j \in \{r,\ldots,d+1\}$, set $\sigma(j) = r+d+1-j$. As $d+2$ is not parallel to any vertex in G_Δ, Remark 2.11 implies that the deletion $\Delta \setminus A_{d+2}$ is $\Delta_0(d,d+1,(a_1,\ldots,a_{d+1}))$. Now use [**13**, Theorem 3.5] with the vertices permuted by σ. That is, we have $h^{\Delta\setminus A_{d+2}} = f(\Gamma')$, with Γ' generated by

$$\begin{array}{lcccccccccccccccccc}
[& 1 & | & 2 & | & \cdots & | & m & | & d+1 & | & d & | & \cdots & | & r+1,r &] \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1 & | & 2 & | & \cdots & | & m & | & d+1,d & | & d-1 & | & \cdots & | & r &] \\
[& 1 & | & 2 & | & \cdots & | & m,d+1 & | & d & | & d-1 & | & \cdots & | & r &] \\
 & & & & & & & & \vdots & & & & & & & & \\
[& 1,2 & | & 3 & | & \cdots & | & d+1 & | & d & | & d-1 & | & \cdots & | & r &],
\end{array}$$

for some m which plays no role in the proof. Inductively construct Γ'' such that $h^{\mathrm{link}_\Delta A_{d+2}} = f(\Gamma'')$. Assume that Γ'' was constructed using the same strategy of permuting and applying Lemma 2.1 just with $(r-1)$-compatibility. For each $j = r+1,\ldots,d+1$, there are $r-1$ generators of Γ'' which have been added at the jth step. This is due to the fact that the simplification of $\Delta|_{A_1,\ldots,A_{j-1}}$ is dual to the discrete matroid on $j-1$ vertices with $j-r$ loops, thus its h-vector is obtained from $\Gamma_{j-r}(j-1,j,(a_{\sigma(1)},\ldots,a_{\sigma(j)}))$. After applying the gluing from Lemma 2.1, the generators are

$$\begin{array}{cccccccccccccccc} [& 1 & | & 2 & | & \cdots & | & m', m & | & d+1 & | & \cdots & | & j, j-1 & | & \cdots & | & r &] \\ & & & & & & & & & \vdots & & & & & & & & & \\ [& 1,2 & | & 3 & | & \cdots & | & m & | & d+1 & | & \cdots & | & j, j-1 & | & \cdots & | & r &], \end{array}$$

where m and m' depend on the cardinality of the parallel class of $r-1$ in G_Δ. Their precise description is not needed, as they take the same values for both Γ'' and Γ'.

To check $(r-1)$-compatibility, let $\mathrm{P} = [1|2|\cdots|\sigma(i), \sigma(i+1)|\cdots|r]$ be a generator of Γ'. If $i < r-1$, then choose Q among the generators added at the $(d+1)$th step, namely

$$\mathrm{Q} = [1|2|\cdots|\sigma(i), \sigma(i+1)|\cdots|m|d+1, d|\cdots|r].$$

If $i > r-1$, then choose Q among the generators added at the $\sigma(i)$th step, namely

$$\mathrm{Q} = [1|2|\cdots|m', m|d+1|\cdots|\sigma(i), \sigma(i+1)|\cdots|r].$$

It is easy to see that in both cases $\mathrm{P} \leqslant_{\mathbf{a}}^{r-1} \mathrm{Q}$ for any vector $\mathbf{a}$. Finally, the proof of Case 1 works identically also if σ is applied to the inductive hypothesis. $\square$

Propositions 2.12, 2.14, and 2.17, together with the $(d+1)$-partite case [**13**, Corollary 3.9], imply the main theorem of this section.

THEOREM 2.18. *If Δ is a rank d matroid with at most $d+2$ parallel classes, then the h-vector of the quotient by its cover ideal is a pure O-sequence.*

3. *Small type*

If $h^\Delta = h_{\Delta^c}$ is the h-vector of the cover ideal of a matroid Δ, then its last entry is the Cohen–Macaulay type of $\Bbbk[\Delta^c]$. If it is small, then the parallel classes of the matroid must be few thanks to [**13**, Remark 4.4]: Precisely, if a matroid is of rank d and has p parallel classes, then its type is at least $p-d+1$. Theorem 3.3 exploits this fact to prove that h^Δ is a pure O-sequence whenever the type is at most 5. We start with a proposition that shows that among the simple matroids there is only one of rank d with p parallel classes and whose type is $p-d+1$.

PROPOSITION 3.1. *Let Δ be a p-partite matroid of rank d. Then* $\operatorname{type}(S/J(\Delta)) = p-d+1$ *if and only if* ${}^{\mathrm{si}}\Delta = \Delta_{d-2}(d, p, \mathbf{1})$.

Proof. By [**13**, Proposition 2.8], we can assume that Δ is simple. Constantinescu and Varbaro [**13**, Remark 4.4] show that $\operatorname{type}(S/J(\Delta)) \geqslant p-d+1$, and equality holds if $\Delta = \Delta_{d-2}(d, p, \mathbf{1})$. Assume that Δ satisfies $\operatorname{type}(S/J(\Delta)) = p-d+1$. The proof is by induction on $p-d$. The base case is when $d = p$ in which case ${}^{\mathrm{si}}\Delta$ is a simplex. Now assume that $p-d$ is positive. Without loss of generality, assume that the vertex p is not a cone point (otherwise relabel the vertices). By [**13**, Remark 1.7], we have

$$h_k^\Delta = h_{k-1}^{\Delta\setminus p} + h_k^{\operatorname{link}_\Delta p} \quad \forall\, k \in \mathbb{Z}.$$

Again by [**13**, Remark 4.4] and since $\operatorname{type}(S/J(\Delta)) = p-d+1$, we get $\operatorname{type}(S/J(\Delta \setminus p)) = p-d$ and $\operatorname{type}(S/J(\operatorname{link}_\Delta p)) = 1$. The matroid $\operatorname{link}_\Delta p$ is $(d-1)$-partite and, by the induction hypothesis, $\Delta \setminus p = \Delta_{d-2}(d, p-1, \mathbf{1})$. After potentially relabeling the vertices, $\{1, 2, \ldots, d-2, i, j\}$ is a face of Δ for all $i, j \in \{d-1, \ldots, p-1\}$. If $\{1, 2, \ldots, d-2, p\}$ was not a face of Δ, then there is some $k \in \{1, \ldots, d-2\}$ such that $\{1, \ldots, \hat{k}, \ldots, d-2, i, j, p\}$ is a face of Δ for all i

and j in $\{d-1,\dots,p-1\}$. This would imply that $\{i,j\} \in \operatorname{link}_\Delta p$ for all $i,j \in \{1,\dots,p-1\} \setminus \{k\}$ and $\operatorname{link}_\Delta p$ would be $(p-2)$-partite, a contradiction. Therefore, $\{1,2,\dots,d-2,p\}$ is a face of Δ. We now show that, for fixed $i \in \{d-1,\dots,p-1\}$, the set $\{i,k\}$ is a face of $\operatorname{link}_\Delta p$ for all $k \in \{1,\dots,d-2\}$. If not, then $\{1,\dots,d-2,j,p\}$ is a facet of Δ for all $j \in \{d-1,\dots,p-1\} \setminus \{i\}$. Pick $r,s \in \{d-1,\dots,p-1\} \setminus \{i\}$. Certainly, $B = \{1,\dots,d-2,r,s\}$ is a facet of Δ. Since i is parallel to some $k \in \{1,\dots,d-2\}$ (in $\operatorname{link}_\Delta p$), also $B' = (\{1,\dots,d-2,r,p\} \setminus \{k\}) \cup \{i\}$ is a facet of Δ. Removing k from B, the only way to satisfy basis exchange among B and B' is that $\{r,s,p\}$ is a face of Δ. In this case, however, $\operatorname{link}_\Delta p$ would be d-partite, since the restriction of its 1-skeleton to the vertices $\{1,\dots,d-2,r,s\}$ would be a complete graph. □

Remark 3.2. Theorem 4.3 in [**13**] says that $h^{\Delta_{d-2}(d,p,\mathbf{1})}$ is a componentwise lower bound for all simple matroids of rank d on p vertices.

Theorem 3.3. *Let Δ be a matroid and $h^\Delta = (1,h_1,\dots,h_s)$ be its h-vector. If $h_s \leqslant 5$, then h^Δ is a pure O-sequence.*

Remark 3.4. By duality, Theorem 3.3 also holds for Stanley–Reisner ideals.

Proof of Theorem 3.3. By [**13**, Remark 4.4], $\operatorname{type}(S/J(\Delta)) \geqslant p-d+1$ which in our case implies $p \leqslant d+4$. The cases $p=d$ and $p=d+1$ are trivial, and $p=d+2$ is the content of Theorem 2.18. By Proposition 3.1, if $p=d+4$, then ${}^{\mathrm{si}}\Delta = \Delta_{d-2}(d,p,\mathbf{1})$ and the result follows from [**13**, Theorem 3.7]. It remains to check the case $p=d+3$, however, there are no simple matroids with cover ideal of type 5 such that $p=d+3$. To see this, assume that Δ is such a matroid and consider its dual Δ^c. The simplification ${}^{\mathrm{si}}\Delta$ has the same type, so we can assume that Δ is simple and consequently Δ^c is of rank 3. Let G be the complete q-partite graph which is the 1-skeleton of Δ^c. Since Δ^c is of rank 3, $q \geqslant 3$. Let $b_1 \geqslant \ldots \geqslant b_q$ be the sizes of the parallel classes in G which we can assume ordered non-increasingly. Let $h^{\Delta^c} = (1,h_1,h_2,5)$ be the h-vector. By the Brown–Colbourn inequalities [**7**, Theorem 3.1], $1-h_1+h_2 \leqslant 5$. If $n \leqslant d+3$ is the number of vertices of G and e the number of edges, then $h_1 = n-3$ and $h_2 = 3-2n+e$. It follows that $e \leqslant 3n-2$. Now, if $q=3$, then $b_i \geqslant 3$ for $i=1,\dots,q$ and $e > 3n-2$. If $q=4$, then $b_i \geqslant 2$ for $i=1,\dots,q$, except for one graph in which $b_4 = 1$ and $b_2 = b_3 = b_4 = 2$. If $q=5$, there are five possible graphs. If $q=6$, then K_6 the complete graph is the only possible graph. When the graph is fixed, the h-vector of Δ^c is fixed. Table 1 summarizes the possible graphs and their h-vectors.

Using the database of Mayhew and Royle [**31**], a simple for-loop in Sage enumerates all matroids of rank 3, filters those with the given h-vectors, computes their duals, and confirms that none is simple. □

Table 1. *Possible q-partite graphs in the proof of Theorem* 3.3.

q	$(b_1,\dots,b_q)$	h^Δ
4	$(2,2,2,1)$	$(1,4,7,5)$
5	$(1,1,1,1,1)$	$(1,2,3,5)$
5	$(2,1,1,1,1)$	$(1,3,5,5)$
5	$(2,2,1,1,1)$	$(1,4,8,5)$
5	$(3,1,1,1,1)$	$(1,4,7,5)$
5	$(4,1,1,1,1)$	$(1,5,9,5)$
6	$(1,1,1,1,1,1)$	$(1,3,6,5)$

REMARK 3.5. The matroid $\Delta_{d-2}(d, p, \mathbf{1})$ is the only matroid of type t which satisfies $p = d + t - 1$ and in the proof of Theorem 3.3 we showed that, if $t = 5$, then there is no matroid of type 5 such that $p = d + 3$. It would be interesting to understand for which t there is a such a gap in the allowable number of parallelism classes.

4. *The search for counterexamples*

As soon as the number of variables d, the socle degree s, and the type t are fixed, one can enumerate all pure O-sequences with these characteristics. A pure order ideal with these data is generated by t monomials of degree s. Let $N_{d,s} = \binom{s+(d-1)}{d-1}$ be the number of monomials of degree s in d variables. *A priori*, there are $\binom{N_{d,s}}{t}$ generating sets of order ideals to consider and our program loops over these, computing their f-vectors. Naturally, many of those socles will be equivalent after relabeling the variables, or have the same f-vector even if they are not equivalent. One may hope to reduce the number of combinations by exploiting this symmetry. However, it is not clear how to do so. Checking if, after permuting the variables, two socles are equivalent is computationally more expensive than just computing the f-vectors of the order ideals they generate. One shortcut that is easy to implement is to require the lexicographically first monomial in each socle to have weakly increasing exponent vector. This can be achieved by a permutation of the variables and is quick to check. Further improvements are possible if one is not interested in all pure O-sequences, but just wants to check a particular example. The computation of the face numbers of an order ideal descends degree by degree. In each step, the program searches for monomials that divide the given monomials in the previous degree. If a candidate h-vector is given, then one can stop the degree descent as soon as there is disagreement between the candidate vector and the number of monomials in the current degree. Our software implements all of these shortcuts.

EXAMPLE 4.1. By Theorem 3.3 and [**15**], any candidate counterexample for Stanley's conjecture must be on at least ten vertices and of Cohen–Macaulay type 6. Assume that Δ is of rank 4. For h-vectors of cover ideals, checking an example with these data amounts to enumerating order ideals generated by six monomials of degree 6 in four variables. Our implementation handles approximately 30 000 order ideals per second on a standard laptop. Checking all $\binom{84}{6} = 406\,481\,544$ potential socles would take approximately 4 h. However, this number grows quickly. If a counterexample exists and was of rank 5 on twelve vertices and type 7, then a back-of-the-envelope calculation estimates the computational time as around 173 CPU years.

Lemma 2.1 inspires a method to search for pure order ideals.

METHOD 4.2. Let Δ be a p-partite matroid of rank d with parallel classes $A_1, \ldots, A_p$ which we may choose ordered such that $A_1 \cdots A_d \in \Delta$, that is, $\{v_1, \ldots, v_d\}$ is a facet whenever $v_i \in A_i$ for all $i = 1, \ldots, d$. To find a pure order ideal whose f-vector equals h^Δ, instead of enumeration, one may proceed as follows.

(1) For each $i \in \{d, \ldots, p\}$, let G_i be the set of generators of $\Gamma_0(d-1, i-1, (a_1, \ldots, a_{i-1}))$.
(2) Compute c_i, the last entry of the h-vector of $\operatorname{link}_{\Delta|_{A_1 \cup \ldots \cup A_{i-1}}} A_i$.
(3) For every $i \in \{d, \ldots, p\}$, choose a c_i-subset H_i of G_i.
(4) Define $\Gamma = \langle \bar{H}_d \cup \ldots \cup \bar{H}_p \rangle$, where the collection of partitions $\bar{H}_j$ is obtained by adding the set $\{j, \ldots, p\}$ to every $(d-1)$-partition of $[j-1]$ contained in H_j.
(5) Check if $h^\Delta = f(\Gamma)$.

The gist of this method is, instead of searching all socles, to search only order ideal generators among the monomials that could potentially arise from a repeated application of Lemma 2.1. The method starts at the complete matroid $\Delta|_{A_1\cup\ldots\cup A_d}$ and reconstructs Δ by gluing the remaining parallel classes. In this process, it mimics the construction of Lemma 2.1 in many different ways. The compatibility condition is never checked. It is faster to just confront the f-vector of the final result with h^Δ.

The choice of ordering of the A_i fixes the order in which Lemma 2.1 would be applied (and one may try different orderings). Step (1) creates lists of candidates for the generators of Γ'' (in the notation of the lemma). Steps (2) and (3) enumerate the sets of order ideal generators that may result from the choices. Finally, Step (4) implements the gluing in Lemma 2.1. Evidently, if the procedure does not find an order ideal whose f-vector is h^Δ, then we have not found a counterexample.

Example 4.3. In specific examples, the number of orderings of the parallel classes can be reduced using symmetries of the matroid. For instance, in Example 2.8 the pairs (A_1, A_2) and (A_3, A_4), and also the classes in each pair, could be exchanged. Given that $A_1A_2A_5$ and $A_3A_4A_5$ are not in Δ, the only orderings to check in this case are A_1, A_2, A_3, A_4, A_5 and A_1, A_3, A_5, A_2, A_4.

Example 4.4. Let Δ be the simple rank 4 matroid on eight vertices with the following facets:

$$1235, 1236, 1237, 1238, 1245, 1246, 1247, 1248, 1256, 1257, 1268, 1278, 1345, 1346, 1347,$$
$$1348, 1357, 1358, 1367, 1368, 1456, 1458, 1467, 1478, 1567, 1568, 1578, 1678, 2356, 2357,$$
$$2358, 2456, 2457, 2458, 2568, 2578, 3456, 3457, 3458, 3567, 3568, 4567, 4578, 5678.$$

Precisely, Δ is a series-extension (15 is a cocircuit) of the Fano matroid. The largest example that we tried our method on is the rank 4 matroid $\Delta_{\mathbf{a}}$ on 20 vertices whose simplification is Δ and whose parallel classes have sizes $(1, 2, 3, 4, 1, 3, 4, 2)$. We have

$$h^{\Delta_{\mathbf{a}}} = (1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 112, 116, 111, 96, 70, 40, 14),$$

which means that enumeration of order ideals is entirely pointless. However, using Method 4.2 we found that this vector is a pure O-sequence. It equals the f-vector of the order ideal

$$\Gamma = \langle bc^2d^{13}, bc^6d^9, b^4c^3d^9, bc^{10}d^5, b^8c^3d^5, bc^{12}d^3, b^4c^9d^3,$$
$$a^9b^3c^4, a^5b^9c^2, bc^{15}, a^5b^3c^8, b^{14}c^2, a^2b^{12}c^2, a^2b^{10}c^4\rangle.$$

The Artinian monomial level algebra with $\Bbbk$-basis Γ is $\Bbbk[a, b, c, d]/I$ where

$$I = (a^{10}, a^6b^4, a^3b^{10}, ab^{13}, b^{15}, a^3b^4c^3, b^{11}c^3, a^6c^5, ab^4c^5, b^5c^5, ac^9, b^2c^{10},$$
$$c^{16}, ad, b^9d, b^5c^4d, c^{13}d, b^2c^4d^4, c^{11}d^4, b^5d^6, c^7d^6, b^2d^{10}, c^3d^{10}, d^{14}).$$

Remark 4.5. The number of different h-vectors of coloop-free matroids is equal to the number of different f-vectors of coloop-free matroids. Since matroids are very particular pure multicomplexes, the number of their f-vectors is smaller than the number of pure O-sequences (which are f-vectors of pure multicomplexes). Therefore, it seems plausible that the probability of finding a pure O-sequence equal to the h-vector of a matroid tends to zero as the parameters grow. This limits the usefulness of random search for order ideals in larger examples.

Acknowledgements. We are grateful to Fondazione Bruno Kessler for funding a two-week stay at CIRM Trento where this project was started, and to MSRI, where this paper took shape. We thank Alex Fink for valuable suggestions and comments.

References

1. A. Aramova, J. Herzog and T. Hibi, 'Ideals with stable Betti numbers', *Adv. Math.* 152 (2000) 72–77.
2. D. Bayer, H. Charalambous and S. Popescu, 'Extremal Betti numbers and applications to monomial ideals', *J. Algebra* 221 (1999) 497–512.
3. D. Bayer and M. Stillman, 'A criterion for detecting m-regularity', *Invent. Math.* 87 (1987) 1–11.
4. L. J. Billera and C. W. Lee, 'Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes', *Bull. Amer. Math. Soc* 2 (1980) 181–185.
5. L. J. Billera and C. W. Lee, 'A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes', *J. Combin. Theory Ser. A* 31 (1981) 237–255.
6. M. Boij, J. Migliore, R. Mirò-Roig, U. Nagel and F. Zanello, 'On the shape of a pure o-sequence', *Mem. Amer. Math. Soc.*, Preprint, 2010, http://arxiv.org/abs/1003.3825.
7. J. I. Brown and C. J. Colbourn, 'Roots of the reliability polynomials', *SIAM J. Discrete Math.* 5 (1992) 571–585.
8. T. Brylawski, 'Constructions', *Theory of matroids*, Encyclopedia of Mathematics and Its Applications 26 (ed. N. White; Cambridge University Press, Cambridge, 1986) 127–223.
9. G. Caviglia and M. Varbaro, 'Componentwise regularity (I)', Preprint, 2013, arXiv:1308.2034.
10. M. K. Chari, 'Two decompositions in topological combinatorics with applications to matroid complexes', *Trans. Amer. Math. Soc.* 349 (1997) 3925–3943.
11. CoCoATeam, CoCoA: *a system for doing computations in commutative algebra*, `cocoa.dima.unige.it`.
12. A. Conca and J. Herzog, 'Castelnuovo-Mumford regularity of products of ideals', *Collect. Math* 54 (2003) 137–152.
13. A. Constantinescu and M. Varbaro, 'h-vectors of matroid complexes', Preprint, 2012, arXiv:1212.3226.
14. C. DeConcini and C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Universitext (Springer, New York, 2011).
15. J. DeLoera, Y. Kemper and S. Klee, 'h-vectors of small matroid complexes', *Electron. J. Combin.* 19 (2012) P14.
16. H. Derksen and A. Fink, 'Valuative invariants for polymatroids', *Adv. Math.* 225 (2010) 1840–1892.
17. D. Eisenbud and S. Goto, 'Linear free resolutions and minimal multiplicity', *J. Algebra* 88 (1984) 89–133.
18. S. Eliahou and M. Kervaire, 'Minimal resolutions of some monomial ideals', *J. Algebra* 129 (1990) 1–25.
19. A. Fink, 'Matroid polytope subdivisions and valuations', PhD Thesis, University of California, Berkeley, 2010.
20. D. Grayson and M. Stillman, 'Macaulay2, a software system for research in algebraic geometry', http://www.math.uiuc.edu/Macaulay2/.
21. H. T. Hà, E. Stokes and F. Zanello, 'Pure O-sequences and matroid h-vectors', Preprint, 2010, arXiv:1006.0325.
22. J. Herzog and T. Hibi, 'Componentwise linear ideals', *Nagoya Math. J.* 153 (1999) 141–153.
23. J. Herzog and T. Hibi, *Monomial ideals*, Graduate Texts in Mathematics 260 (Springer, London, 2011).
24. T. Hibi, 'What can be said about pure O-sequences?', *J. Combin. Theory Ser. A* 50 (1989) 319–322.
25. O. Holtz and A. Ron, 'Zonotopal algebra', *Adv. Math.* 227 (2011) 847–894.
26. J. Huh, 'h-vectors of matroids and logarithmic concavity', Preprint, 2012, arXiv:1201.2915.
27. J. Huh, 'Milnor numbers of projective hyper surfaces and the chromatic polynomial of graphs', *J. Amer. Math. Soc.* 25 (2012) 907–927.
28. G. Kalai, 'Many triangulated spheres', *Discrete Comput. Geom.* 3 (1988) 1–14.
29. M. Lenz, 'The f-vector of a representable-matroid complex is log-concave', *Adv. Appl. Math.* 51 (2013) 543–545.
30. F. S. Macaulay, 'Some properties of enumeration in the theory of modular systems', *Proc. London Math. Soc.* 26 (1927) 531–555.
31. D. Mayhew and G. F. Royle, 'Matroids with nine elements', *J. Combin. Theory Ser. B* 98 (2008) 415–431.
32. C. Merino, *The chip firing game and matroid complexes*, Discrete Mathematics & Theoretical Computer Science Proceedings (AA, Maison Inform. Math. Discrete (MIMD), Paris, 2001) 245–255.
33. C. Merino, S. D. Noble, M. Ramírez-Ibá nez and R. Villarroel-Flores, 'On the structure of the h-vector of a paving matroid', *European J. Combin.* 33 (2012) 787–1799.
34. E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics (Springer, New York, 2005).
35. L. Moci, 'A Tutte polynomial for toric arrangements', *Trans. Amer. Math. Soc.* 364 (2012) 1067–1088.
36. S. Oh, 'Generalized permutohedra, h-vector of cotransversal matroids and pure O-sequences', Preprint, 2010, arXiv:1005.5586.
37. J. Oxley, *Matroid theory*, 2nd edn, Oxford Graduate Texts in Mathematics 21 (Oxford University Press, Oxford, 2011).
38. R. P. Stanley, 'Cohen–Macaulay complexes', *Higher combinatorics* 31 (1977) 51–62.

39. R. P. Stanley, 'The number of faces of a simplicial convex polytope', *Adv. Math.* 35 (1980) 236–238.
40. R. P. Stanley, *Combinatorics and commutative algebra* (Birkhäuser, Boston, 1996).
41. W. A. Stein *et al.* 'Sage mathematics software (Version 5.4.1)', The Sage Development Team, 2012, http://www.sagemath.org.
42. H. Whitney, 'On the abstract properties of linear dependence', *Amer. J. Math.* 57 (1935) 509–533.
43. G. M. Ziegler, 'Lectures on 0/1-polytopes', *Polytopes—combinatorics and computation* (eds G. Kalai and Günter M. Ziegler; Birkhäuser, Boston, 2000) 1–41.

Alexandru Constantinescu
Mathematisches Institut
Freie Universität Berlin
Arnimallee 3
14195 Berlin
Germany

aconstant@math.fu-berlin.de
http://userpage.fu-berlin.de/aconstant

Thomas Kahle
Fakultät für Mathematik
Otto-von-Guericke-Universität
Universitätsplatz 2
39106 Magdeburg
Germany

thomas.kahle@ovgu.de
http://www.thomas-kahle.de

Matteo Varbaro
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
Genova 16146
Italy

varbaro@dima.unige.it
http://www.dima.unige.it/~varbaro/